ECE 536 – Integrated Optics and Optoelectronics
From Lecture 5

Spring 2022

Tu-Th 11:00am-12:20pm

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Time-dependent perturbation theory

Consider a physical system described by a time-independent Hamiltonian (assumed to be discrete and non-degenerate)

$$H_0\, \varphi_n = E_n \varphi_n$$

Suppose that at $t = 0$ a time-dependent perturbation is applied to the system

$$H(t \geq 0) = H_0 + \lambda\, H'$$

where the parameter $\lambda \ll 1$. The system is initially in the state φ_i which is an eigenstate of H_0 with eigenvalue E_i.

We are looking for the first-order approximation of the probability $P_{ij}(t)$ of finding the system in another eigenstate φ_f of H_0 at time t.

The Schrödinger equation is

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{r},t) = H\,\psi(\mathbf{r},t) = (H_0 + \lambda H')\psi(\mathbf{r},t)$$

We assume to know the time-dependent solution for the unperturbed Hamiltonian

$$i\hbar \frac{\partial}{\partial t}\varphi_n(\mathbf{r},t) = H_0\,\varphi_n(\mathbf{r},t)$$

$$\varphi_n(\mathbf{r},t) = \varphi_n(\mathbf{r})e^{-iE_n t/\hbar}$$

Expand $\psi(\mathbf{r},t)$ in terms of the unperturbed eigensolutions

$$\psi(\mathbf{r},t) = \sum_n a_n(t)\,\varphi_n(\mathbf{r})e^{-iE_n t/\hbar}$$

dove $|a_n(t)|^2$ is probability for the electron to be in state n at t

Substitute the expansion in Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\left(\sum_n a_n(t)\,\varphi_n(\mathbf{r})\,e^{-iE_nt/\hbar}\right)$$

$$= (H_0 + \lambda H')\left(\sum_n a_n(t)\,\varphi_n(\mathbf{r})\,e^{-iE_nt/\hbar}\right)$$

First term of equation above

$$i\hbar \sum_n \frac{da_n(t)}{dt}\varphi_n(\mathbf{r})\;e^{-iE_nt/\hbar} + \underbrace{i\hbar \sum_n a_n(t)\frac{d}{dt}\left(\varphi_n(\mathbf{r})\;e^{-iE_nt/\hbar}\right)}_{\boxed{H_0\,\varphi_n(\mathbf{r},t)}}$$

$$\Downarrow$$

$$\sum_n \frac{da_n(t)}{dt}\varphi_n(\mathbf{r})\;e^{-iE_nt/\hbar} = -\frac{i}{\hbar}\sum_n \lambda H'(\mathbf{r},t)a_n(t)\,\varphi_n(\mathbf{r})\;e^{-iE_nt/\hbar}$$

$$\sum_n \frac{da_n(t)}{dt} \varphi_n(\mathbf{r})\ e^{-iE_n t/\hbar} = -\frac{i}{\hbar} \sum_n \lambda H'(\mathbf{r}, t) a_n(t)\ \varphi_n(\mathbf{r})\ e^{-iE_n t/\hbar}$$

Take inner product with $\varphi_m^*(\mathbf{r})$

$$\frac{da_m(t)}{dt} = -\frac{i}{\hbar}\ \lambda \sum_n a_n(t)\ H'_{mn}(t)\ e^{-i(E_m - E_n)t/\hbar}$$

$$H'_{mn}(t) = \int \varphi_m^*(\mathbf{r})\ H'(\mathbf{r}, t) \varphi_n(\mathbf{r})\ \boldsymbol{d^3 r}$$

Now write the coefficients in the form of a power series

$$a_n(t) = a_n^{(0)}(t) + \lambda a_n^{(1)}(t) + \lambda^2 a_n^{(2)}(t) + \cdots$$

We seek the solution to first order in λ.

$$\frac{da_m(t)}{dt} = -\frac{i}{\hbar}\,\lambda \sum_n a_n(t)\, H'_{mn}(t)\, e^{-i(E_m - E_n)t/\hbar}$$

$$a_n(t) = a_n^{(0)}(t) + \lambda a_n^{(1)}(t) + \lambda^2 a_n^{(2)}(t) + \cdots$$

We have

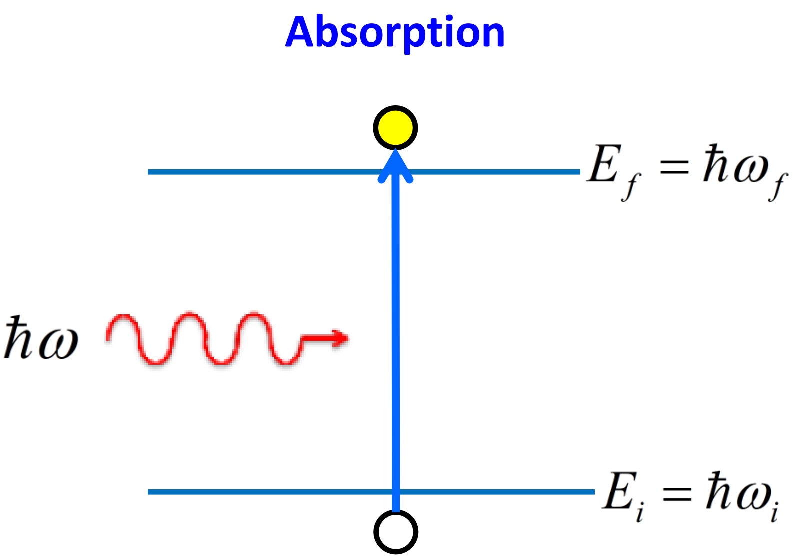
$$\frac{da_m^{(0)}}{dt} = 0$$

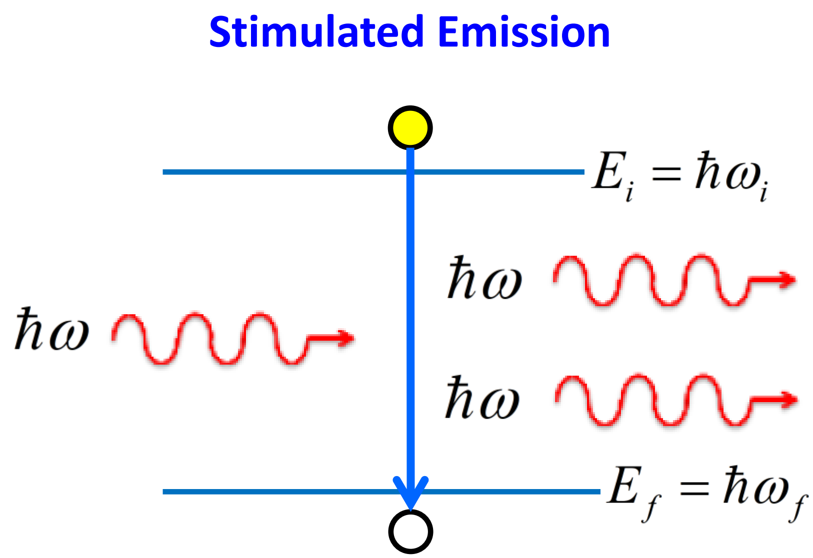
$$\frac{da_m^{(1)}(t)}{dt} = -\frac{i}{\hbar} \sum_n a_n^{(0)}(t)\, H'_{mn}(t)\, e^{-i(E_m - E_n)t/\hbar}$$

$$\frac{da_m^{(2)}(t)}{dt} = -\frac{i}{\hbar} \sum_n a_n^{(1)}(t)\, H'_{mn}(t)\, e^{-i(E_m - E_n)t/\hbar}$$

Fermi Golden Rule

We will use the main results of time-dependent perturbation theory to determine the **transition probability from one state to another**, due to an external perturbation.





The electron is at state " i " initially. The zeroth-order solutions are constant and electron stays in that state in absence of perturbation

$$a_i^{(0)}(t) = 1$$

$$a_m^{(0)}(t) = 0 \qquad m \neq i$$

The first order solution is obtained from

$$\boxed{\omega_{mi} = \frac{E_m - E_i}{\hbar}}$$

$$\frac{d a_m^{(1)}(t)}{dt} = -\frac{i}{\hbar} H'_{mi}(t)\; e^{-i(E_m - E_i)t/\hbar} = -\frac{i}{\hbar} H'_{mi}(t)\; e^{-i\omega_{mi} t}$$

Assume time-dependent perturbation (e.g., photons) with form

$H'(\mathbf{r}, t) = H'_m(\mathbf{r}) e^{-i\omega t} + H'^{+}_m(\mathbf{r}) e^{i\omega t}$

$$H'_{mi}(t) = \int \varphi_m^*(\mathbf{r})\, H'(\mathbf{r}, t) \varphi_i(\mathbf{r})\, d^3\boldsymbol{r} = H'_{mi} e^{-i\omega t} + H'^{+}_{mi} e^{i\omega t}$$

$$\boldsymbol{\omega_{fi} = \frac{E_f - E_i}{\hbar}}$$

Initial state $n = i$

$$\frac{da_m^{(1)}(t)}{dt} = -\frac{i}{\hbar} H'_{mi}(t)\ e^{-i\omega_{mi}t}$$
$$= -\frac{i}{\hbar}\left(H'_{mi}\ e^{i(\omega_{mi}-\omega)t} + H'^{+}_{mi}\ e^{i(\omega_{mi}+\omega)t}\right)$$

Integrate equation between **0** and ***t*** for a final state $m = f$

$$a_f^{(1)}(t) = -\frac{1}{\hbar}\left[H'_{fi}\frac{e^{i(\omega_{fi}-\omega)t} - 1}{\omega_{fi} - \omega} + H'^{+}_{fi}\frac{e^{i(\omega_{fi}+\omega)t} - 1}{\omega_{fi} + \omega}\right]$$

The associated probability is

$$\left|a_f^{(1)}(t)\right|^2 = \frac{1}{\hbar^2}\left[H'_{fi}\frac{e^{i(\omega_{fi}-\omega)t} - 1}{\omega_{fi} - \omega} + H'^{+}_{fi}\frac{e^{i(\omega_{fi}+\omega)t} - 1}{\omega_{fi} + \omega}\right]^2$$

$$\left|a_f^{(1)}(t)\right|^2 = \frac{1}{\hbar^2}\left[H'_{fi}\,\frac{e^{i(\omega_{fi}-\omega)t}-1}{\omega_{fi}-\omega} + H'^{+}_{fi}\,\frac{e^{i(\omega_{fi}+\omega)t}-1}{\omega_{fi}+\omega}\right]^2$$

Using

$$\sin x = \frac{1}{2i}\left(e^{ix}-e^{-ix}\right)$$

$$e^{-i(\omega_{fi}-\omega)t}-1 = 2i\,e^{i\frac{(\omega_{fi}-\omega)t}{2}}\sin\frac{(\omega_{fi}-\omega)t}{2}$$

$$\left|a_f^{(1)}(t)\right|^2 = \frac{4\left|H'_{fi}\right|^2}{\hbar^2}\,\frac{\sin^2\frac{(\omega_{fi}-\omega)t}{2}}{\left(\omega_{fi}-\omega\right)^2} + \frac{4\left|H'^{+}_{fi}\right|^2}{\hbar^2}\,\frac{\sin^2\frac{(\omega_{fi}+\omega)t}{2}}{\left(\omega_{fi}+\omega\right)^2} + \cancel{\cdots}$$

$\boxed{\text{drop cross term}}$

$$\left|a_f^{(1)}(t)\right|^2 = \frac{4\left|H'_{fi}\right|^2}{\hbar^2}\frac{\sin^2\frac{\left(\omega_{fi}-\omega\right)t}{2}}{\left(\omega_{fi}-\omega\right)^2} + \frac{4\left|H'^{+}_{fi}\right|^2}{\hbar^2}\frac{\sin^2\frac{\left(\omega_{fi}+\omega\right)t}{2}}{\left(\omega_{fi}+\omega\right)^2}$$

For a sufficiently long interaction time

$$\frac{\sin^2\left(\frac{x}{2}t\right)}{x^2} \rightarrow \frac{\pi t}{2}\,\delta(x)$$

$$\left|a_f^{(1)}(t)\right|^2 = \frac{2\pi t}{\hbar^2}\left|H'_{fi}\right|^2\delta\left(\omega_{fi}-\omega\right) + \frac{2\pi t}{\hbar^2}\left|H'^{+}_{fi}\right|^2\delta\left(\omega_{fi}+\omega\right)$$

$$\left|a_f^{(1)}(t)\right|^2 = \frac{2\pi t}{\hbar^2}\left|H'_{fi}\right|^2 \delta\left(\omega_{fi} - \omega\right) + \frac{2\pi t}{\hbar^2}\left|H'^{+}_{fi}\right|^2 \delta\left(\omega_{fi} + \omega\right)$$

Using the property $\delta(\hbar\omega) = \delta(\omega)/\hbar$ the transition rate is given by

$$W_{i\to f} = \frac{d}{dt}\left|a_f^{(1)}(t)\right|^2$$

$$W_{i\to f} = \frac{2\pi}{\hbar}\left|H'_{fi}\right|^2 \delta\left(E_f - E_i - \hbar\omega\right) + \frac{2\pi}{\hbar}\left|H'^{+}_{fi}\right|^2 \delta\left(E_f - E_i + \hbar\omega\right)$$

$$\left|a_f^{(1)}(t)\right|^2 = \frac{2\pi t}{\hbar^2}\left|H'_{fi}\right|^2 \delta(\omega_{fi} - \omega) + \frac{2\pi t}{\hbar^2}\left|H'^{+}_{fi}\right|^2 \delta(\omega_{fi} + \omega)$$

Using the property $\delta(\hbar\omega) = \delta(\omega)/\hbar$ the transition rate is given by

$$W_{i\to f} = \frac{d}{dt}\left|a_f^{(1)}(t)\right|^2$$

energy conserving delta functions

$$W_{i\to f} = \frac{2\pi}{\hbar}\left|H'_{fi}\right|^2 \delta(E_f - E_i - \hbar\omega) + \frac{2\pi}{\hbar}\left|H'^{+}_{fi}\right|^2 \delta(E_f - E_i + \hbar\omega)$$

$E_f = E_i + \hbar\omega$

$E_f = E_i - \hbar\omega$

absorption of a photon

emission of a photon

ECE 536 – Integrated Optics and Optoelectronics
Lecture 6 – February 3, 2022

Spring 2022

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Lecture 6 Outline

- Total reflection at a dielectric interface
- Optical waveguides
- The symmetric dielectric slab waveguide
- TE and TM mode behavior
- Effective index

Optical waveguides

Short distance (device and circuit level)

dielectric slab waveguide

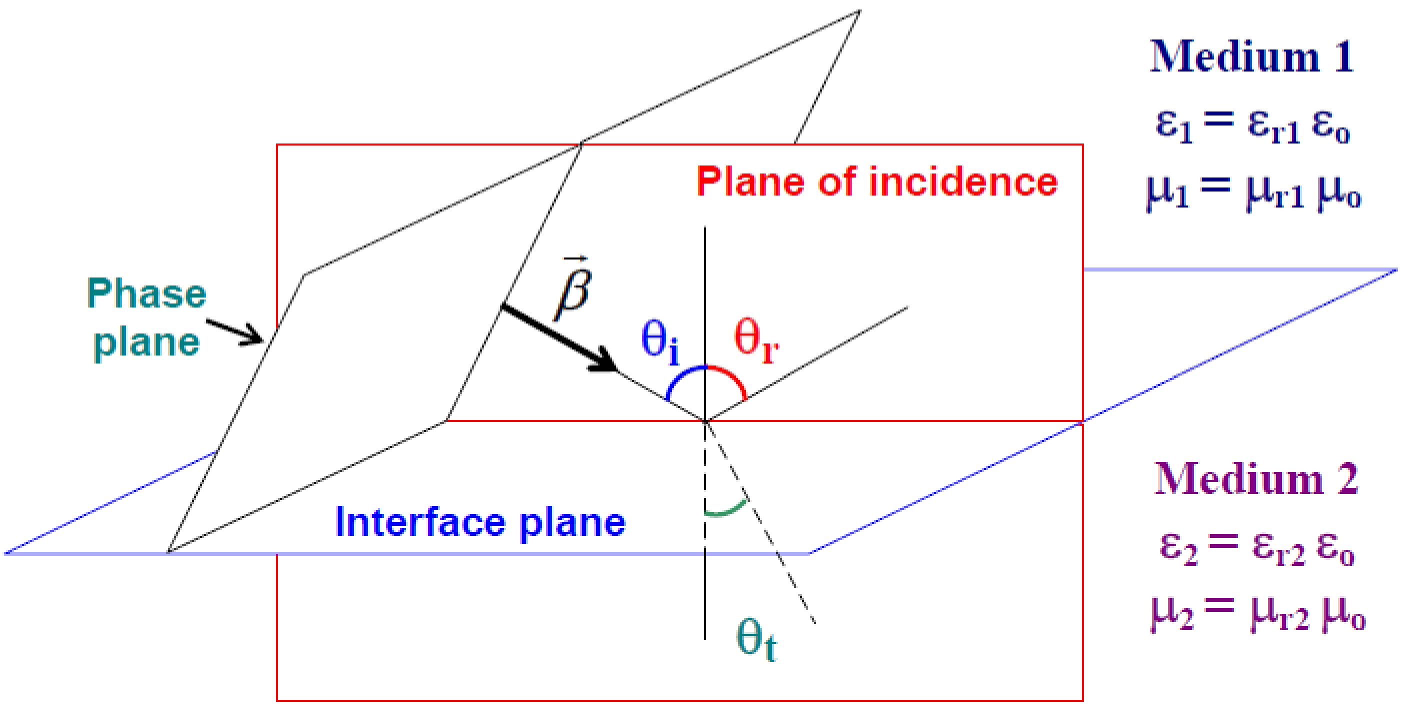
Short and medium distance

multimode optical fiber

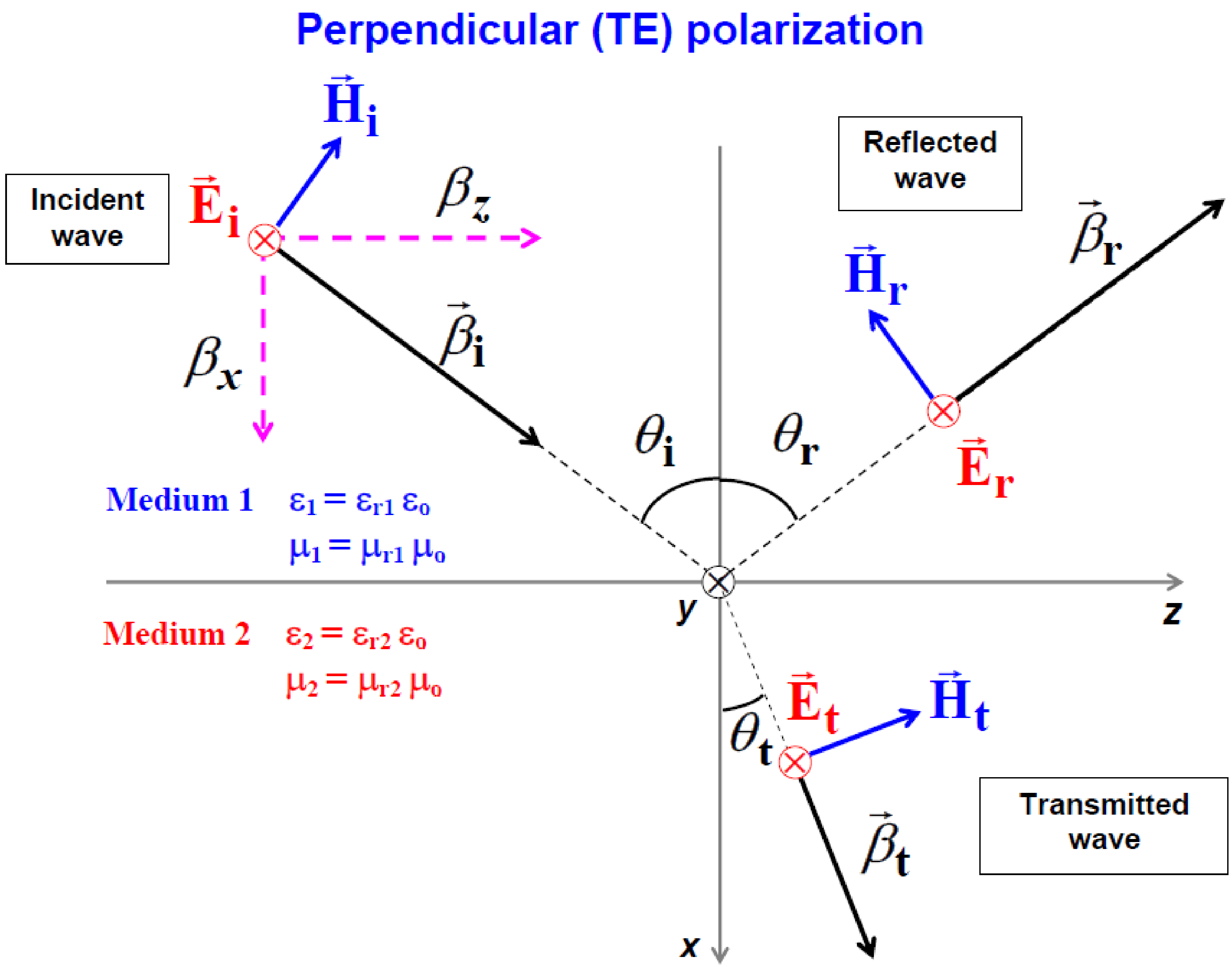
Long distance

monomode optical fiber

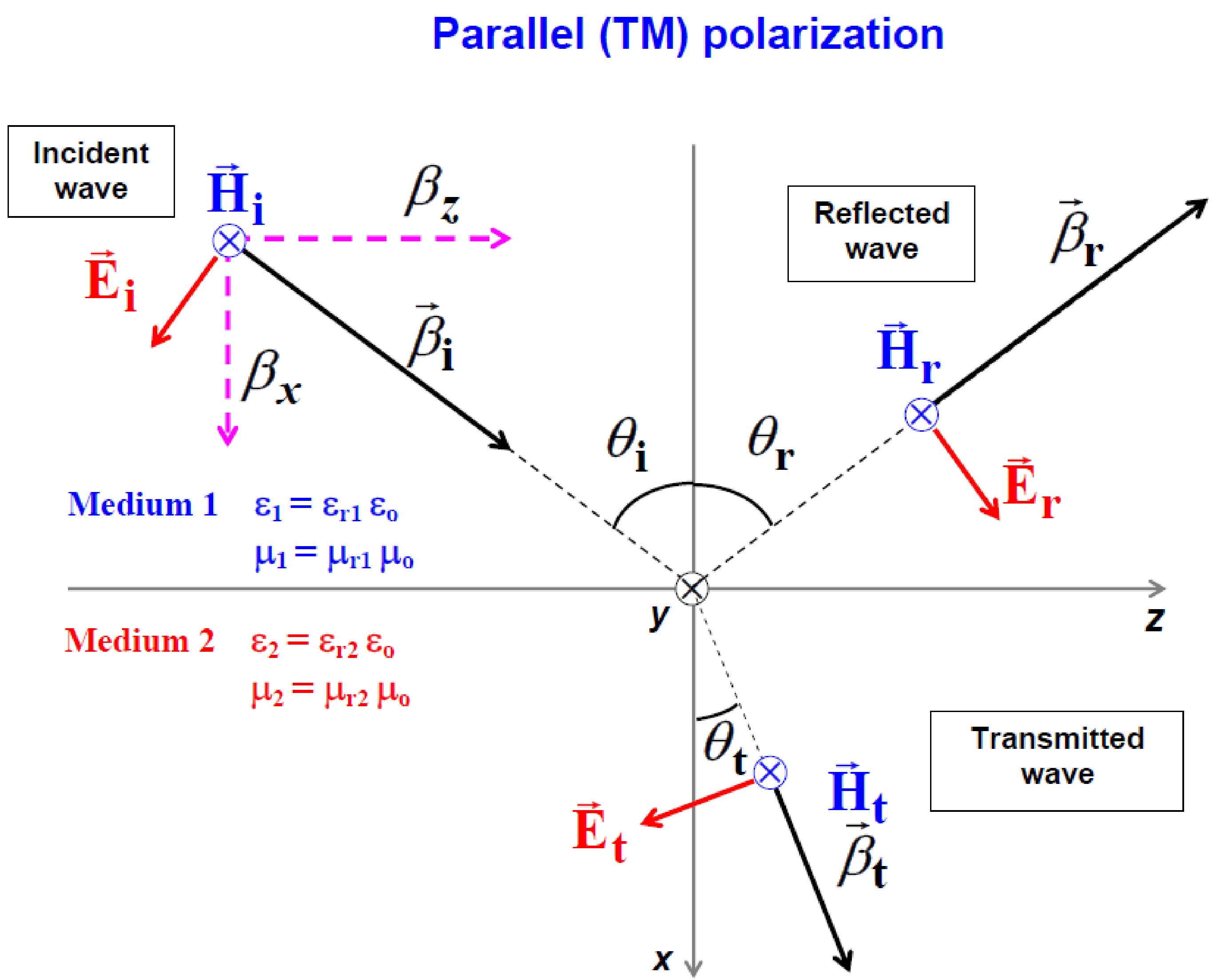
Reflection at dielectric interface



Wave Polarization



Wave Polarization



Angle of refraction

Snell's law

$$\overbrace{\mu_1 = \mu_2 = \mu_o}$$

$$\theta_t = \sin^{-1}\left(\sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} \sin\theta_i\right) = \sin^{-1}\left(\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \sin\theta_i\right)$$

non-magnetic dielectric medium

$$\boxed{\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} \qquad (n = \text{index of refraction})}$$

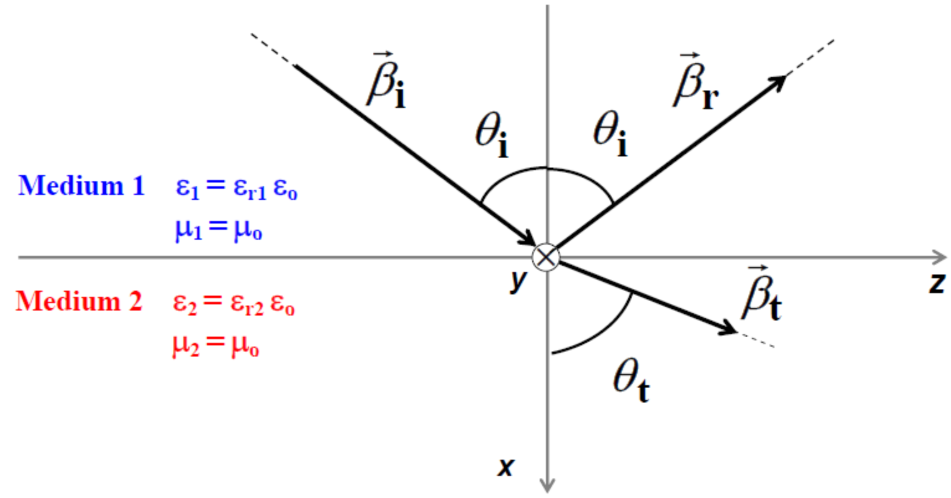
Reflection coefficients for dielectric media

$$\Gamma_\perp(E) = -\Gamma_\perp(H) = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$\Gamma_{||}(E) = -\Gamma_{||}(H) = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

Total reflection

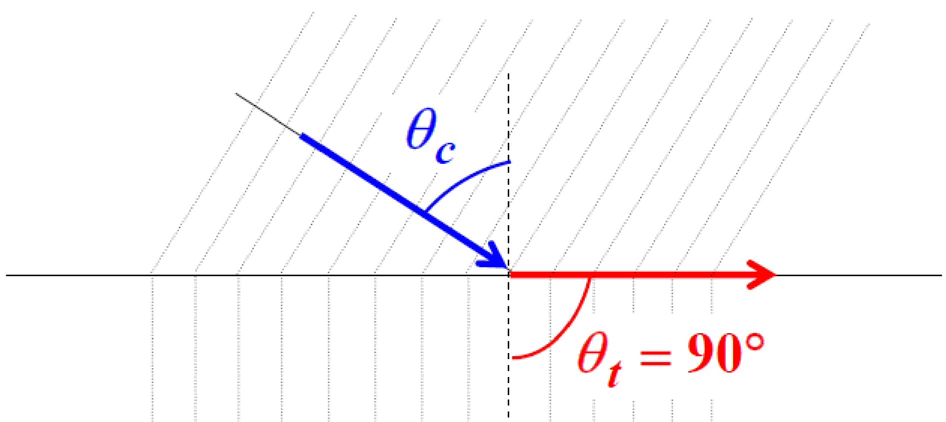
$\varepsilon_2 < \varepsilon_1 \Rightarrow \theta_t > \theta_i$



When

$$\sin\theta_i = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\sin\theta_t = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \Rightarrow \sin\theta_t = 1 \Rightarrow \theta_t = 90^\circ$$

critical angle $\theta_i = \theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$



Total reflection

The reflection and transmission coefficients become complex

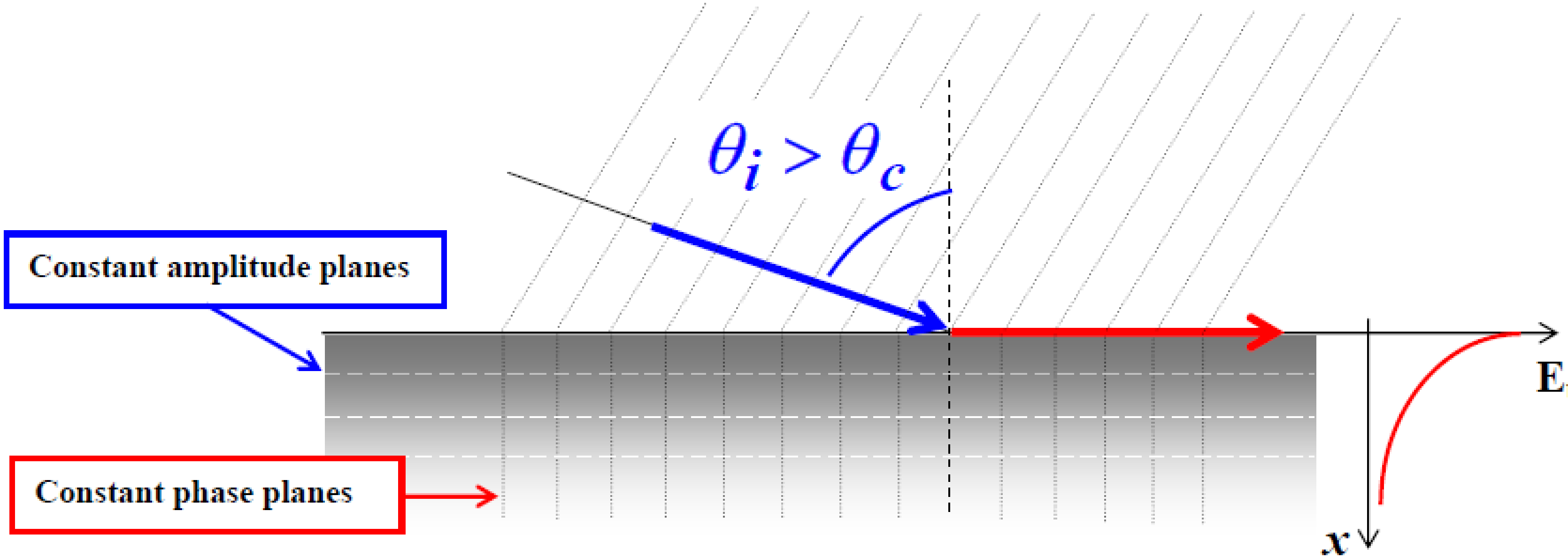
$$\Gamma_{\perp}(E) = -\Gamma_{\perp}(H) = \frac{\cos\theta_i + j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}{\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}$$

$$\Gamma_{||}(E) = -\Gamma_{||}(H) = \frac{-\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}$$

$$\tau_{\perp}(E) = \tau_{\perp}(H)\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} = \frac{2\cos\theta_i}{\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}$$

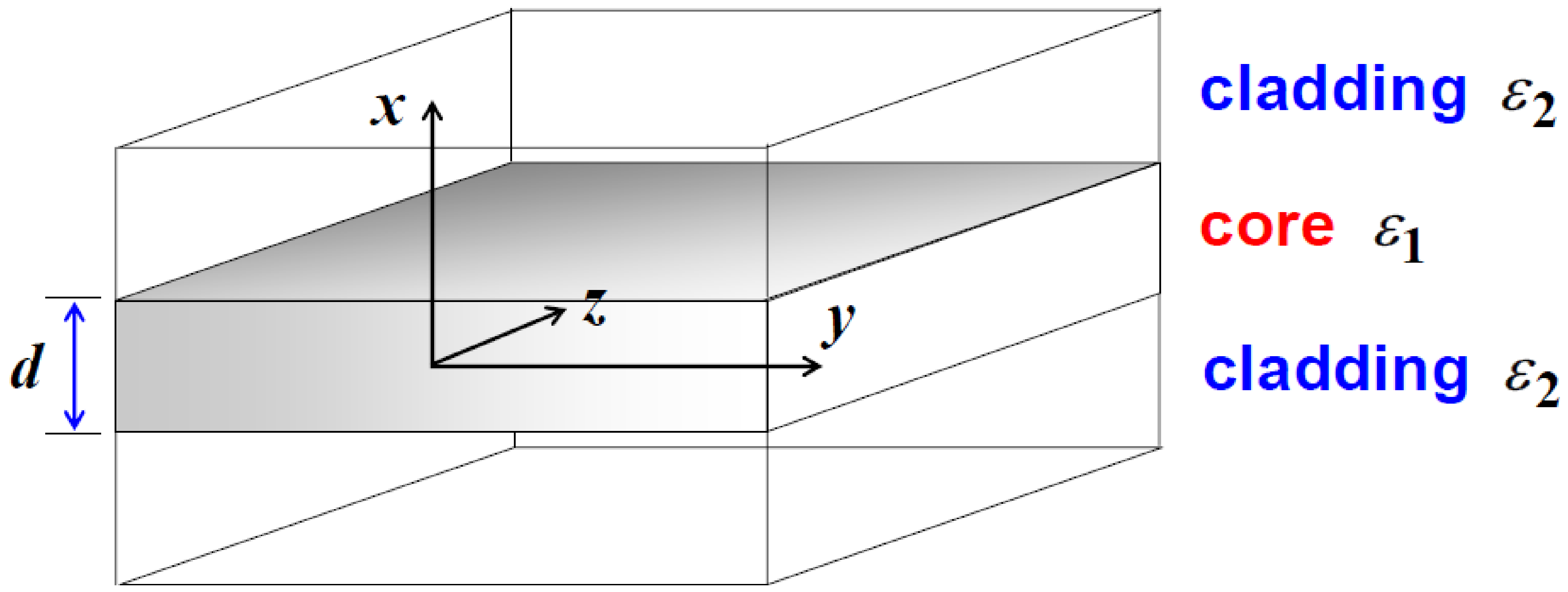
$$\tau_{||}(E) = \tau_{||}(H)\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} = \frac{2\sqrt{\varepsilon_2/\varepsilon_1}\cos\theta_i}{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}$$

Total reflection – surface wave



$$\mathrm{E_t} = E_t\, e^{-j(-j\alpha_t \cdot x)} e^{-j\beta_{iz} \cdot z} = E_t\, e^{-\alpha_t \cdot x} e^{-j\beta_{iz} \cdot z}$$

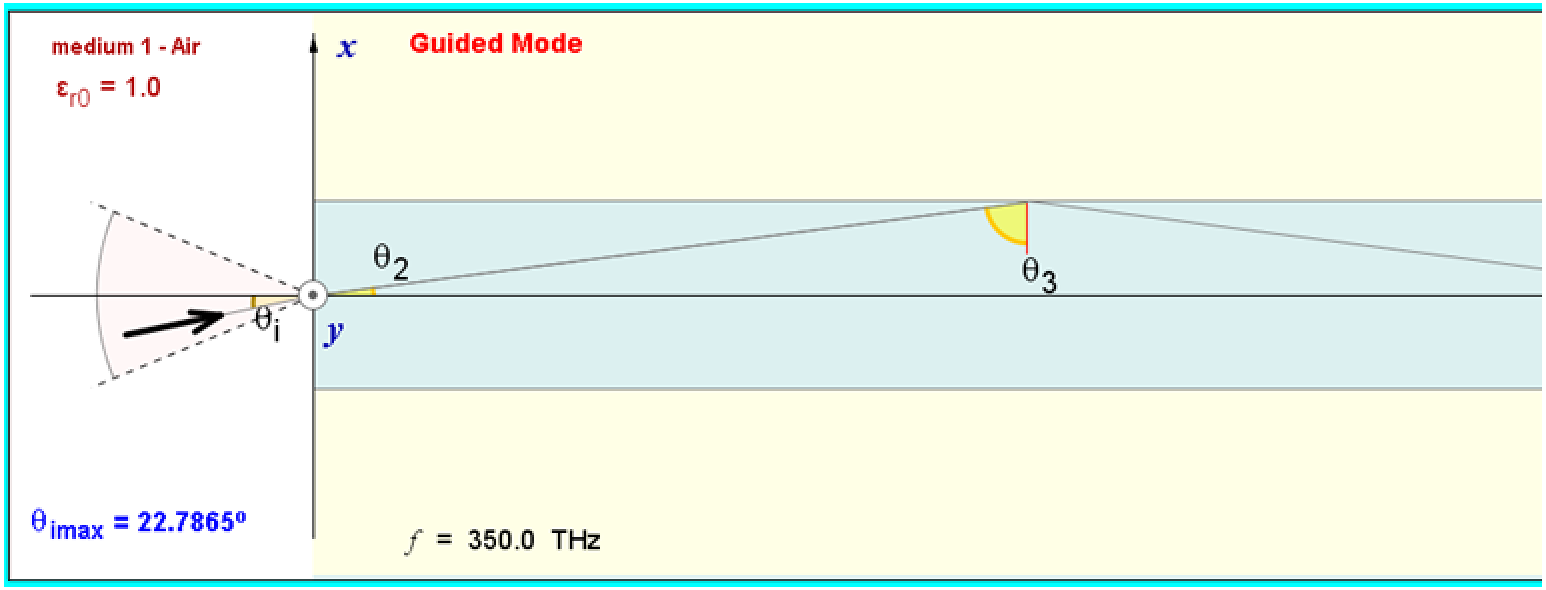
Symmetric dielectric slab waveguide

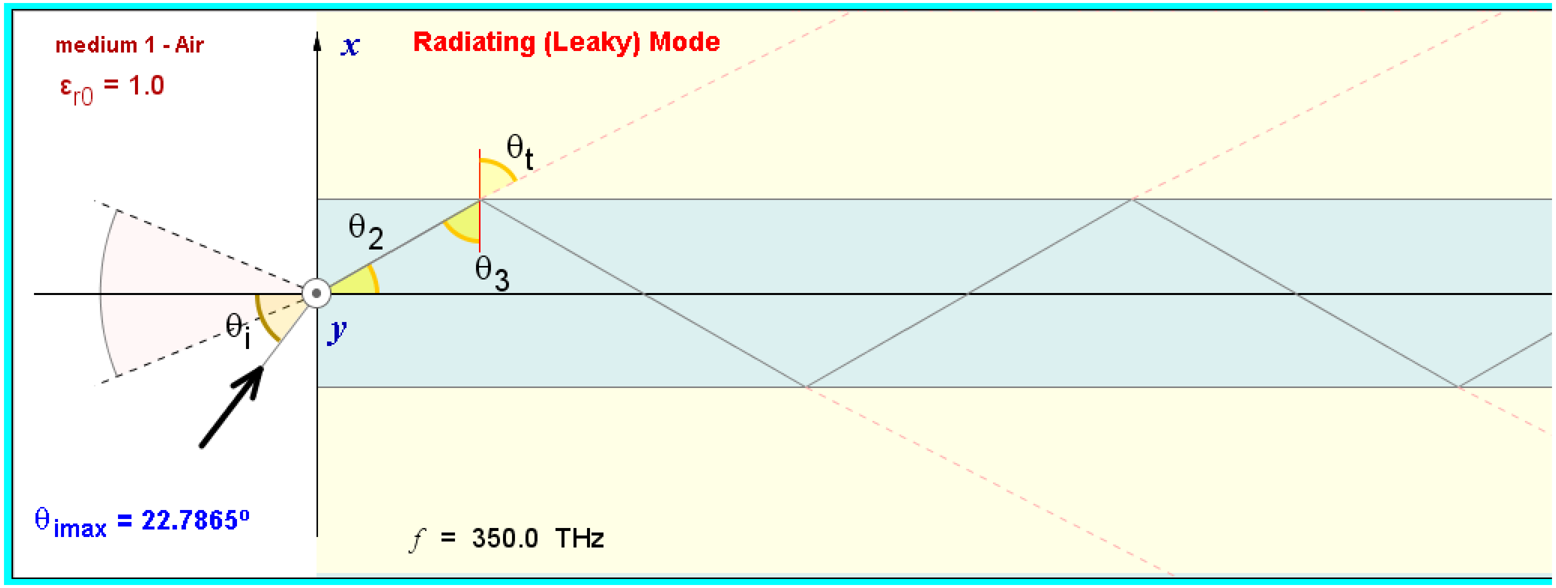


For guidance one must have

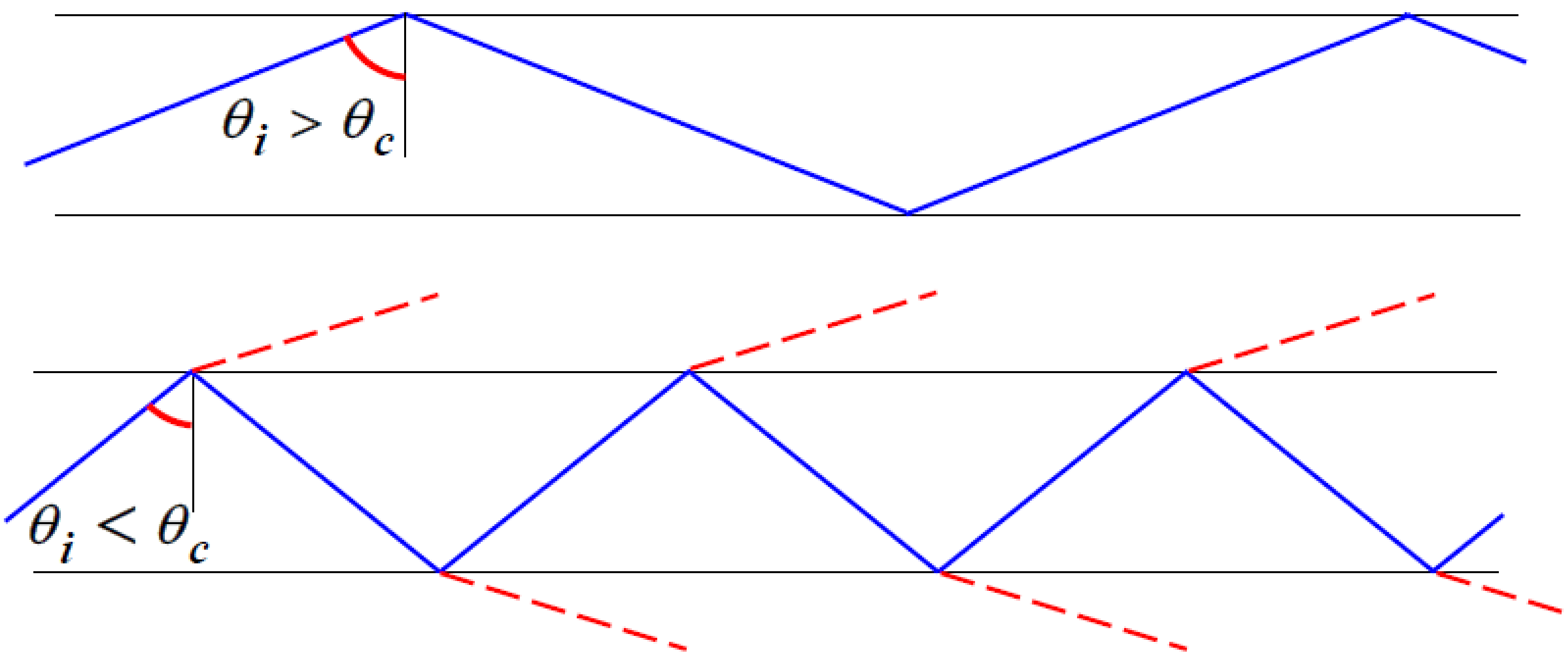
$$\boxed{\varepsilon_1 > \varepsilon_2}$$

Symmetric dielectric slab waveguide

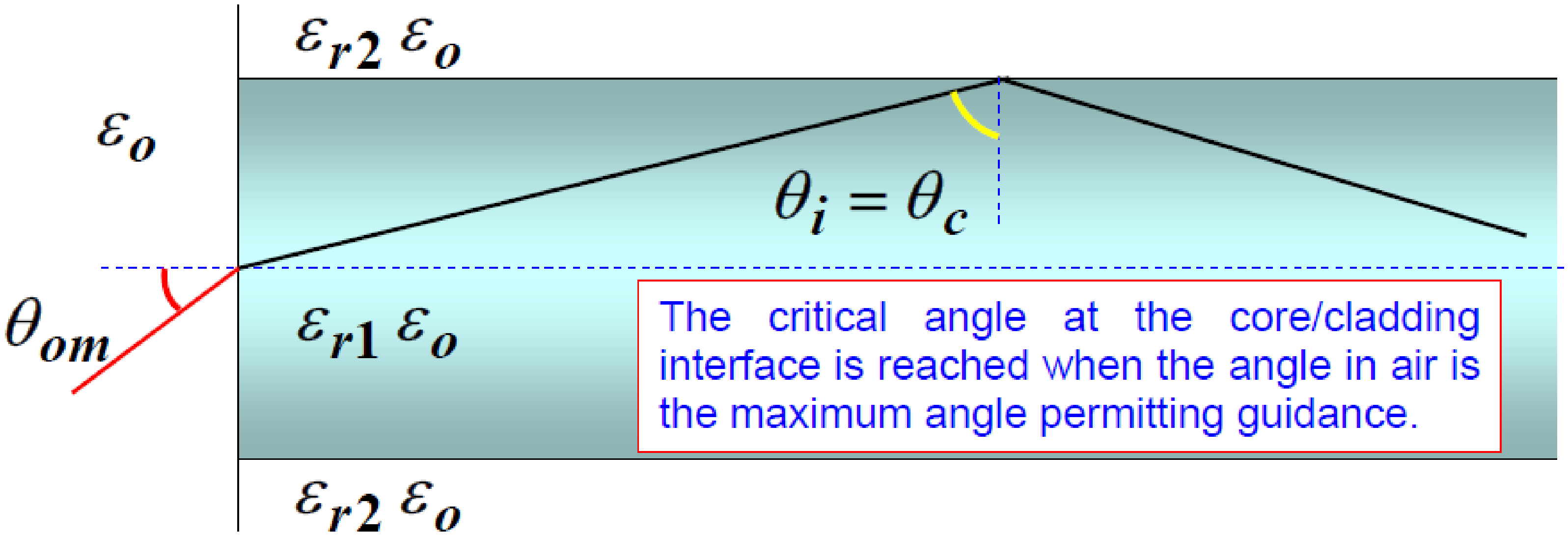




Symmetric dielectric slab waveguide



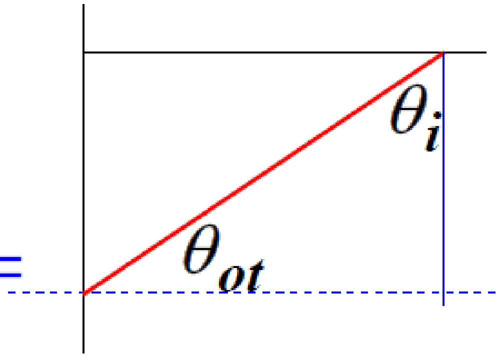
Symmetric dielectric slab waveguide – Numerical aperture



At the air–core interface

$$\sin\theta_{ot} = \sqrt{\frac{\varepsilon_o}{\varepsilon_{r1}\varepsilon_o}}\sin\theta_o = \sqrt{\frac{1}{\varepsilon_{r1}}}\sin\theta_o$$

$$\theta_i + \theta_{ot} = 90^\circ \quad \Rightarrow \quad \cos\theta_{ot} = \sin\theta_i \quad \Leftarrow$$



Symmetric dielectric slab waveguide – Numerical aperture

At the critical angle

$$\sin\theta_i = \sin\theta_c = \sqrt{\frac{\varepsilon_{r2}}{\varepsilon_{r1}}}$$

$$\sin^2\theta_{otm} = 1-\cos^2\theta_{otm} = 1-\sin^2\theta_c = 1-\frac{\varepsilon_{r2}}{\varepsilon_{r1}} = \frac{\sin^2\theta_{om}}{\varepsilon_{r1}}$$

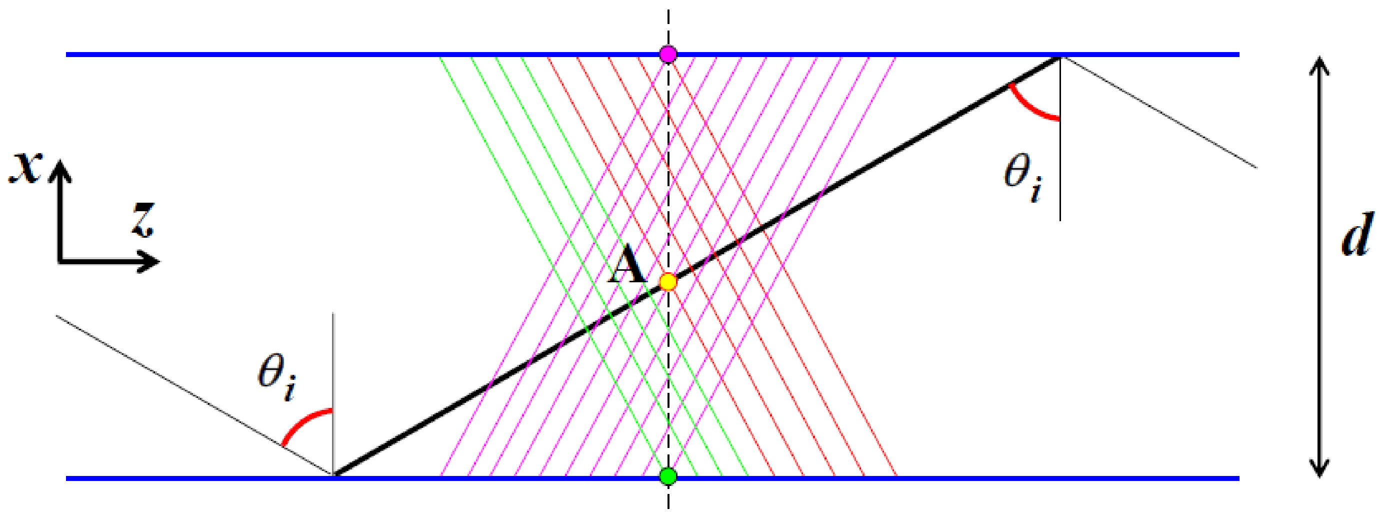
$$\Rightarrow \quad \sin\theta_{om} = \sqrt{\varepsilon_{r1}-\varepsilon_{r2}} = \text{numerical aperture}$$

$$\theta_{om} = \sin^{-1}\sqrt{\varepsilon_{r1}-\varepsilon_{r2}}$$

Symmetric dielectric slab waveguide

We consider again TE and TM modes. Only certain angles of incidence are allowed, but here the reflection coefficient for total reflection is a complex quantity, introducing a phase shift in the reflected field, which depends on the angle of incidence. In the case of metal plates, instead, there is always a phase shift of 180° for the tangential electric field.

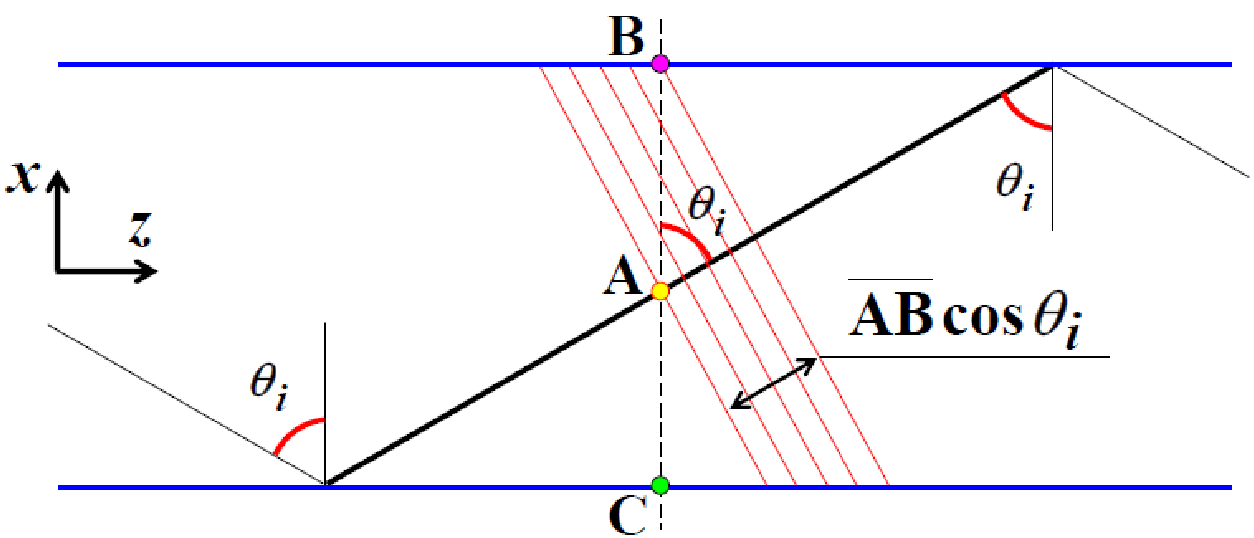
In order for the angle to be accepted, the wave needs to establish a self–consistent constructive interference pattern for any point inside the core, as indicated in the figure below



Symmetric dielectric slab waveguide

Consider a point A in the core of the wave guide and a wave front moving from it reaching point B. The phase shift for the phase planes moving from A to B is

$$\Delta\varphi_1 = -\beta_1 \cdot \overline{\mathrm{AB}}\cos\theta_i = -\frac{2\pi}{\lambda_o / \sqrt{\varepsilon_{r1}}}\overline{\mathrm{AB}}\cos\theta_i$$



λ_o is the wavelength in vacuum at the given frequency of operation.

Symmetric dielectric slab waveguide

The wave front reflected at point B experiences a phase jump equal to the phase of the complex reflection coefficient. Assuming a TE wave, or perpendicular polarization,

$$\Delta\varphi_2 = \angle\Gamma_\perp(E)_\mathrm{B} = \angle\left(\frac{\sqrt{\varepsilon_1}\cos\theta_i + j\sqrt{\varepsilon_2}\sqrt{\varepsilon_1/\varepsilon_2\sin^2\theta_i - 1}}{\sqrt{\varepsilon_1}\cos\theta_i - j\sqrt{\varepsilon_2}\sqrt{\varepsilon_1/\varepsilon_2\sin^2\theta_i - 1}}\right)$$

$$= \angle\left(\frac{\cos\theta_i + j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}{\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}\right)$$

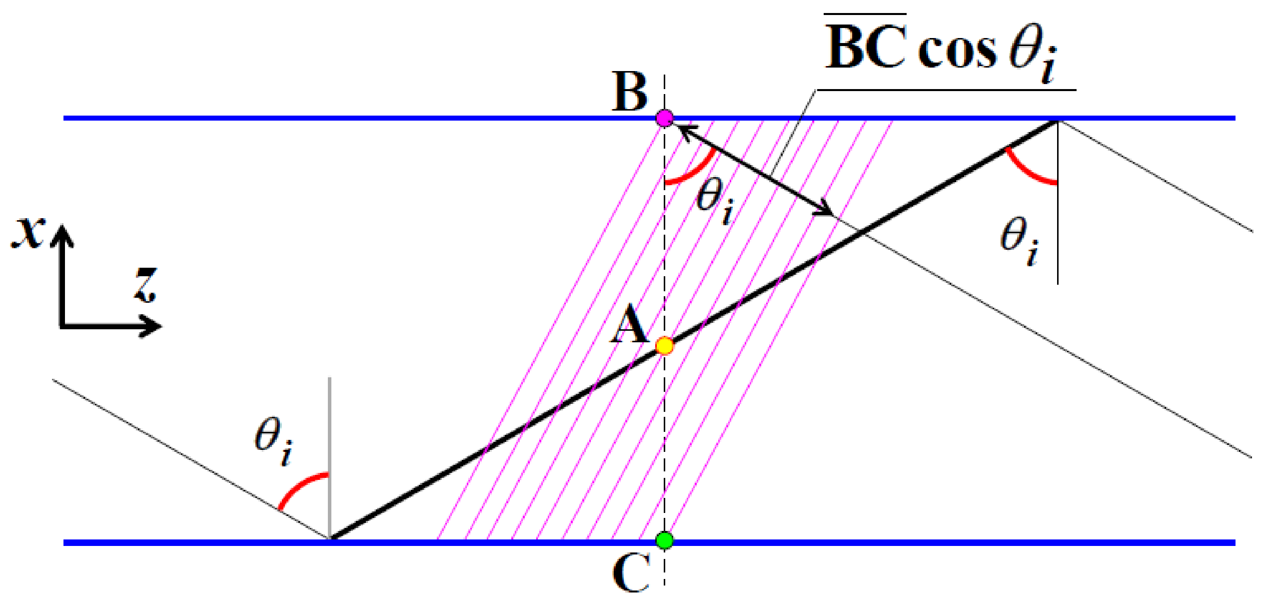
$$= 2\angle\left(\cos\theta_i + j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}\right)$$

$$= 2\tan^{-1}\frac{\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}{\cos\theta_i}$$

Symmetric dielectric slab waveguide

Then, the reflected wave experiences a phase shift when moving from B to C

$$\Delta\varphi_3 = -\beta_1 \cdot \overline{\mathrm{BC}}\cos\theta_i = -\frac{2\pi}{\lambda_o / \sqrt{\varepsilon_{r1}}}\overline{\mathrm{BC}}\cos\theta_i$$



Symmetric dielectric slab waveguide

The wave front reflected at point C experiences again a phase jump equal to the phase of the complex reflection coefficient. For a symmetric waveguide

$$\Rightarrow \quad \Delta\varphi_4 = \Delta\varphi_2$$

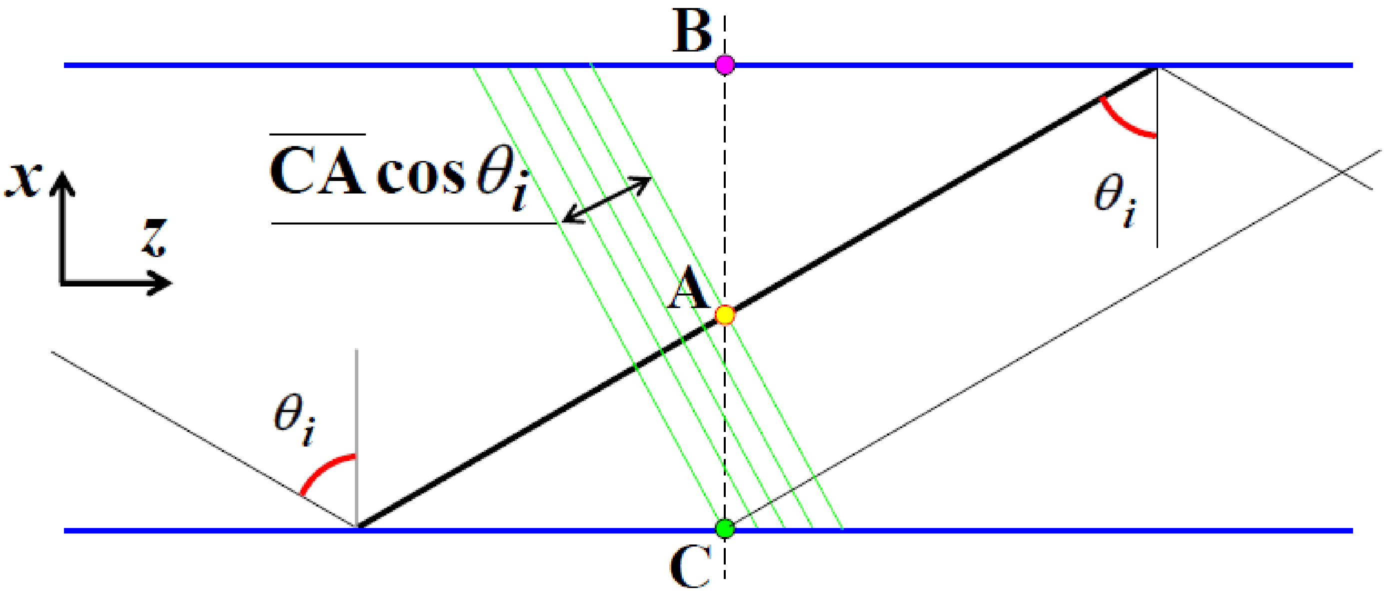
$$\Delta\varphi_4 = \angle\Gamma_{\perp}(E)_C = \angle\left(\frac{\sqrt{\varepsilon_1}\cos\theta_i + j\sqrt{\varepsilon_2}\sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1}}{\sqrt{\varepsilon_1}\cos\theta_i - j\sqrt{\varepsilon_2}\sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1}}\right)$$

$$= 2\tan^{-1}\frac{\sqrt{\sin^2\theta_i - \dfrac{\varepsilon_2}{\varepsilon_1}}}{\cos\theta_i}$$

Symmetric dielectric slab waveguide

The reflected wave experiences a phase shift moving from C back to A

$$\Delta\varphi_5 = -\beta_1 \cdot \overline{\mathrm{CA}}\cos\theta_i = -\frac{2\pi}{\lambda_o / \sqrt{\varepsilon_{r1}}}\overline{\mathrm{CA}}\cos\theta_i$$



Symmetric dielectric slab waveguide

For constructive interference (self–consistency), the sum of all the phase shift components must be equal to a multiple of 2π

$$-\frac{2\pi}{\lambda_o/\sqrt{\varepsilon_{r1}}}\left(\overline{AB}+\overline{BC}+\overline{CA}\right)\cos\theta_i+\Delta\varphi_2+\Delta\varphi_4=-2m\pi,$$

$$m=0,1,2\ldots$$

with $\left(\overline{AB}+\overline{BC}+\overline{CA}\right)=2d$

$$\Rightarrow \boxed{\frac{2\pi d}{\lambda_o/\sqrt{\varepsilon_{r1}}}\cos\theta_i-m\pi=2\tan^{-1}\frac{\sqrt{\sin^2\theta_i-\dfrac{\varepsilon_2}{\varepsilon_1}}}{\cos\theta_i} \qquad m=0,1,2\ldots}$$

Symmetric dielectric slab waveguide TE modes

Taking the tangent of all terms we obtain the characteristic equation for the TE modes.

$$\tan\left(\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_o}\cos\theta_i-\frac{m\pi}{2}\right)=\frac{\sqrt{\sin^2\theta_i-\dfrac{\varepsilon_2}{\varepsilon_1}}}{\cos\theta_i}\quad,\quad m=0,1,2\ldots$$

In terms of even and odd solutions, we can rewrite

$$\tan\left(\underbrace{\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_o}\cos\theta_i}_{f(\cos\theta_i)}\right)=\begin{cases}\dfrac{\sqrt{\sin^2\theta_i-\dfrac{\varepsilon_2}{\varepsilon_1}}}{\cos\theta_i}=g(\cos\theta_i) & \boxed{\text{Even modes}}\quad m=0,2,\ldots\\[2ex] -\dfrac{\cos\theta_i}{\sqrt{\sin^2\theta_i-\dfrac{\varepsilon_2}{\varepsilon_1}}}=-\dfrac{1}{g(\cos\theta_i)} & \boxed{\text{Odd modes}}\quad m=1,3,\ldots\end{cases}$$

Symmetric dielectric slab waveguide TM modes

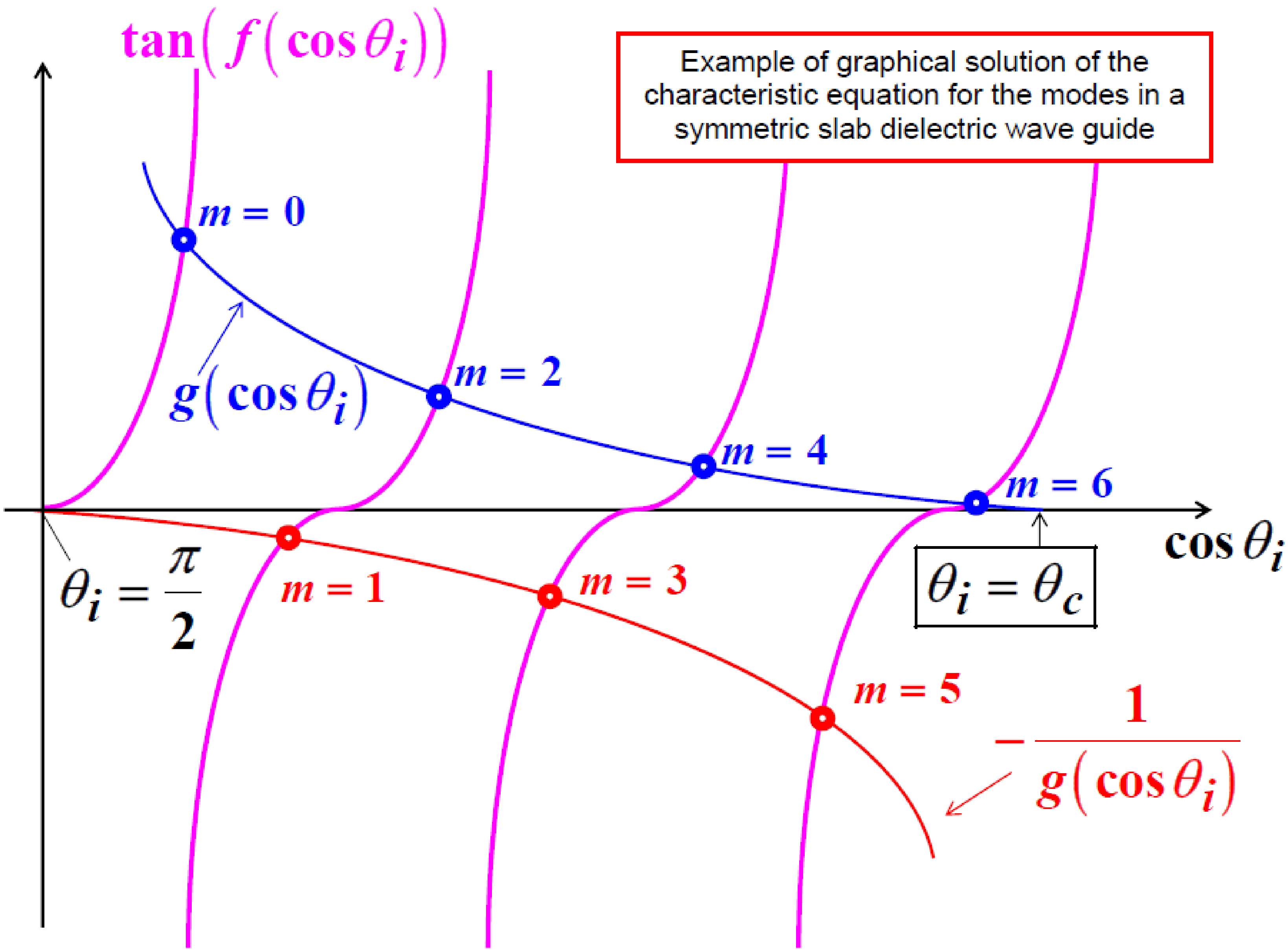
The characteristic equation for TM modes is obtained by using the reflection coefficient for parallel polarization in the derivation

$$\tan\left(\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_o}\cos\theta_i - \frac{m\pi}{2}\right) = \frac{\sqrt{\sin^2\theta_i - \dfrac{\varepsilon_2}{\varepsilon_1}}}{(\varepsilon_2/\varepsilon_1)\cos\theta_i} \quad , \quad m = 0, 1, 2\ldots$$

or, in terms of even and odd solutions

$$\tan\Bigg(\underbrace{\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_o}\cos\theta_i}_{f(\cos\theta_i)}\Bigg) = \begin{cases} \dfrac{\sqrt{\sin^2\theta_i - \dfrac{\varepsilon_2}{\varepsilon_1}}}{(\varepsilon_2/\varepsilon_1)\cos\theta_i} = g(\cos\theta_i) & \boxed{\text{Even modes}} \quad m = 0, 2, \ldots \\ -\dfrac{(\varepsilon_2/\varepsilon_1)\cos\theta_i}{\sqrt{\sin^2\theta_i - \dfrac{\varepsilon_2}{\varepsilon_1}}} = -\dfrac{1}{g(\cos\theta_i)} & \boxed{\text{Odd modes}} \quad m = 1, 3, \ldots \end{cases}$$

Symmetric dielectric slab waveguide



Symmetric dielectric slab waveguide

The cut-off frequencies for the modes are obtained by observing that at cut-off the angle of incidence is minimum (critical angle). At the critical angle, the characteristic equation is

$$\text{TE)}\ \tan\left(\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_{oc}}\cos\theta_c - \frac{m\pi}{2}\right) = \frac{\sqrt{\sin^2\theta_c - \dfrac{\varepsilon_2}{\varepsilon_1}}}{\cos\theta_c} = 0$$

$$\text{TM)}\ \tan\left(\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_{oc}}\cos\theta_c - \frac{m\pi}{2}\right) = \underbrace{\frac{\sqrt{\sin^2\theta_c - \dfrac{\varepsilon_2}{\varepsilon_1}}}{(\varepsilon_2/\varepsilon_1)\cos\theta_c} = 0}$$

since $\theta_c = \sin^{-1}\sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}$

for both TE and TM modes $\Rightarrow$ $$\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_{oc}}\cos\theta_c = \frac{m\pi}{2}$$

Symmetric dielectric slab waveguide

The cut-off wavelengths (referenced to free space as usual in optical wave guides) and the corresponding cut-off frequencies for the guided modes are

$$\lambda_{oc} = \frac{2d\sqrt{\varepsilon_{r1}}}{m}\cos\theta_c = \frac{2d\sqrt{\varepsilon_{r1}}}{m}\sqrt{1-\sin^2\theta_c}$$

$$= \frac{2d\sqrt{\varepsilon_{r1}}}{m}\sqrt{1-\frac{\varepsilon_{r2}}{\varepsilon_{r1}}} = \frac{2d}{m}\sqrt{\varepsilon_{r1}-\varepsilon_{r2}}$$

$$\boxed{f = \frac{c}{\lambda_{oc}} = \frac{mc}{2d\sqrt{\varepsilon_{r1}-\varepsilon_{r2}}}} \quad , \quad \boxed{m = 0, 1, 2\ldots}$$

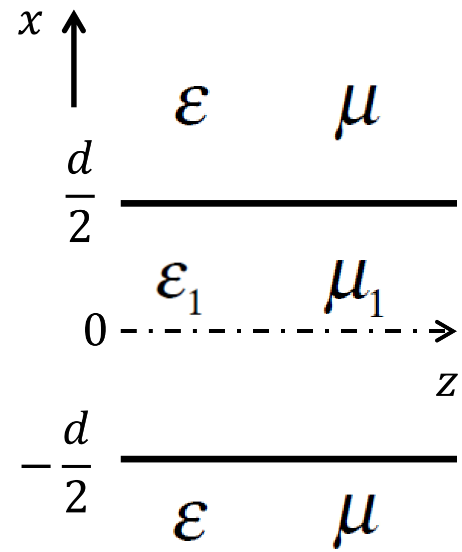
The fundamental modes are the TE_0 and the TM_0 with zero cut-off frequency.

TE and TM modes with the same index form degenerate pairs with identical cut-off frequencies.

Field Solutions – TE modes case

wave equation

$$\left(\nabla^2 + \omega^2 \mu\varepsilon\right)\mathbf{E} = 0$$



even solutions

$$E_y = e^{ik_z z}\begin{cases} C_0 e^{-\alpha(|x|-d/2)} & |x| \geq d/2 \\ C_1 \cos k_x x & |x| \leq d/2 \end{cases}$$

propagation condition along z

wave vectors

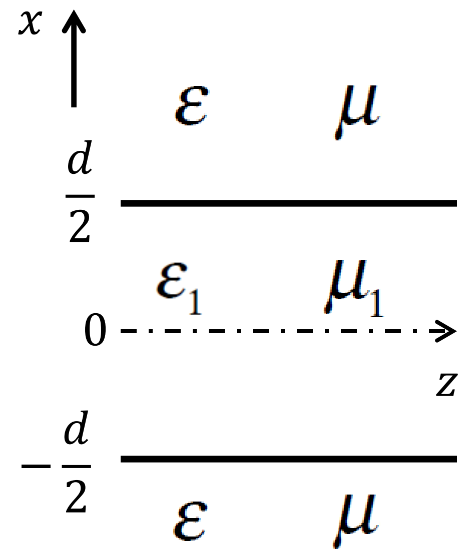
$$k_x^2 + k_z^2 = \omega^2 \mu_1 \varepsilon_1$$

$$-\alpha^2 + k_z^2 = \omega^2 \mu\varepsilon$$

Field Solutions – TE modes case

Boundary conditions

E_y and H_z continuous at $x = d/2$ and $x = -d/2$



$$C_0 = C_1 \cos\left(k_x \frac{d}{2}\right)$$

$$\frac{\alpha}{\mu} C_0 = C_1 \frac{k_x}{\mu_1} \sin\left(k_x \frac{d}{2}\right)$$

$$\boxed{\alpha = \frac{\mu}{\mu_1} k_x \tan\left(k_x \frac{d}{2}\right)}$$

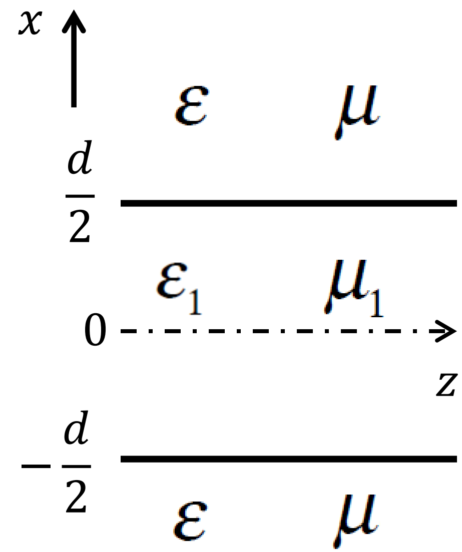
From Faraday's law

$$\frac{\partial}{\partial x}\mathbf{E}_y = -j\omega\,\mu_o \mathbf{H}_z$$

Field Solutions – TE modes case

wave equation

$$\left(\nabla^2 + \omega^2 \mu \varepsilon\right)\mathbf{E} = 0$$



odd solutions

$$E_y = e^{ik_z z}\begin{cases} C_0 e^{-\alpha(|x|-d/2)} & x \geq d/2 \\ C_1 \sin k_x x & |x| \leq d/2 \\ -C_0 e^{\alpha(|x|+d/2)} & x \leq -d/2 \end{cases}$$

propagation condition along z

wave vectors

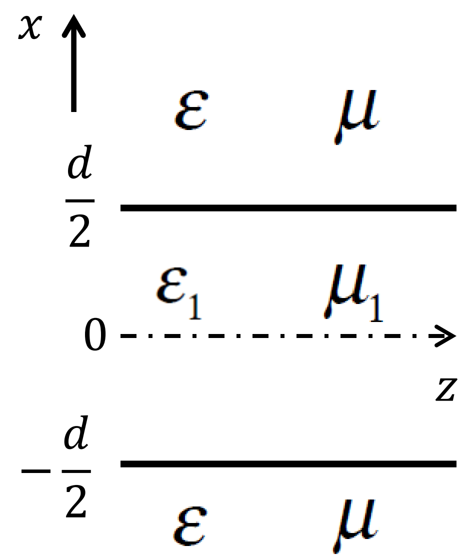
$$k_x^2 + k_z^2 = \omega^2 \mu_1 \varepsilon_1$$

$$-\alpha^2 + k_z^2 = \omega^2 \mu \varepsilon$$

Field Solutions – TE modes case

Boundary conditions

$\boldsymbol{E_y}$ **and** $\boldsymbol{H_z}$ **continuous at** $\boldsymbol{x = d/2}$ **and** $\boldsymbol{x = -d/2}$



$$C_0 = C_1 \sin\left(k_x \frac{d}{2}\right)$$

$$-\frac{\alpha}{\mu} C_0 = C_1 \frac{k_x}{\mu_1} \cos\left(k_x \frac{d}{2}\right)$$

$$\boxed{\alpha = -\frac{\mu}{\mu_1} k_x \cot\left(k_x \frac{d}{2}\right)}$$

Graphical solution

We have

$$k_x^2 + k_z^2 = \omega^2 \mu_1 \varepsilon_1$$

$$-\alpha^2 + k_z^2 = \omega^2 \mu \varepsilon$$

$$\Rightarrow \boxed{k_x^2 + \alpha^2 = \omega^2 \mu_1 \varepsilon_1 - \omega^2 \mu \varepsilon}$$

Coordinate transformation

$$X = k_x \frac{d}{2} \quad \text{and} \quad Y = \alpha \frac{d}{2}$$

$$\left(\frac{2}{d}\right)^2 X^2 + \left(\frac{2}{d}\right)^2 Y^2 = \omega^2 (\mu_1 \varepsilon_1 - \mu \varepsilon) \Rightarrow X^2 + Y^2 = \underbrace{\omega^2 \left(\frac{d}{2}\right)^2 (\mu_1 \varepsilon_1 - \mu \varepsilon)}_{R^2}$$

$$k_0 = \omega \sqrt{\mu_o \varepsilon_o}$$

$$n_1 = \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_o \varepsilon_o}}$$

$$n = \sqrt{\frac{\mu\ \varepsilon}{\mu_o \varepsilon_o}}$$

$$R = \omega \left(\frac{d}{2}\right) (\mu_1 \varepsilon_1 - \mu \varepsilon) = \left(k_0 \frac{d}{2}\right) \sqrt{n_1^2 - n^2}$$

Graphical solution

Coordinate transformation

$$X = k_x \frac{d}{2} \quad \text{and} \quad Y = \alpha \frac{d}{2}$$

even modes

$$\boxed{\alpha = \frac{\mu}{\mu_1} k_x \tan\left(k_x \frac{d}{2}\right)}$$

odd modes

$$\boxed{\alpha = -\frac{\mu}{\mu_1} k_x \cot\left(k_x \frac{d}{2}\right)}$$

$$Y = \begin{cases} \dfrac{\mu}{\mu_1} X \tan X & \text{TE even} \\ -\dfrac{\mu}{\mu_1} X \cot X & \text{TE odd} \end{cases}$$

Cut-off condition

$$R = \omega\left(\frac{d}{2}is\right)(\mu_1\varepsilon_1 - \mu\varepsilon) = \left(k_0\frac{d}{2}\right)\sqrt{n_1^2 - n^2} = m\frac{\pi}{2}$$

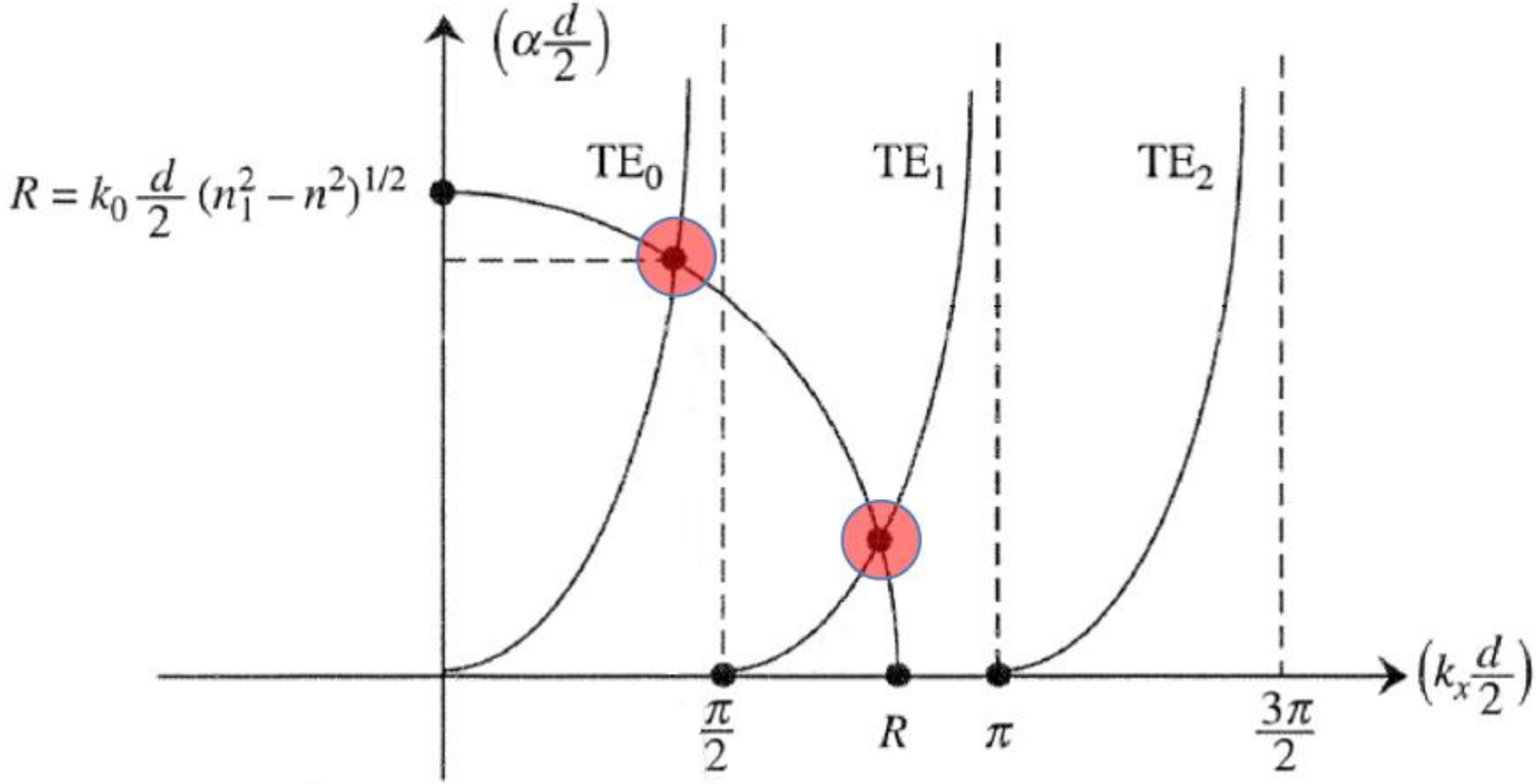
$$[m = 0, 1, 2, \dots]$$

For single mode operation:

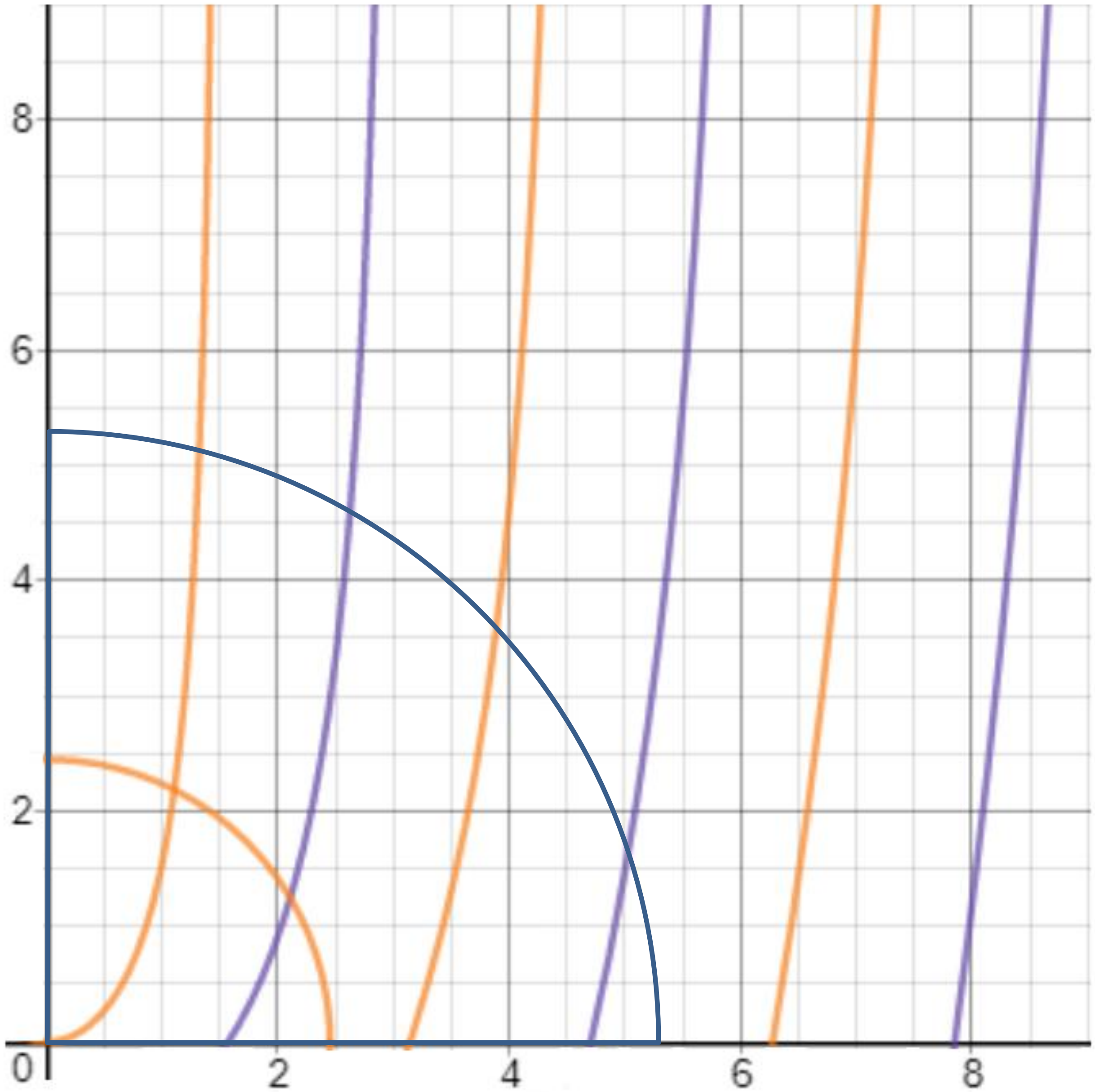
$$\left(k_0\frac{d}{2}\right)\sqrt{n_1^2 - n^2} < \frac{\pi}{2}$$

$$n_1^2 - n^2 < \frac{(\pi/2)^2}{(k_0 d/2)^2} = \left(\frac{\lambda_0}{2d}\right)^2$$

Graphical solution



Graphical solution



Low Frequency limits

At cut-off condition $\boldsymbol{R = m\pi/2}$ $\quad (m = 0,1,2, \dots)$

$$k_0 \frac{d}{2}\sqrt{n_1^2 - n^2} = m\frac{\pi}{2}$$

$$\alpha = 0$$

$$\boxed{\begin{aligned} k_x^2 + k_z^2 &= \omega^2 \mu_1 \varepsilon_1 \\ -\alpha^2 + k_z^2 &= \omega^2 \mu \varepsilon \end{aligned}}$$

$$k_x \frac{d}{2} = m\frac{\pi}{2}$$

$$k_z = \omega\sqrt{\mu\varepsilon} \quad (\text{since } \alpha = 0)$$

At low frequency the propagation constant approaches that of the cladding medium outside the waveguide

High Frequency limit

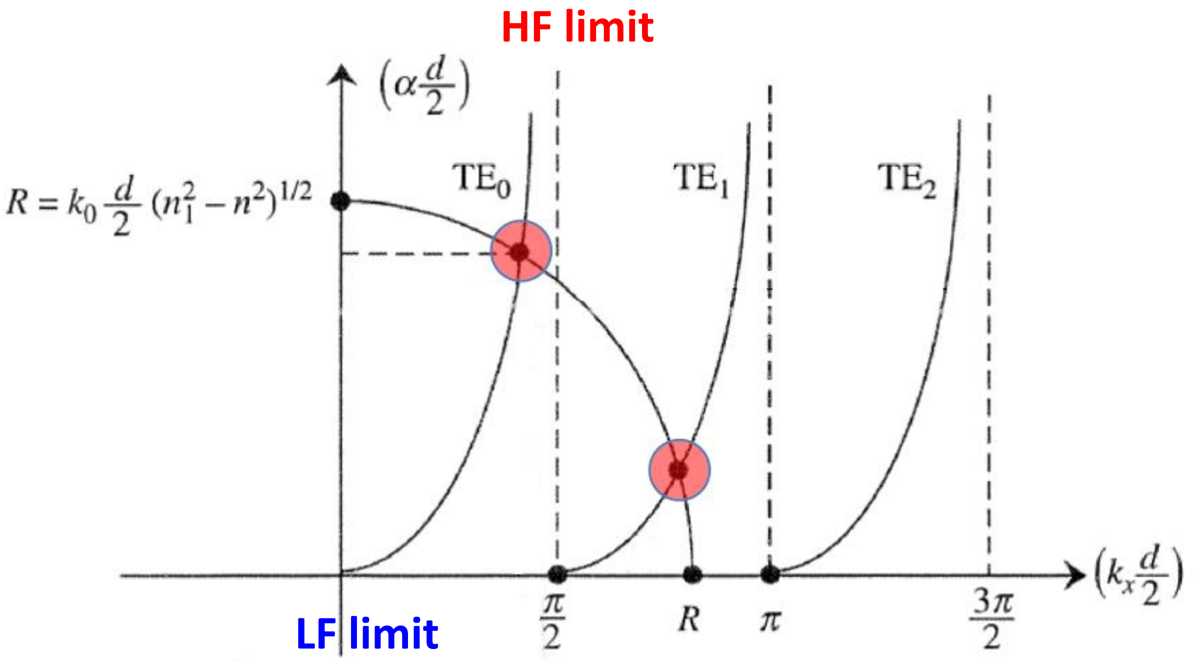
$$k_x^2 + k_z^2 = \omega^2 \mu_1 \varepsilon_1$$
$$-\alpha^2 + k_z^2 = \omega^2 \mu \varepsilon$$

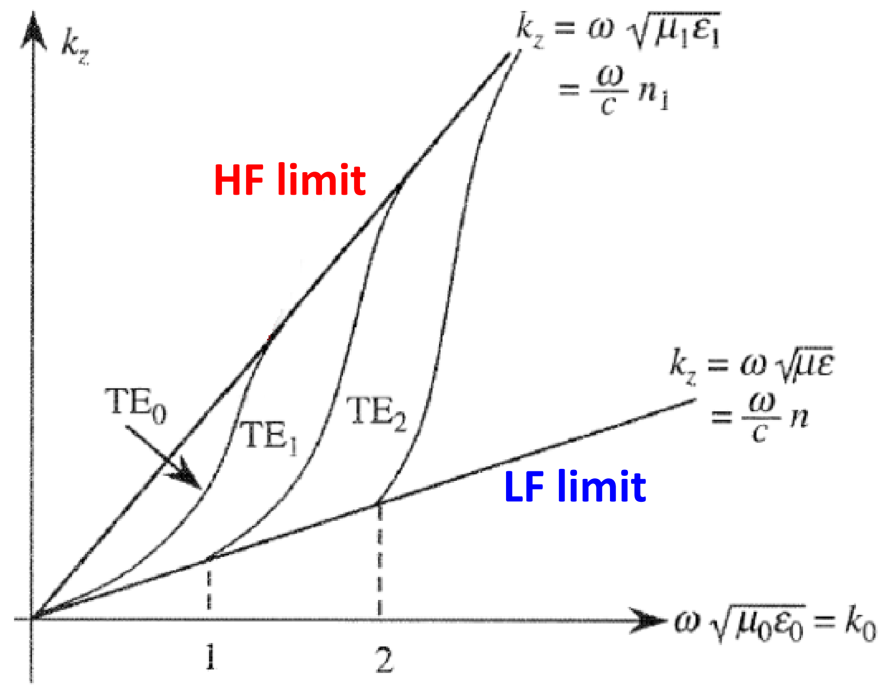
At high frequency $R \to \infty$

$$k_x \frac{d}{2} \to (m+1)\frac{\pi}{2}$$

$$k_z \to \omega\sqrt{\mu_1 \varepsilon_1} = \omega \frac{n_1}{c}$$

At high frequency the propagation constant approaches that of the core medium inside the waveguide. There is very little power in the cladding.

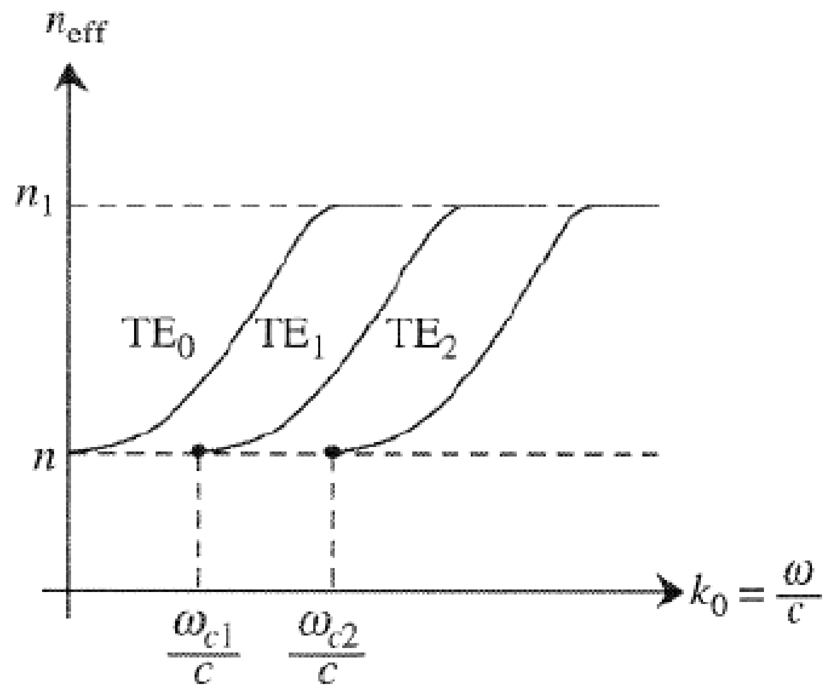

HF limit
$(\alpha\frac{d}{2})$
$R = k_0\frac{d}{2}(n_1^2 - n^2)^{1/2}$
TE$_0$
TE$_1$
TE$_2$
$(k_x\frac{d}{2})$
LF limit
$\frac{\pi}{2}$
R
π
$\frac{3\pi}{2}$


k_z
$k_z = \omega\sqrt{\mu_1\varepsilon_1} = \frac{\omega}{c}n_1$
HF limit
TE$_0$
TE$_1$
TE$_2$
$k_z = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}n$
LF limit
1
2
$\omega\sqrt{\mu_0\varepsilon_0} = k_0$

Effective index for the guided modes

$$n_{eff} = \frac{k_z}{k_0}$$

$$n_{eff} = \frac{2\pi}{\lambda_z}\frac{\lambda_0}{2\pi} = \frac{f}{v_{pz}}\frac{c}{f} = \frac{c}{v_{pz}}$$



$$n_{eff} = n_1 \sin\theta$$
$$n \le n_{eff} \le n_1$$

$$\lambda_z = \frac{\lambda_0}{n_{eff}}$$

Normalized frequency

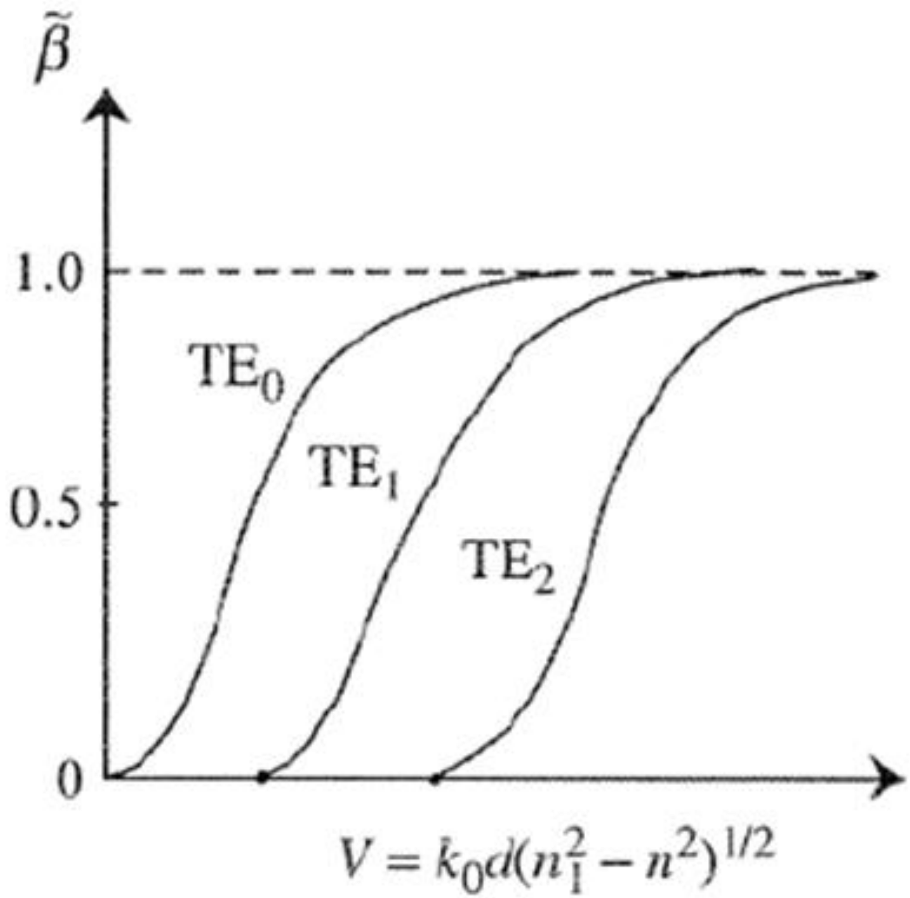
$$V = k_0 d\sqrt{n_1^2 - n^2}$$

$$V = \frac{2\pi}{\lambda_0} d\sqrt{n_1^2 - n^2} = f\,\underbrace{\frac{2\pi}{c} d\sqrt{n_1^2 - n^2}}$$

contains all the information on wave guide geometry and materials

Normalized propagation parameter

$$\tilde{\beta} = \frac{k_z^2 - \omega^2 \mu \varepsilon}{\omega^2 \mu_1 \varepsilon_1 - \omega^2 \mu \varepsilon} = \frac{n_{eff}^2 - n^2}{n_1^2 - n^2}$$



Validity of refraction index model

The theoretical analysis provides always good insight but it has assumed that the refractive indices are independent of frequency.

This is true only in ideal dielectrics. In real media, particularly in semiconductors, this is generally not true.

Symmetric dielectric slab waveguide

Magnetic field components for TE modes are obtained from Faraday's law

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\vec{\mathbf{H}}$$

$$\Downarrow$$

$$\det\begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \cancel{\frac{\partial}{\partial y}} & \frac{\partial}{\partial z} \\ \mathbf{E}_x = 0 & \mathbf{E}_y & \mathbf{E}_z = 0 \end{bmatrix} \Rightarrow \begin{array}{c} -\frac{\partial}{\partial z}\mathbf{E}_y = -j\omega\mu_o\mathbf{H}_x \\ \frac{\partial}{\partial z}\mathbf{E}_x - \frac{\partial}{\partial x}\mathbf{E}_z = -j\omega\mu_o\mathbf{H}_y = 0 \\ \frac{\partial}{\partial x}\mathbf{E}_y = -j\omega\mu_o\mathbf{H}_z \end{array}$$

Symmetric dielectric slab waveguide

For example, the transverse magnetic field component is proportional to the (transverse) electric field. In the guide core:

$$\boxed{-\frac{\partial}{\partial z}\mathrm{E}_y = -j\omega\mu_o \mathrm{H}_x}$$

$$\left.\begin{array}{l}\text{Even)}\ -\dfrac{\partial}{\partial z}E_o\cos\left(\beta_{x1}\cdot x\right)e^{-j\beta_z z}\\ \text{Odd)}\ -\dfrac{\partial}{\partial z}E_o\sin\left(\beta_{x1}\cdot x\right)e^{-j\beta_z z}\end{array}\right. = -j\omega\mu_o \mathrm{H}_x$$

$$\Rightarrow \quad \mathrm{H}_x = -\frac{\beta_z}{\omega\mu_o}\mathrm{E}_y = \begin{array}{ll} -\dfrac{\beta_z E_o}{\omega\mu_o}\cos\left(\beta_{x1}\cdot x\right)e^{-j\beta_z z} & \text{(Even)} \\ -\dfrac{\beta_z E_o}{\omega\mu_o}\sin\left(\beta_{x1}\cdot x\right)e^{-j\beta_z z} & \text{(Odd)} \end{array}$$

Symmetric dielectric slab waveguide

Electric field components for TM modes are obtained from Ampere's law

$$\nabla \times \vec{\mathbf{H}} = j\omega\varepsilon\vec{\mathbf{E}}$$

$$\Downarrow$$

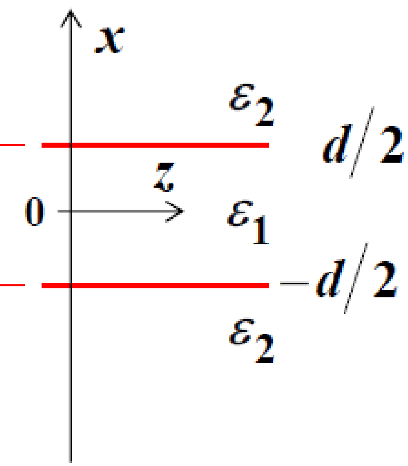
$$\det\begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \dfrac{\partial}{\partial x} & \cancel{\dfrac{\partial}{\partial y}} & \dfrac{\partial}{\partial z} \\ \mathrm{H}_x=0 & \mathrm{H}_y & \mathrm{H}_z=0 \end{bmatrix} \Rightarrow \begin{array}{c} -\dfrac{\partial}{\partial z}\mathrm{H}_y = j\omega\varepsilon\mathrm{E}_x \\ \dfrac{\partial}{\partial z}\mathrm{H}_x - \dfrac{\partial}{\partial x}\mathrm{H}_z = 0 = j\omega\varepsilon\mathrm{E}_y \\ \dfrac{\partial}{\partial x}\mathrm{H}_y - = j\omega\varepsilon\mathrm{E}_z \end{array}$$

Next, is a summary of all the field components parallel to the plane of incidence.

Symmetric dielectric slab waveguide – Field Expressions

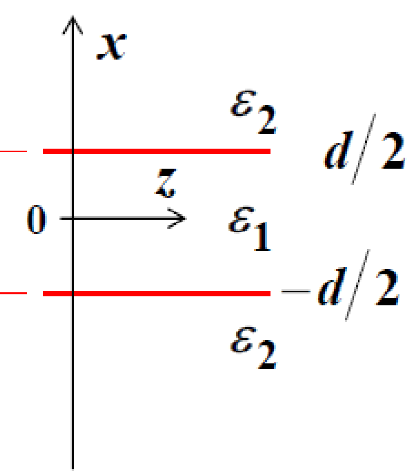
Even TE modes

$$E_y = \begin{cases} E_o \cos\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ E_o \cos\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ E_o \cos\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



Odd TE modes

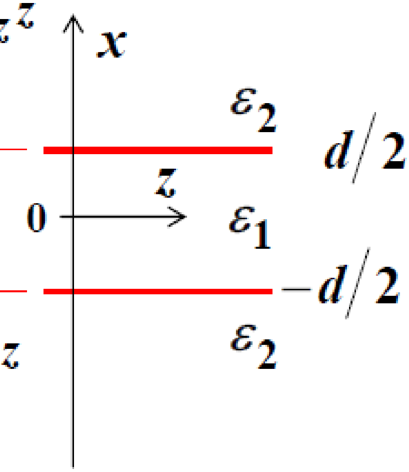
$$E_y = \begin{cases} E_o \sin\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ E_o \sin\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ E_o \sin\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



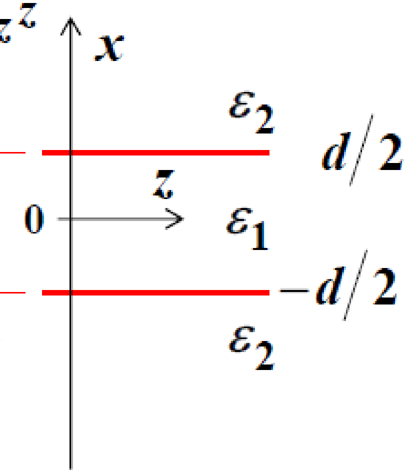
Symmetric dielectric slab waveguide – Field Expressions

Even TE modes

$$H_x = \begin{cases} -\dfrac{\beta_z}{\omega \mu_o} E_o \cos\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ -\left(\beta_z / \omega \mu_o\right) E_o \cos\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ -\dfrac{\beta_z}{\omega \mu_o} E_o \cos\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



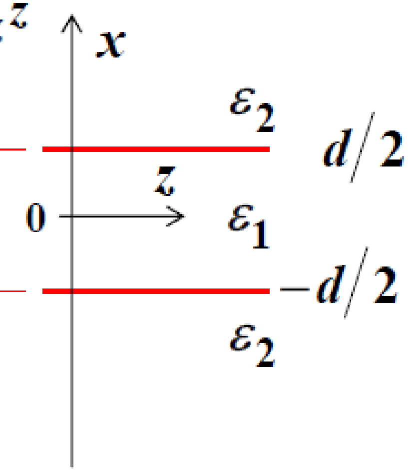
$$H_z = \begin{cases} -\dfrac{j\alpha_{x2}}{\omega \mu_o} E_o \cos\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ -j\left(\beta_{x1} / \omega \mu_o\right) E_o \sin\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ \dfrac{j\alpha_{x2}}{\omega \mu_o} E_o \cos\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



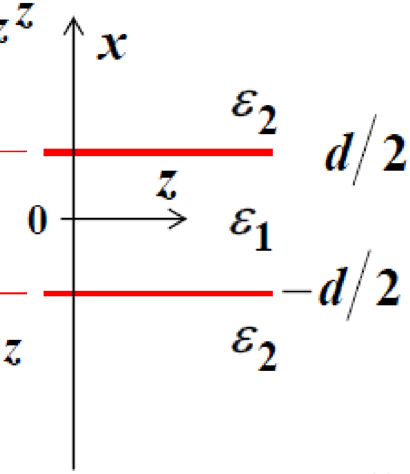
Symmetric dielectric slab waveguide – Field Expressions

Odd TE modes

$$H_x = \begin{cases} -\dfrac{\beta_z}{\omega\mu_o} E_o \sin\left(\beta_{x1}\cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ -\left(\beta_z / \omega\mu_o\right) E_o \sin\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} \\ \dfrac{\beta_z}{\omega\mu_o} E_o \sin\left(\beta_{x1}\cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



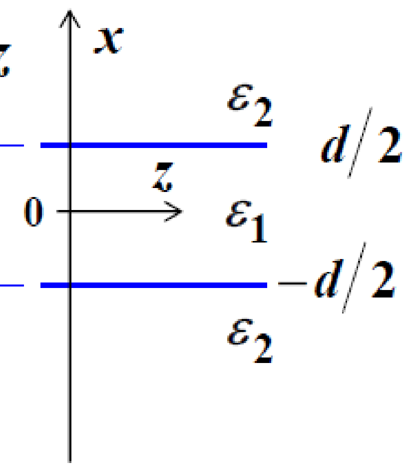
$$H_z = \begin{cases} -\dfrac{j\alpha_{x2}}{\omega\mu_o} E_o \sin\left(\beta_{x1}\cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ j\left(\beta_{x1} / \omega\mu_o\right) E_o \cos\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} \\ -\dfrac{j\alpha_{x2}}{\omega\mu_o} E_o \sin\left(\beta_{x1}\cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



Symmetric dielectric slab waveguide – Field Expressions

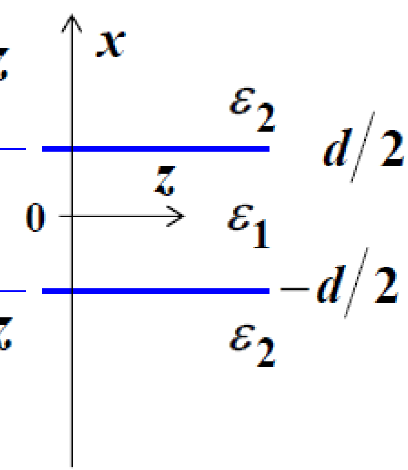
Even TM modes

$$H_y = \begin{cases} H_o \cos\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ H_o \cos\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ H_o \cos\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



Odd TM modes

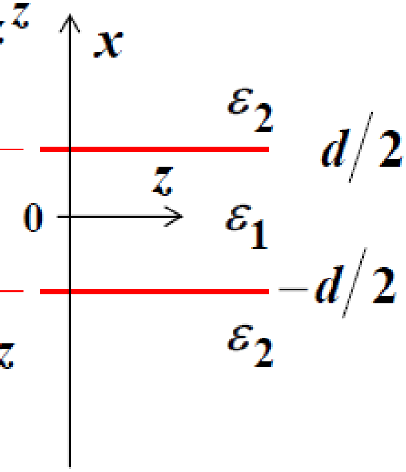
$$H_y = \begin{cases} H_o \sin\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ H_o \sin\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ -H_o \sin\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



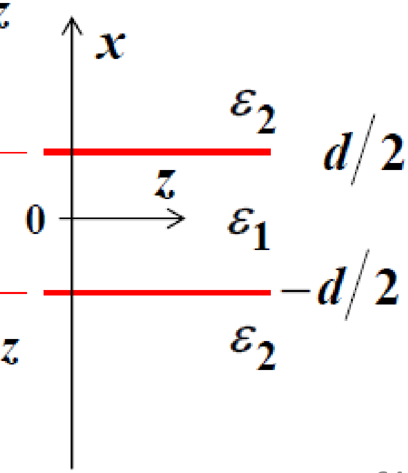
Symmetric dielectric slab waveguide – Field Expressions

Even TM modes

$$
E_x = \begin{cases} -\dfrac{\beta_z}{\omega \varepsilon_2} H_o \cos\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ -\left(\beta_z / \omega \varepsilon_1\right) H_o \cos\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ -\dfrac{\beta_z}{\omega \varepsilon_2} H_o \cos\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}
$$



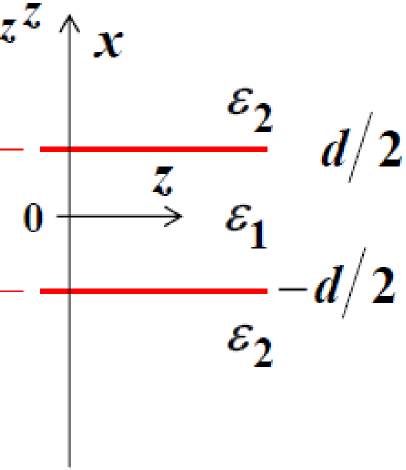
$$
E_z = \begin{cases} \dfrac{j\alpha_{x2}}{\omega \varepsilon_2} H_o \cos\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ j\left(\beta_{x1} / \omega \varepsilon_1\right) H_o \sin\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ -\dfrac{j\alpha_{x2}}{\omega \varepsilon_2} H_o \cos\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}
$$



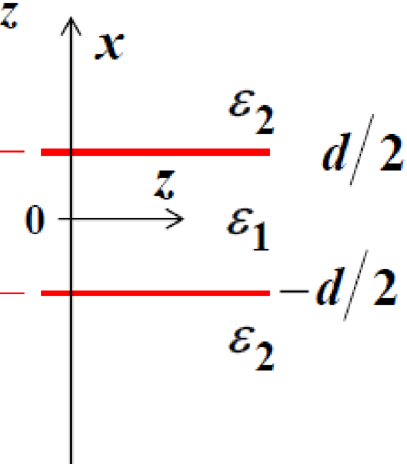
Symmetric dielectric slab waveguide – Field Expressions

Odd TM modes

$$E_x = \begin{cases} -\dfrac{\beta_z}{\omega \varepsilon_2} H_o \sin\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ -\left(\beta_z / \omega \varepsilon_1\right) H_o \sin\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ \dfrac{\beta_z}{\omega \varepsilon_2} H_o \sin\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



$$E_z = \begin{cases} \dfrac{j\alpha_{x2}}{\omega \varepsilon_2} H_o \sin\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ -j\left(\beta_{x1} / \omega \varepsilon_1\right) H_o \cos\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ \dfrac{j\alpha_{x2}}{\omega \varepsilon_2} H_o \sin\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



Reading Assignments:

Chapter 7 of Chuang’s book

Chapter 7 of Coldren & Corzine’s book (supplement)