ECE 536 – Integrated Optics and Optoelectronics
Lecture 9 – February 15, 2022

Spring 2022

Tu-Th 11:00am-12:20pm

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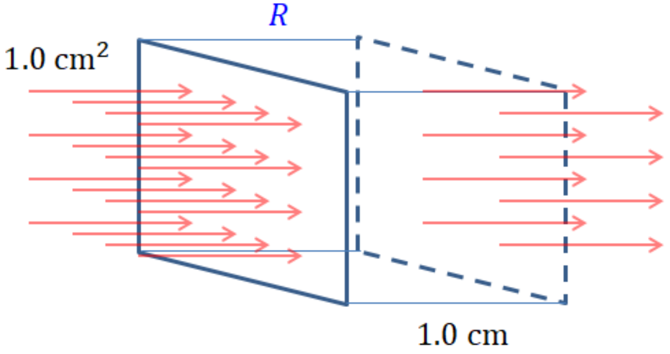
Lecture 9 Outline

- Optical Absorption
- Optical Gain

Quick Recap – Absorption Coefficient

Net Upward Transition
$$R = \frac{2}{V}\sum_{k_a}\sum_{k_b}\frac{2\pi}{\hbar}\left|H'_{ba}\right|^2 \delta\left(E_b - E_a - \hbar\omega\right)\left(f_a - f_b\right)$$

$$\frac{\text{optical intensity}}{\text{energy per photon}} = \frac{|\langle P\rangle|}{\hbar\omega} = \frac{n_r c\varepsilon_0\omega^2 A_0^2}{2\ \hbar\omega}$$



$$\alpha = \frac{R}{(P/\hbar\omega)} = \frac{\hbar\omega}{\left(\dfrac{n_r c\varepsilon_o\omega^2 A_o^2}{2}\right)}\frac{2}{V}\sum_{k_a}\sum_{k_b}\frac{2\pi}{\hbar}\left|H'_{ba}\right|^2 \delta\left(E_b - E_a - \hbar\omega\right)\left(f_a - f_b\right)$$

Perturbation Hamiltonian
$$H'(\mathbf{r}) = -\frac{eA_o e^{i\mathbf{k}_{op}\cdot\mathbf{r}}}{2m_o}(\hat{e}\cdot\mathbf{p})$$

Long λ (dipole) Approximation
$$\mathbf{A}(\mathbf{r}) = \mathbf{A}e^{i\mathbf{k}_{op}\cdot\mathbf{r}} \simeq \mathbf{A}$$

$$H'_{ba} \simeq -\frac{e}{m_o}\mathbf{A}\cdot\langle b|\mathbf{p}|a\rangle = -\frac{eA_o}{2m_o}\hat{e}\cdot\mathbf{p}_{ba}$$

Quick Recap – Absorption Coefficient

$$\boxed{\alpha = C_o \frac{2}{V} \sum_{k_a} \sum_{k_b} \left| \hat{e} \cdot \mathbf{p}_{ba} \right|^2 \delta\left(E_b - E_a - \hbar\omega\right)\left(f_a - f_b\right)}$$

$$C_o = \frac{\pi e^2}{n_r c \varepsilon_0 m_0^2 \omega} \qquad p_{ba} = \left\langle b \right| \frac{\hbar}{i} \nabla \left| a \right\rangle = \int \Psi_b^*(\boldsymbol{r}) \frac{\hbar}{i} \nabla \Psi_a(\boldsymbol{r})$$

Expressed in terms of the electric dipole moment

$$\boxed{\alpha(\hbar\omega) = \frac{\pi\omega}{n_r c \varepsilon_0} \frac{2}{V} \sum_{k_a} \sum_{k_b} \left| \hat{e} \cdot \boldsymbol{\mu}_{ba} \right|^2 \delta\left(E_b - E_a - \hbar\omega\right)\left(f_a - f_b\right)}$$

$$\boldsymbol{\mu}_{ba} = e \left\langle b \middle| \mathbf{r} \middle| a \right\rangle = e \mathbf{r}_{ba}$$

Then we have asked ourselves how to calculate the optical matrix element we have in the net upward transition

$$R = R_{a \to b} - R_{b \to a} = \frac{2}{V} \sum_{k_a} \sum_{k_b} \frac{2\pi}{\hbar} \left| H'_{ba} \right|^2 \delta\left(E_b - E_a - \hbar\omega\right)\left(f_a - f_b\right)$$

$$\boxed{H'_{ba} = \left\langle b \left| \frac{-eA(\mathbf{r})}{m_0} \cdot \mathbf{p} \right| a \right\rangle}$$

Vector potential for the optical field

$$\mathbf{A}(\mathbf{r}) = \mathbf{A} e^{i\mathbf{k}_{op} \cdot \mathbf{r}} = \frac{\hat{e} A_0}{2} e^{i\mathbf{k}_{op} \cdot \mathbf{r}}$$

valence band state

$$\psi_a = \frac{u_v(\mathbf{r}) e^{i\mathbf{k}_v \cdot \mathbf{r}}}{\sqrt{Vol}}$$

conduction band state

$$\psi_b = \frac{u_c(\mathbf{r}) e^{i\mathbf{k}_c \cdot \mathbf{r}}}{\sqrt{Vol}}$$

$$H'_{ba} = -\frac{eA_o}{2m_o} \hat{e} \cdot \int \frac{u_c^*(\mathbf{r}) e^{-i\mathbf{k}_c \cdot \mathbf{r}}}{\sqrt{Vol}} e^{i\mathbf{k}_{op} \cdot \mathbf{r}} \left(-i\hbar\nabla\right) \frac{u_v(\mathbf{r}) e^{i\mathbf{k}_v \cdot \mathbf{r}}}{\sqrt{Vol}} d^3\mathbf{r}$$

Then we have asked ourselves how to calculate the optical matrix element we have in the net upward transition

$$R = R_{a\to b} - R_{b\to a} = \frac{2}{V}\sum_{k_a}\sum_{k_b}\frac{2\pi}{\hbar}\left|H'_{ba}\right|^2 \delta\left(E_b - E_a - \hbar\omega\right)\left(f_a - f_b\right)$$

$$\boxed{H'_{ba} = \left\langle b\right|\frac{-eA(\mathbf{r})}{m_0}\bullet\mathbf{p}\left|a\right\rangle}$$

Vector potential for the optical field	valence band state	conduction band state
$\mathbf{A}(\mathbf{r}) = \mathbf{A}e^{i\mathbf{k}_{op}\bullet\mathbf{r}} = \frac{\hat{e}A_0}{2}e^{i\mathbf{k}_{op}\bullet\mathbf{r}}$	$\psi_a = \frac{u_v(\mathbf{r})e^{i\mathbf{k}_v\bullet\mathbf{r}}}{\sqrt{Vol}}$	$\psi_b = \frac{u_c(\mathbf{r})e^{i\mathbf{k}_c\bullet\mathbf{r}}}{\sqrt{Vol}}$

$$H'_{ba} = -\frac{eA_o}{2m_o}\hat{e}\bullet\int\frac{u_c^*(\mathbf{r})e^{-i\mathbf{k}_c\bullet\mathbf{r}}}{\sqrt{Vol}}\, e^{i\mathbf{k}_{op}\bullet\mathbf{r}}\,\left(-i\hbar\nabla\right)\frac{u_v(\mathbf{r})e^{i\mathbf{k}_v\bullet\mathbf{r}}}{\sqrt{Vol}}\, d^3\mathbf{r}$$

$e^{-i\mathbf{k}_c\cdot\mathbf{r}}$ $e^{i\mathbf{k}_v\cdot\mathbf{r}}$	slowly varying over a unit cell	$u_c^*(\mathbf{r})\left(-i\hbar\nabla u_v(r)\right)$ $u_c^*(\mathbf{r})\, u_v(r)$	fast varying over a unit cell

We separated the integral into the product of two integrals

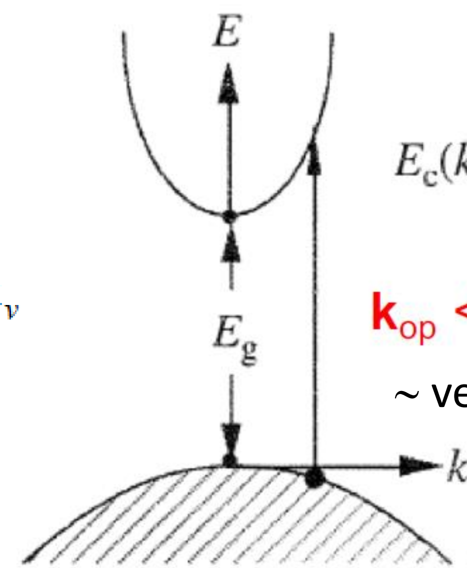
$$H'_{ba} \simeq -\frac{eA_0}{2m_0}\hat{e}\cdot\int_{\Omega} u_c^*(\mathrm{r})\left(-i\hbar\nabla u_v(r)\right)\frac{d^3\mathrm{r}}{\Omega}\int_V e^{i(-\mathrm{k}_c+\mathrm{k}_v+\mathrm{k}_{op})\cdot\mathrm{r}}\frac{d^3\mathrm{r}}{V} + 0$$

$$= -\frac{eA_0}{2m_0}\hat{e}\cdot\mathrm{p}_{cv}\,\delta(k_c, k_v + k_{op})$$

$$\boxed{\mathrm{p}_{cv} = \int_{\Omega} u_c^*(\mathrm{r})\left(-i\hbar\nabla u_v(r)\right)\frac{d^3\mathrm{r}}{\Omega}}$$

k-selection Rule

$$H'_{ba} \simeq -\frac{eA_o}{2m_o}\hat{e}\bullet\mathbf{p}_{cv}\delta_{\mathbf{k}_c,\mathbf{k}_v}$$



$$E_c(k) = E_g + \frac{\hbar^2 k^2}{2m_e^*}$$

$$E_v = -\frac{\hbar^2 k^2}{2m_h^*}$$

$k_{op} << k_c, k_v$

~ vertical transition

$$E_c - E_v = E_g + \frac{\hbar^2 k^2}{2m_e^*} + \frac{\hbar^2 k^2}{2m_h^*} = E_g + \frac{\hbar^2 k^2}{2m_r^*}$$

$$\boxed{\frac{1}{m_r^*} = \frac{1}{m_e^*} + \frac{1}{m_h^*}}$$

Optical Absorption Spectrum

$$\alpha(\hbar\omega) = \frac{\hbar\omega}{\left(\frac{n_r c\varepsilon_o \omega^2 A^2_o}{2}\right)} \frac{2}{V} \sum_{k_a} \sum_{k_b} \frac{2\pi}{\hbar} \left|H'_{ba}\right|^2 \delta(E_b - E_a - \hbar\omega)(f_a - f_b)$$

$$= \frac{\hbar\omega}{\left(\frac{n_r c\varepsilon_o \omega^2 A^2_o}{2}\right)} \frac{2}{V} \sum_{k_v} \sum_{k_c} \frac{2\pi}{\hbar} \left| \frac{-eA_o}{2m_o} \hat{e} \bullet \mathbf{p}_{cv} \delta_{\mathbf{k}_c, \mathbf{k}_v} \right|^2 \delta(E_c - E_v - \hbar\omega)(f_v - f_c)$$

$$= \frac{\pi e^2}{n_r c\varepsilon_o m_o{}^2 \omega} \frac{2}{V} \sum_{k_v} \sum_{k_c} \left|\hat{e} \bullet \mathbf{p}_{cv}\right|^2 \delta_{\mathbf{k}_c, \mathbf{k}_v} \delta(E_c - E_v - \hbar\omega)(f_v - f_c)$$

$$= C_o \frac{2}{V} \sum_{k} \left|\hat{e} \bullet \mathbf{p}_{cv}\right|^2 \delta(E_c - E_v - \hbar\omega)(f_v(\mathbf{k}) - f_c(\mathbf{k}))$$

k represents both $\mathbf{k}_c$ and $\mathbf{k}_v$
They are equal by the k-selection rule

Optical Absorption Spectrum

Starting with the general expression

$$\alpha(\hbar\omega) = C_o \frac{2}{V} \sum_{\mathbf{k}} |\hat{e} \bullet \mathbf{p}_{cv}|^2 \delta(E_c - E_v - \hbar\omega)(f_v - f_c)$$

we assume an undoped bulk material in thermal equilibrium, with valence band fully occupied and conduction band completely empty

$$F_c = F_v = E_F \qquad f_v = 1 \qquad f_c = 0$$

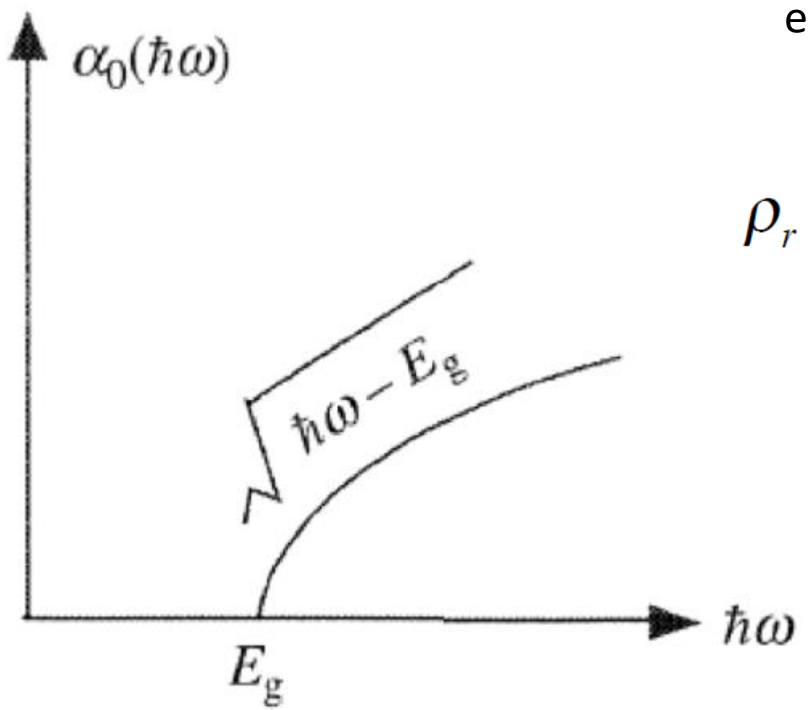
$$\alpha_0(\hbar\omega) = C_0 |\hat{e} \cdot \mathrm{p}_{cv}|^2 \int \frac{2d^2k}{(2\pi)^3} \delta\left(E_g + \frac{\hbar^2 k^2}{2m_r} - \hbar\omega\right)$$

$$\frac{1}{m_r^*} = \frac{1}{m_e^*} + \frac{1}{m_h^*}$$

Optical Absorption Spectrum

The integral can solved analytically arriving at the **bulk absorption coefficient**

$$\alpha_o(\hbar\omega) = C_o \underbrace{\left|\hat{e}\bullet\mathbf{p}_{cv}\right|^2}_{\text{momentum matrix element}} \underbrace{\rho_r\left(\hbar\omega - E_g\right)}_{\text{joint (reduced) density of states}}$$



$$\rho_r\left(\hbar\omega - E_g\right) = \frac{1}{2\pi^2}\left(\frac{2m_r^*}{\hbar^2}\right)^{3/2}\left(\hbar\omega - E_g\right)^{1/2}$$

$$\frac{1}{m_r^*} = \frac{1}{m_e^*} + \frac{1}{m_h^*}$$

Optical Gain Spectra (with carriers)

Under current injection or optical pumping, we have quasi-Fermi levels F_c and F_v

$$\alpha(\hbar\omega) = C_o \frac{2}{V} \sum_k \left| \hat{e} \cdot \mathbf{p}_{cv} \right|^2 \delta\left(E_c - E_v - \hbar\omega\right)\left(f_v(\mathbf{k}) - f_c(\mathbf{k})\right)$$

$$f_v(\mathbf{k}) = \frac{1}{1 + e^{(E_v(\mathbf{k}) - F_v)/\mathrm{kT}}} \qquad f_c(\mathbf{k}) = \frac{1}{1 + e^{(E_c(\mathbf{k}) - F_c)/\mathrm{kT}}}$$

We carry out the integral as before

$$\alpha(\hbar\omega) = C_0 \left| \hat{e} \cdot \mathrm{p}_{cv} \right|^2 \int \frac{2d^2k}{(2\pi)^3} \delta\left(E_g + \frac{\hbar^2 k^2}{2m_r} - \hbar\omega \right) \left[f_v(k) - f_c(k) \right]$$

Optical Gain Spectra (with carriers)

$$\alpha(\hbar\omega) = C_0 \left|\hat{e}\cdot \mathrm{p}_{cv}\right|^2 \int \frac{2d^2k}{(2\pi)^3}\,\delta\left(E_g + \frac{\hbar^2 k^2}{2m_r} - \hbar\omega\right)\,[f_v(k) - f_c(k)]$$

$$\Downarrow$$

$$\alpha(\hbar\omega) = \alpha_0(\hbar\omega)\big(f_v(\mathrm{k}_0) - f_c(\mathrm{k}_0)\big)$$

$$|\mathbf{k}| = \sqrt{\frac{2m_r^*}{\hbar^2}(\hbar\omega - E_g)} \equiv \mathrm{k}_0$$

Optical Gain Spectra (with carriers)

$$\alpha(\hbar\omega) = \alpha_0(\hbar\omega)\left(f_v(\mathrm{k}_0) - f_c(\mathrm{k}_0)\right)$$

We have **Gain** (negative absorption) when

$$f_v(\mathrm{k}_0) < f_c(\mathrm{k}_0)$$

$$\Downarrow$$

$$E_g < \hbar\omega < F_c - F_v$$

$$F_c - F_v > E_c - E_v = \hbar\omega$$

population inversion condition
(Bernard-Duraffourg condition)

Optical Gain Spectra (with carriers)

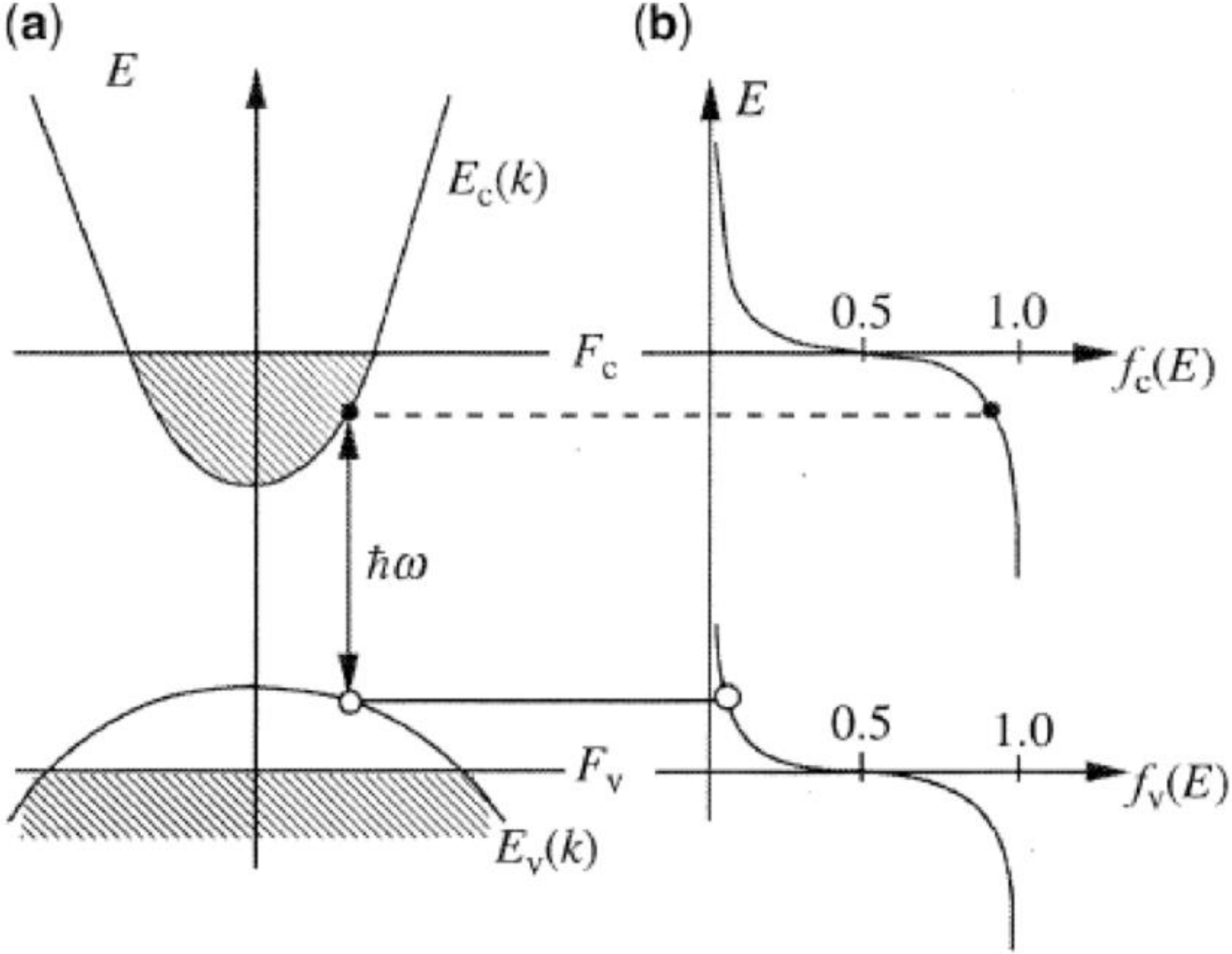
The magnitude of the loss (or gain) depends on wavelength. **If the separation of the quasi-Fermi levels is greater than the band gap, there is gain**.

- Photons with energies greater than the bandgap but less than the energy separation of the quasi-Fermi levels will experience gain
- Photons with energies greater than the separation of the quasi-Fermi levels will experience loss

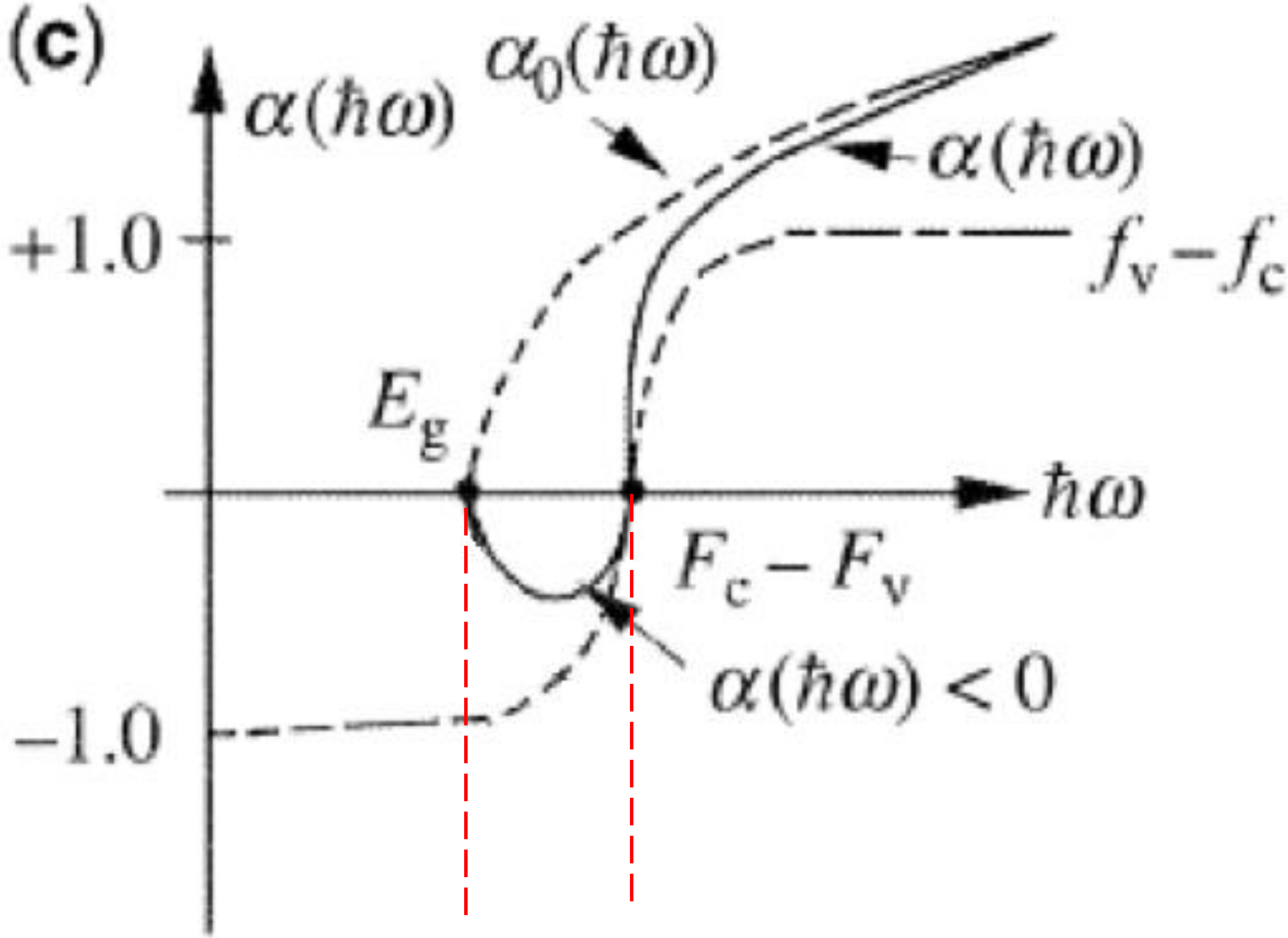
Gain condition

$$F_C - F_V > \hbar\omega > \underbrace{E_C - E_V}_{E_g}$$

Optical Gain Spectra (with carriers)



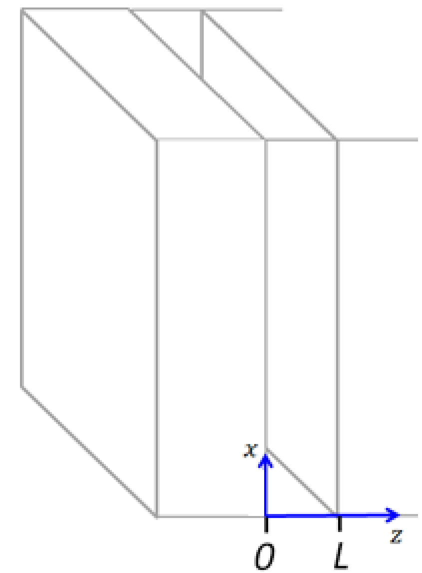
Optical Gain Spectra (with carriers)



Interband Absorption and Gain in Quantum Well

Simple square well potential (infinite barrier)

Consider a 1D well in 3D space quantized along *z*- axis

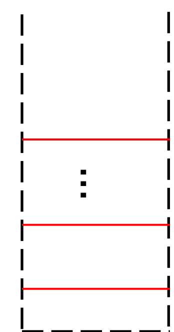


$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(z)\right]\psi(x,y,z) = E\psi(x,y,z)$$

$$\psi_n(x,y,z) = 0 \qquad\qquad z \le 0,\ z \ge L$$

$$\psi_n(x,y,z) = \overbrace{\frac{e^{ik_x x + ik_y y}}{\sqrt{A}}}^{\text{plane wave}}\ \overbrace{\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}z\right)}^{\phi_n(z)} \qquad\qquad \begin{array}{l} 0 < z < L \\ \mathrm{n} = 1, 2, 3, \ldots \end{array}$$

area in (*x*,*y*) plane   well width



$E_1 \ldots \quad E_N$

$$E_n = \frac{\hbar^2}{2m^*}\left(k_x^2 + k_y^2 + \left(\frac{n\pi}{L}\right)^2\right) \qquad\qquad k_z = \frac{n\pi}{L}$$

2-band model – we need to add the valence band

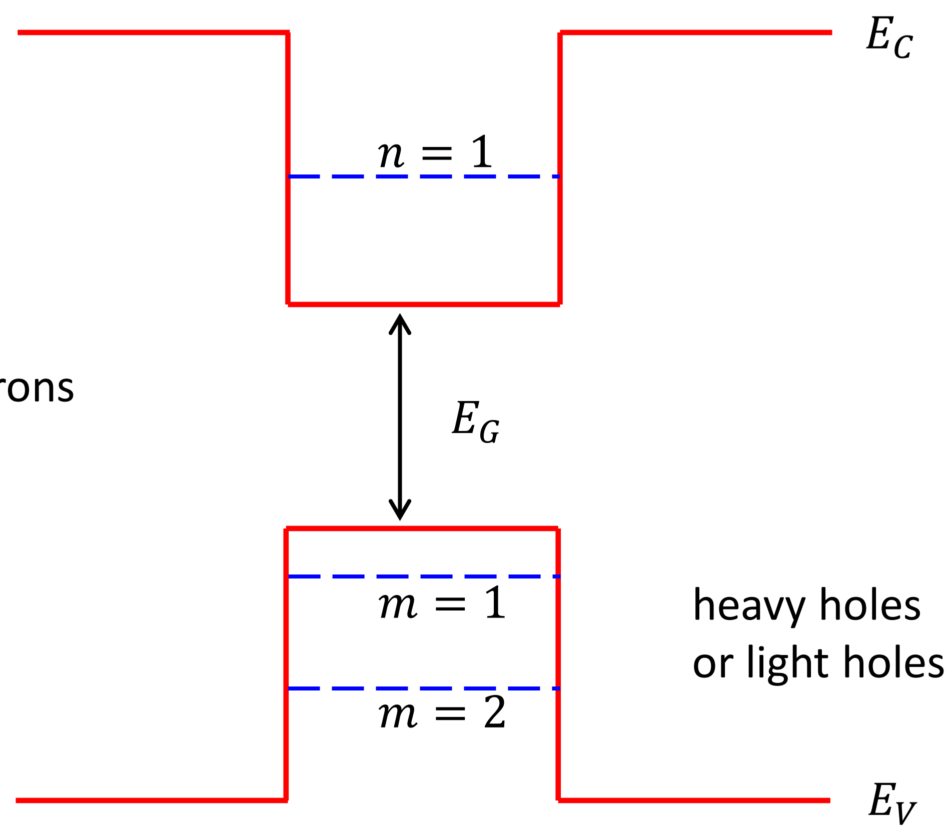
$$\psi_b(\mathbf{r}) = u_c(\mathbf{r}) \frac{e^{i\mathbf{k}_t'\cdot\rho}}{\sqrt{A}} \phi_n(z)$$

$$|\rho| = \sqrt{x^2 + y^2}$$

$\mathbf{k}_t'$= transverse momentum for electrons

$\mathbf{k}_t$= transverse momentum for holes

$$\psi_a(\mathbf{r}) = u_v(\mathbf{r}) \frac{e^{i\mathbf{k}_t\cdot\rho}}{\sqrt{A}} g_m(z)$$



quantized envelope wavefunction

plane wave on (*x*,*y*) plane

periodic Bloch wave

Interband Absorption and Gain in Quantum Well

(We neglect here exciton interactions between electrons and holes)

Starting with the general expression for the absorption coefficient

$$\alpha(\hbar\omega) = C_o \frac{2}{V} \sum_{\mathbf{k}_a} \sum_{\mathbf{k}_b} \left| \hat{e} \bullet \mathbf{p}_{ba} \right|^2 \delta\left(E_b - E_a - \hbar\omega\right)\left(f_a - f_b\right)$$

$$\mathbf{p}_{ba} = \langle \psi_b | \mathbf{p} | \psi_a \rangle = \int \psi_b^* \, \frac{\hbar}{i} \nabla \psi_a \, d^3\mathbf{r}$$

$$= \int u_c^*(\mathbf{r}) \frac{e^{-i\mathbf{k}_t' \cdot \mathbf{r}}}{\sqrt{A}} \phi_n^*(z) \frac{\hbar}{i} \nabla \left(u_v(\mathbf{r}) \frac{e^{i\mathbf{k}_t \cdot \mathbf{r}}}{\sqrt{A}} g_m(z) \right) d^3\mathbf{r}$$

$$\mathbf{p}_{ba} = \langle \psi_b | \mathbf{p} | \psi_a \rangle = \int \psi_b^* \frac{\hbar}{i} \nabla \psi_a \, d^3\mathbf{r}$$

$$= \int u_c^*(\mathbf{r}) \frac{e^{-i\mathbf{k}_t'\cdot\mathbf{r}}}{\sqrt{A}} \phi_n^*(z) \frac{\hbar}{i} \nabla \left(u_v(\mathbf{r}) \frac{e^{i\mathbf{k}_t\cdot\mathbf{r}}}{\sqrt{A}} g_m(z) \right) d^3\mathbf{r}$$

$$= \frac{\hbar}{iA} \int u_c^*(\mathbf{r}) e^{-i\mathbf{k}_t'\cdot\mathbf{r}} \phi_n^*(z) \left[e^{i\mathbf{k}_t\cdot\mathbf{r}} g_m(z) \nabla u_v(\mathbf{r}) + u_v(\mathbf{r}) g_m(z) \nabla e^{i\mathbf{k}_t\cdot\mathbf{r}} + u_v(\mathbf{r}) e^{i\mathbf{k}_t\cdot\mathbf{r}} \nabla g_m(z) \right] d^3\mathbf{r}$$

$$= \frac{\hbar}{iA} \int \underbrace{u_c^*(\mathbf{r}) \nabla u_v(\mathbf{r})}_{\approx \langle u_c | \mathbf{p} | u_v \rangle} \underbrace{e^{-i\mathbf{k}_t'\cdot\mathbf{r}} e^{i\mathbf{k}_t\cdot\mathbf{r}}}_{\delta_{\mathbf{k}_t,\mathbf{k}_t'}} \underbrace{\phi_n^*(z)\, g_m(z)}_{I_{hm}^{en}} + \cdots$$

electron state n

$$\mathbf{p}_{ba} \approx \langle u_c|\mathbf{p}|u_v\rangle \; \delta_{\mathbf{k}_t,\mathbf{k}_t'} \; \boldsymbol{I}_{hm}^{en}$$

hole state m

overlap integral of quantized wavefunctions in well

$\mathbf{p}_{cv}$

k-selection rule $\rightarrow$ $\mathbf{k}_t = \mathbf{k}_t'$
conservation of transverse momentum from exponentials

$$\boxed{\boldsymbol{I}_{hm}^{en} = \int_{-\infty}^{\infty} \phi_n^*(z)\, g_m(z)\, dz}$$

overlap integral

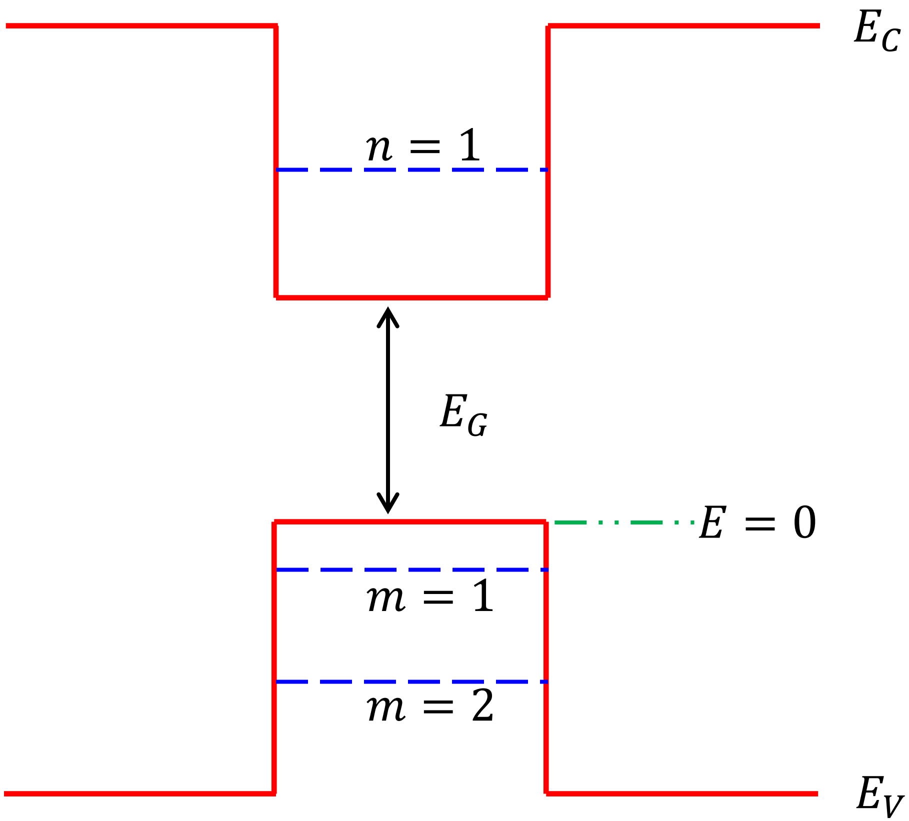
Transition energies (all are relative to top of valence band)

$$E_b = E_g + E_{en} + \frac{\hbar^2 k_t^2}{2m_e^*}$$

Energy level of electron state

$$E_a = E_{hm} - \frac{\hbar^2 k_t^2}{2m_h^*}$$

Energy level of hole state
(NOTE: it's negative)

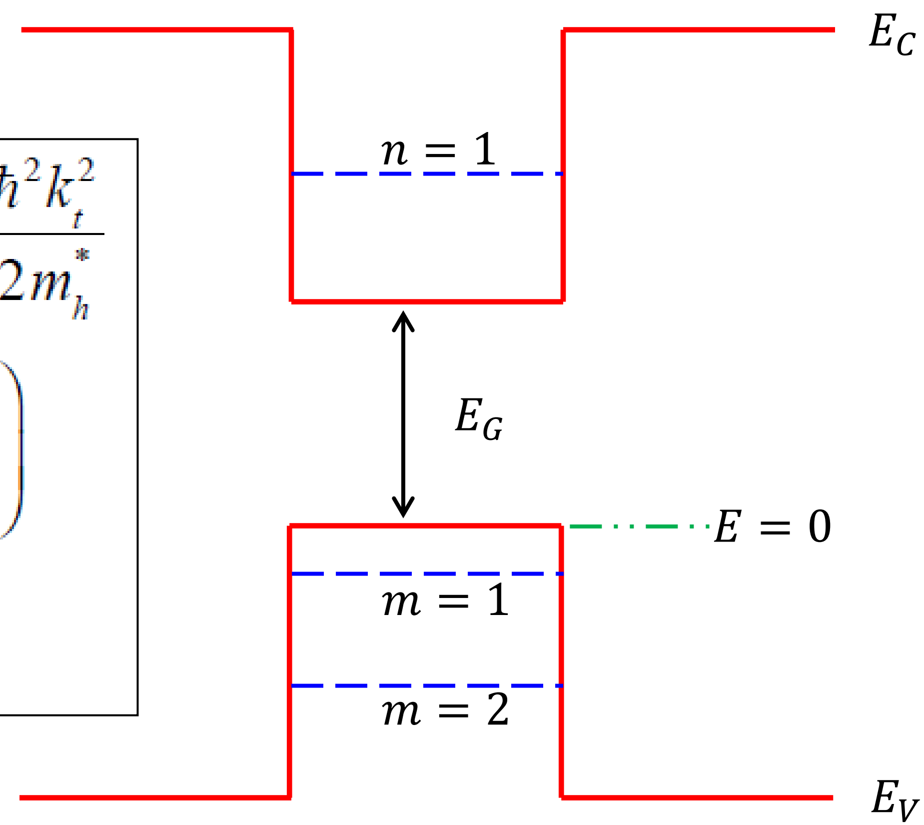


Transition energies (all are relative to top of valence band)

$$E_b - E_a = E_g + E_{en} + \frac{\hbar^2 k_t^2}{2m_e^*} - E_{hm} + \frac{\hbar^2 k_t^2}{2m_h^*}$$

$$= \left(E_g + E_{en} - E_{hm}\right) + \left(\frac{\hbar^2 k_t^2}{2m_h^*} + \frac{\hbar^2 k_t^2}{2m_e^*}\right)$$

$$= E_{hm}^{en} + E_t$$



$$E_{hm}^{en} = E_g + E_{en} - E_{hm}$$

Band edge transition energy

$$E_t = \frac{\hbar^2 k_t^2}{2m_r^*}$$

$$\frac{1}{m_r^*} = \frac{1}{m_e^*} + \frac{1}{m_h^*}$$

Transverse kinetic energy due to in-plane k-vector

Absorption expression for quantum well case

$$\alpha(\hbar\omega) = C_o \frac{2}{V} \sum_{\mathbf{k}_a} \sum_{\mathbf{k}_b} \left| \hat{e} \cdot \mathbf{p}_{ba} \right|^2 \delta\left(E_b - E_a - \hbar\omega\right)\left(f_a - f_b\right)$$

- Summations over k_a and k_b become summations over k_t and k_t'
- The k-selection rule establishes $\mathrm{k}_t = \mathrm{k}_t'$ so the sum becomes a sum over k_t
- We need to sum also over m and n

$$\alpha(\hbar\omega) = C_0 \sum_{n,m} \left| I_{hm}^{en} \right|^2 \frac{2}{V} \sum_{\mathbf{k}_t} \left| \hat{e} \cdot \mathbf{p}_{cv} \right|^2 \delta\left(E_{hm}^{en} + E_t - \hbar\omega\right)\left(f_v^m - f_c^n\right)$$

Joint Density of States

$$\frac{2}{V}\sum_{\mathbf{k_t}} = \frac{2}{V}\int \frac{d^2\mathbf{k_t}}{\left(\frac{2\pi}{L}\right)^2} = \frac{2A}{V}\int \frac{d^2\mathbf{k_t}}{(2\pi)^2} = \frac{1}{\pi L_z}\int_0^\infty k_t dk_t = \int_0^\infty \rho_r^{2D} dE_t$$

$$E_t = \frac{\hbar^2 k_t^2}{2m_r^*} \rightarrow \mathrm{d}E_t = \frac{\hbar^2}{m_r^*} k_t dk_t \rightarrow \boxed{k_t dk_t = \frac{m_r^*}{\hbar^2} dE_t} \rightarrow \boxed{\rho_r^{2D} = \frac{m_r^*}{\pi\hbar^2 L_z}}$$

For an unpumped semiconductor in thermal equilibrium $f_v^m = 1$ $f_c^n = 0$

$$\alpha(\hbar\omega) = C_0 \sum_{m,n} \left|I_{hm}^{en}\right|^2 \left|\hat{e}\bullet\mathbf{p}_{cv}\right|^2 \int_0^\infty \rho_r^{2D} \delta\left(E_{hm}^{en} + E_t - \hbar\omega\right)\left(\cancelto{1}{f_v^m - f_c^n}\right) dE_t$$

$$\boxed{\alpha_0(\hbar\omega) = C_0 \sum_{m,n} \left|I_{hm}^{en}\right|^2 \left|\hat{e}\bullet\mathbf{p_{cv}}\right|^2 \rho_r^{2D} H\left(\hbar\omega - E_{hm}^{en}\right)}$$

Optical Absorption Spectrum

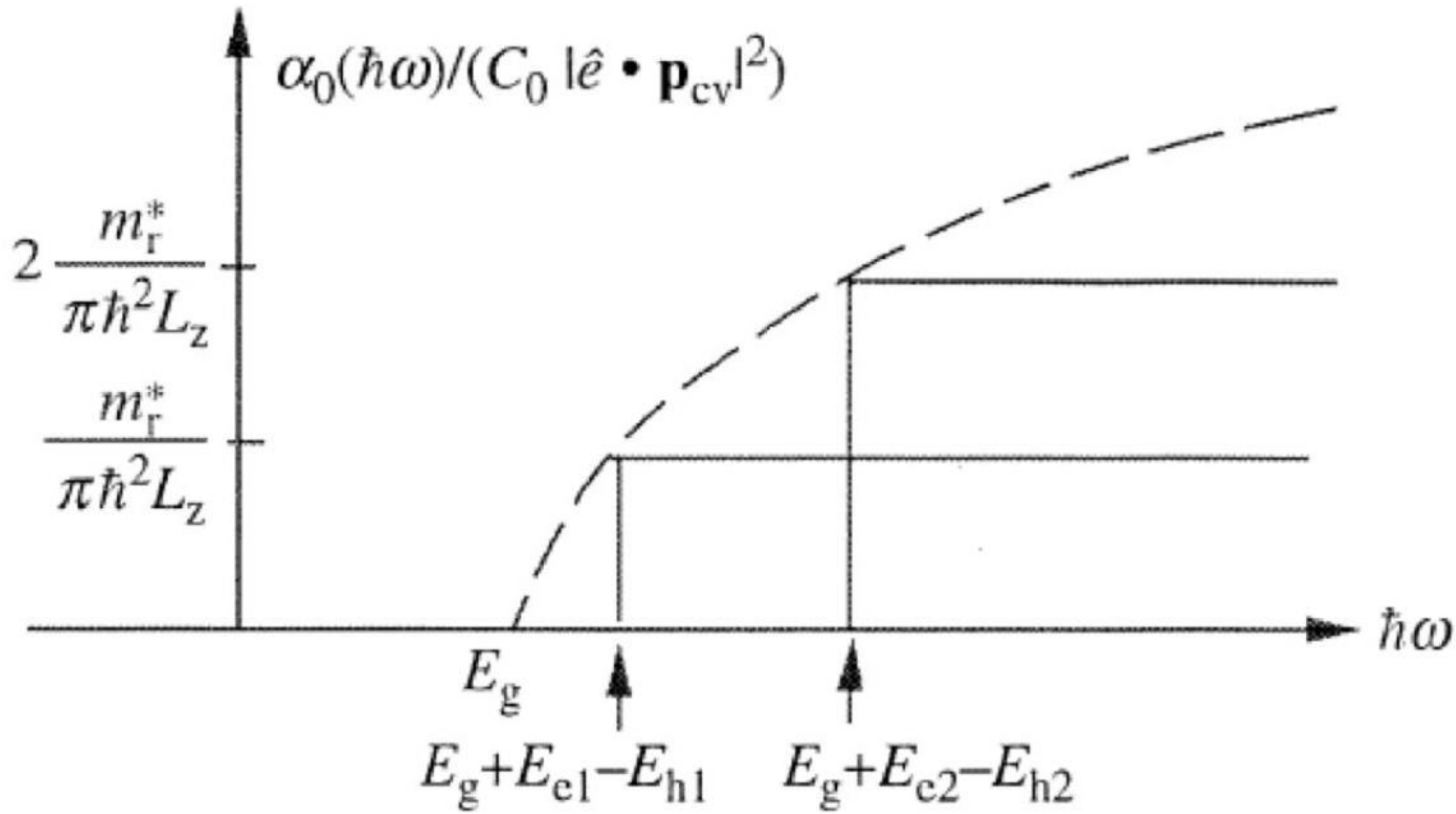
$$\alpha_0(\hbar\omega) = C_0 \left|\hat{e}\bullet\mathbf{p}_{\mathbf{cv}}\right|^2 \begin{cases} \dfrac{m_r^*}{\pi\hbar^2 L_z} & \text{for } E_{h1}^{e1} < \hbar\omega < E_{h2}^{e2} \\ 2\dfrac{m_r^*}{\pi\hbar^2 L_z} & \text{for } E_{h2}^{e2} < \hbar\omega < E_{h3}^{e3} \\ 3\dfrac{m_r^*}{\pi\hbar^2 L_z} & \text{for } E_{h3}^{e3} < \hbar\omega < E_{h4}^{e4} \\ etc & \end{cases}$$

For the case of carrier injection

$$\alpha(\hbar\omega) = \alpha_0(\hbar\omega)\left[f_v^m\left(\hbar\omega - E_{hm}^{en}\right) - f_c^n\left(\hbar\omega - E_{hm}^{en}\right)\right]$$

$$= \alpha_0(\hbar\omega)\left[f_v^m\left(E_t\right) - f_c^n\left(E_t\right)\right]$$

Optical Absorption Spectrum



Reading Assignments:

Sections 9.3 and 9.4 of Chuang's book

Electron Quasi Fermi Levels (when "*n*" electrons have been injected)

$$n = \sum_{\substack{n-occupied \\ subbands}} N_n = \sum_n \int_0^\infty \rho_e^{2D}(E) f_c^n(E)\, dE$$

$$N_n = \int_0^\infty \rho_e^{2D}(E) f_c^n(E)\, dE = \frac{m_e^*}{\pi\hbar^2 L_z} \int_{E_{en}}^\infty f_c^n(E)\, dE$$

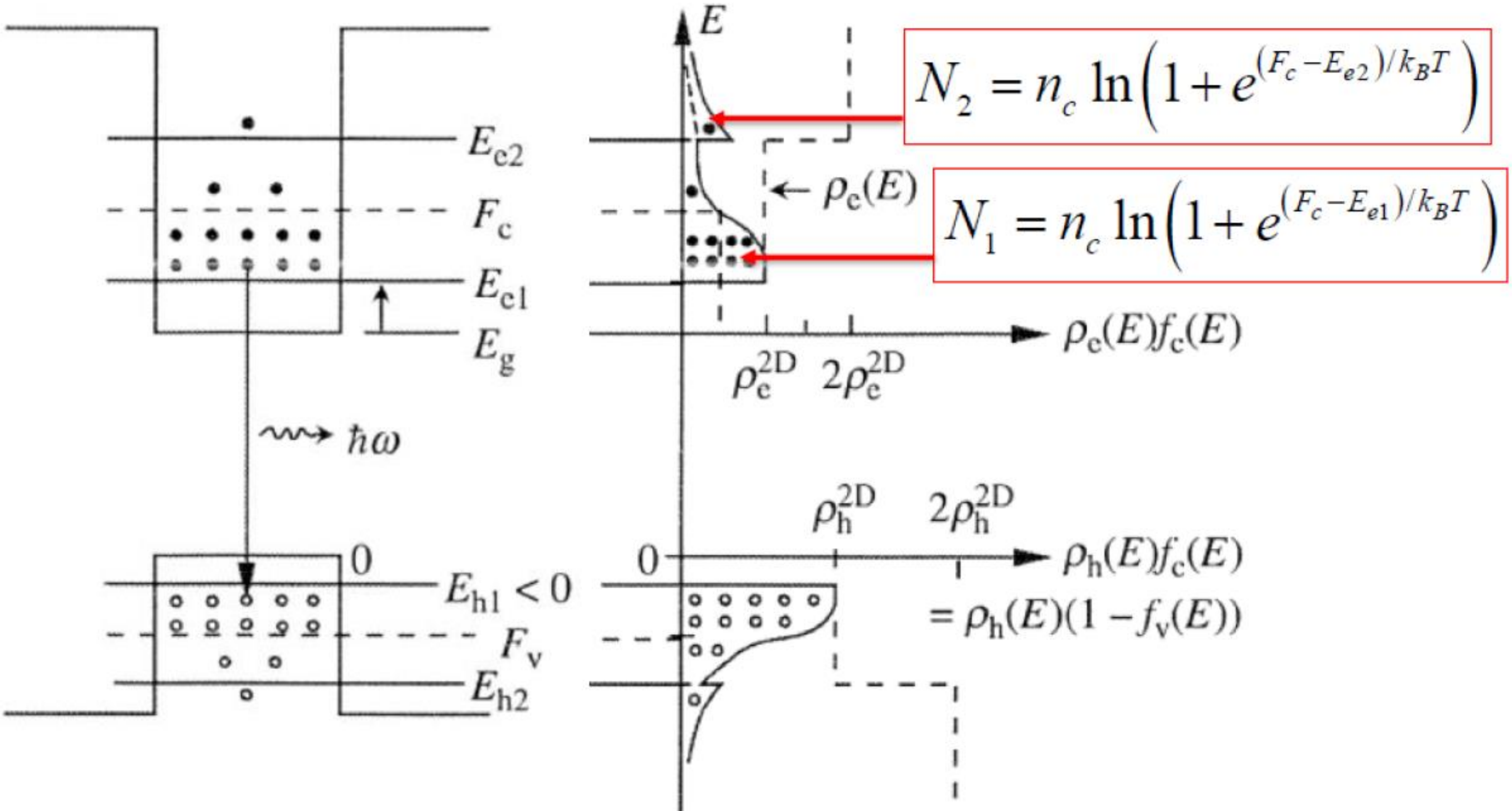
$$\int_{E_{en}}^\infty f_c^n(E)\, dE = \int_{E_{en}}^\infty \frac{1}{1+e^{(E-F_c)/k_B T}}\, dE = -k_B T \ln\left(1+e^{(F_c-E)/k_B T}\right)\Bigg|_{E_{en}}^\infty$$

$$\boxed{N_n = \frac{m_e^* k_B T}{\pi\hbar^2 L_z} \ln\left(1+e^{(F_c-E_{en})/k_B T}\right) = n_c \ln\left(1+e^{(F_c-E_{en})/k_B T}\right)}$$

Electron Quasi Fermi Levels (when "*n*" electrons have been injected)

$$N_n = n_c \ln\left(1 + e^{(F_c - E_{en})/k_B T}\right)$$

$$n_c = \frac{m_e^* k_B T}{\pi \hbar^2 L_z}$$



Holes Quasi Fermi level (when "*p*" holes have been injected)

$$E_a = E_{hm} - \frac{\hbar^2 k_t^2}{2m_h^*} \quad \text{where } E_{hm} < 0$$

$$E_{t,h} = -\frac{\hbar^2 k_t^2}{2m_h^*} \qquad\qquad \mathrm{d}E_t = -\frac{\hbar^2}{m_h^*} k_t dk_t$$

$$\frac{2}{V}\sum_{\mathbf{k_t}} = \frac{2}{V}\int \frac{d^2\mathbf{k_t}}{\left(\dfrac{2\pi}{L}\right)^2} = \frac{2A}{V}\int \frac{d^2\mathbf{k_t}}{(2\pi)^2} = \frac{1}{\pi L_z}\int_0^\infty k_t dk_t$$

$$= -\int_0^{-\infty} \rho_h^{2D} dE_{t,h} = \int_{-\infty}^{0} \rho_h^{2D} dE_{t,h}$$

Holes Quasi Fermi level (when "*p*" holes have been injected)

$$p = n + N_A^- - N_D^+ = \sum_{\substack{occupied \\ subbands}} P_m = \sum_{\substack{occupied \\ subbands}} \int_{-\infty}^{0} \rho_h^{2D}(E)\left[1 - f_v^m(E)\right] dE$$

$$P_m = \int_{-\infty}^{0} \rho_h^{2D}(E) f_h^m(E)\, dE = \frac{m_h^*}{\pi \hbar^2 L_z} \int_{-\infty}^{E_{hm}} f_h^m(E)\, dE$$

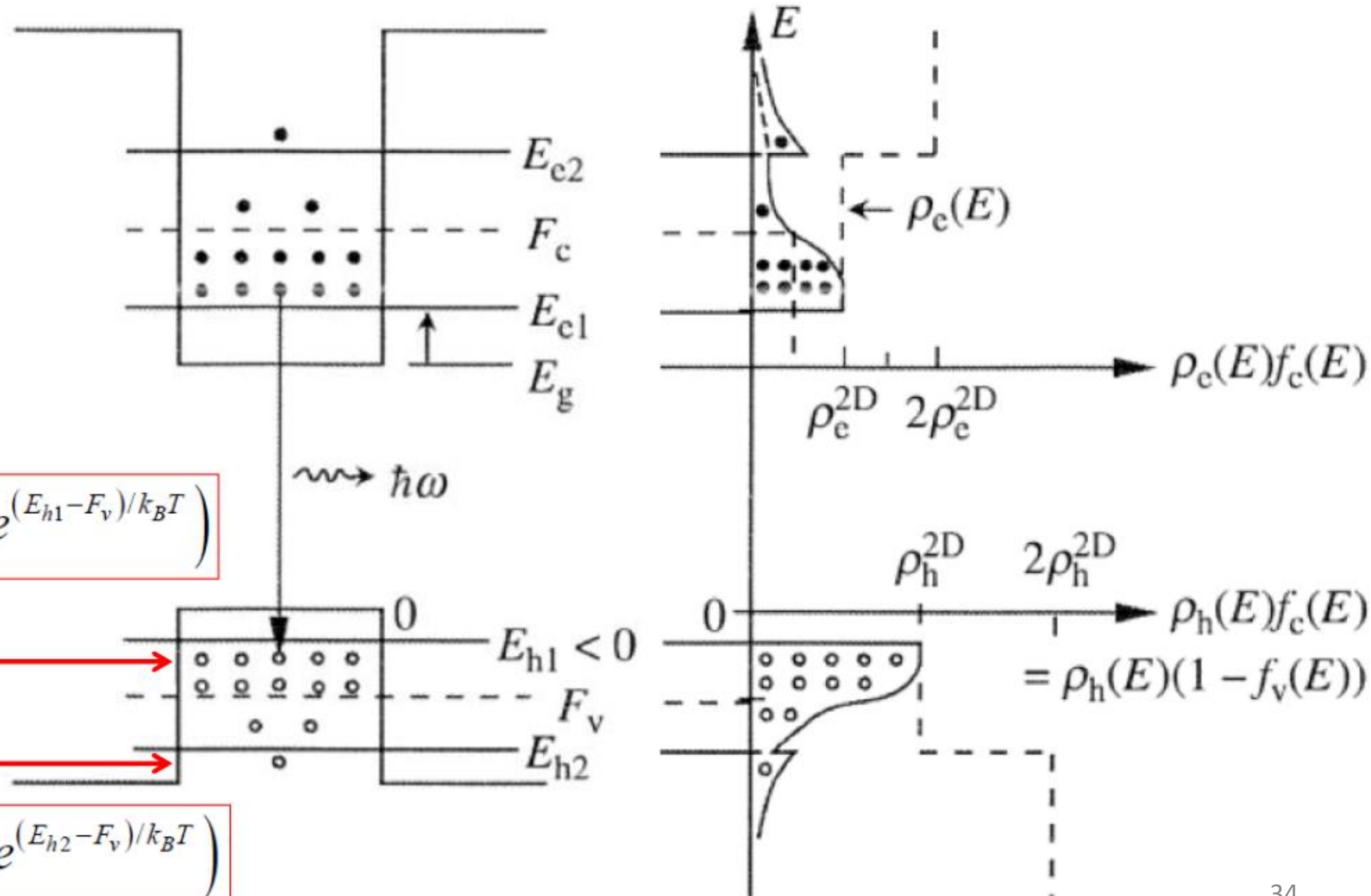
$$\int_{-\infty}^{E_{hm}} f_h^m(E)\, dE = \int_{-\infty}^{E_{hm}} \frac{1}{1 + e^{-(E-F_v)/k_B T}} dE = k_B T \ln\left(1 + e^{(E-F_v)/k_B T}\right)\Bigg|_{-\infty}^{E_{hm}}$$

$$\boxed{P_m = \frac{m_h^* k_B T}{\pi \hbar^2 L_z} \ln\left(1 + e^{(E_{hm}-F_v)/k_B T}\right) = n_v \ln\left(1 + e^{(E_{hm}-F_v)/k_B T}\right)}$$

Holes Quasi Fermi level (when "*p*" holes have been injected)

$$P_m = n_v \ln\left(1 + e^{(E_{hm} - F_v)/k_B T}\right)$$

$$n_v = \frac{m_h^* k_B T}{\pi \hbar^2 L_z}$$



$$P_1 = n_v \ln\left(1 + e^{(E_{h1} - F_v)/k_B T}\right)$$

$$P_2 = n_v \ln\left(1 + e^{(E_{h2} - F_v)/k_B T}\right)$$

Gain Spectrum

Zero Linewidth Case:

$$g(\hbar\omega) = C_0 \sum_{m,n} \left| I_{hm}^{en} \right|^2 \left| \hat{e} \bullet \mathbf{p}_{\mathbf{cv}} \right|^2 \left[f_c^n(E_t) - f_v^m(E_t) \right] \rho_r^{2D} H\left(\hbar\omega - E_{hm}^{en}\right)$$

$$\boxed{C_o = \frac{\pi e^2}{n_r c \varepsilon_0 m_0^2 \omega}}$$

$$\boxed{I_{hm}^{en} = \int_{-\infty}^{\infty} \phi_n^*(z)\, g_m(z)\, dz}$$

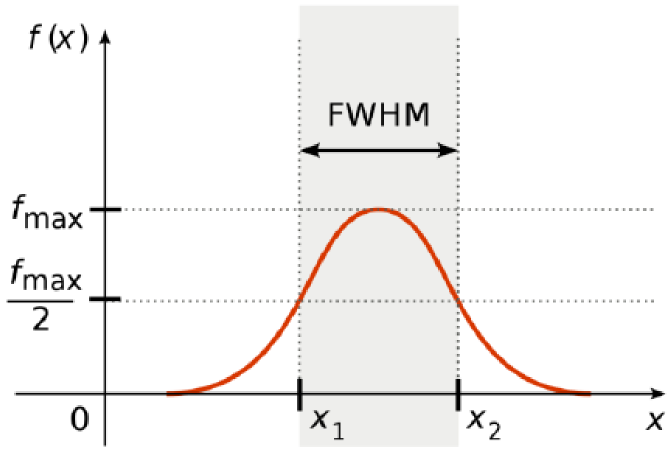
$$\boxed{\rho_r^{2D} = \frac{m_r^*}{\pi \hbar^2 L_z}}$$

$$\boxed{f_c^n(E_t) = \frac{1}{1 + \exp\left[\left(E_g + E_{en} + \frac{m_r^*}{m_e^*} E_t - F_c\right) / k_B T\right]}}$$

$$\boxed{f_v^m(E_t) = \frac{1}{1 + \exp\left[\left(E_{hm} - \frac{m_r^*}{m_h^*} E_t - F_v\right) / k_B T\right]}}$$

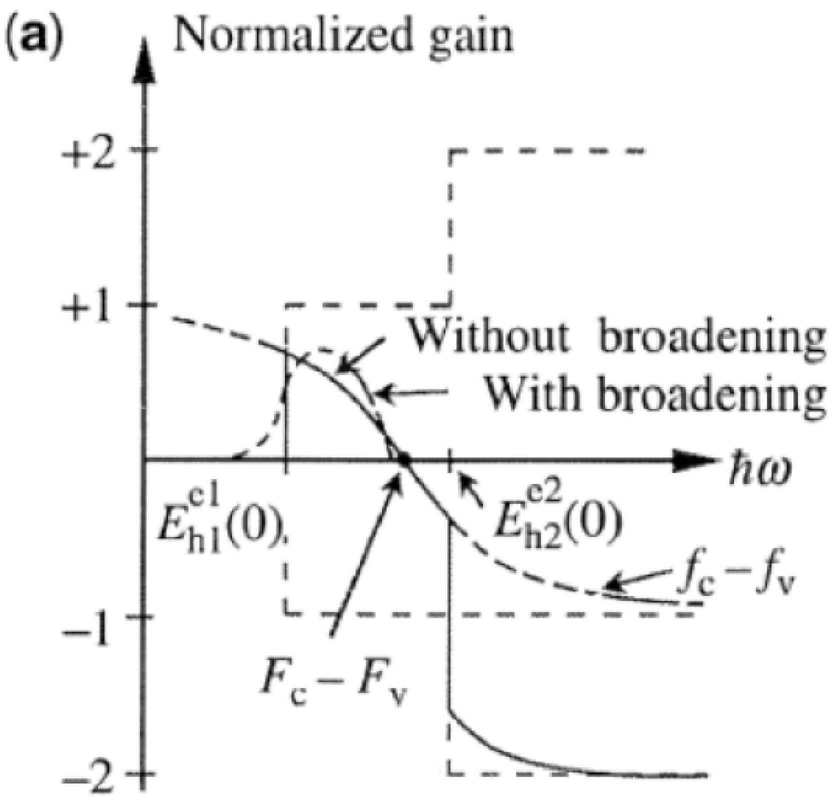
Gain Spectrum

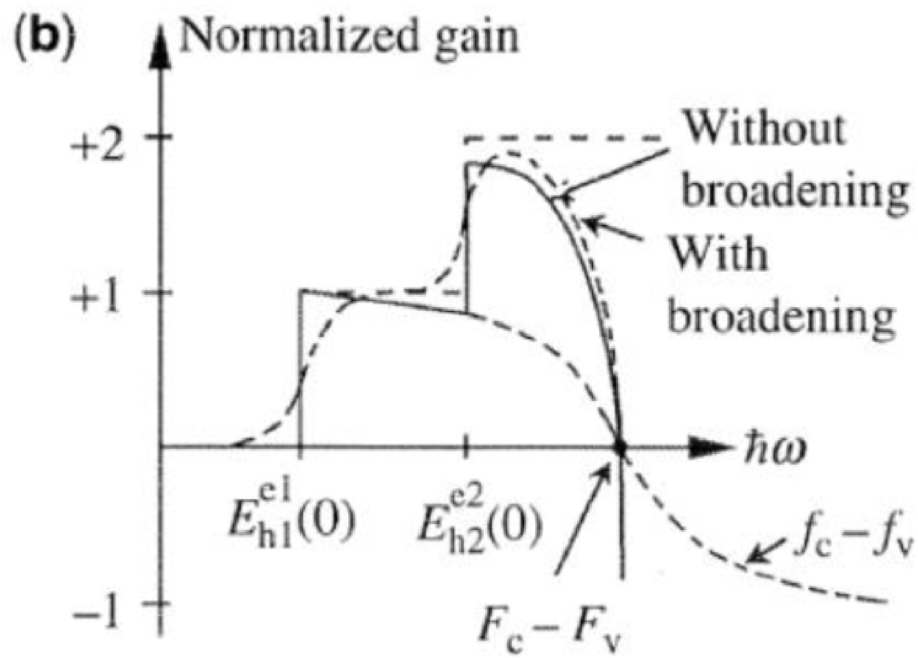
Linewidth Broadening Case
Full Width at Half Maximum (FWHM = 2γ)



$$g(\hbar\omega) = C_0 \sum_{m,n} \left| I_{hm}^{en} \right|^2 \int_0^\infty dE_t \rho_r^{2D} \left| \hat{e} \bullet \mathbf{p}_{\mathbf{cv}} \right|^2 \frac{\gamma / \pi}{\left[E_{hm}^{en} + E_t - \hbar\omega \right]^2 + \gamma^2} \left[f_c^n(E_t) - f_v^m(E_t) \right]$$

Gain Spectra with and without broadening

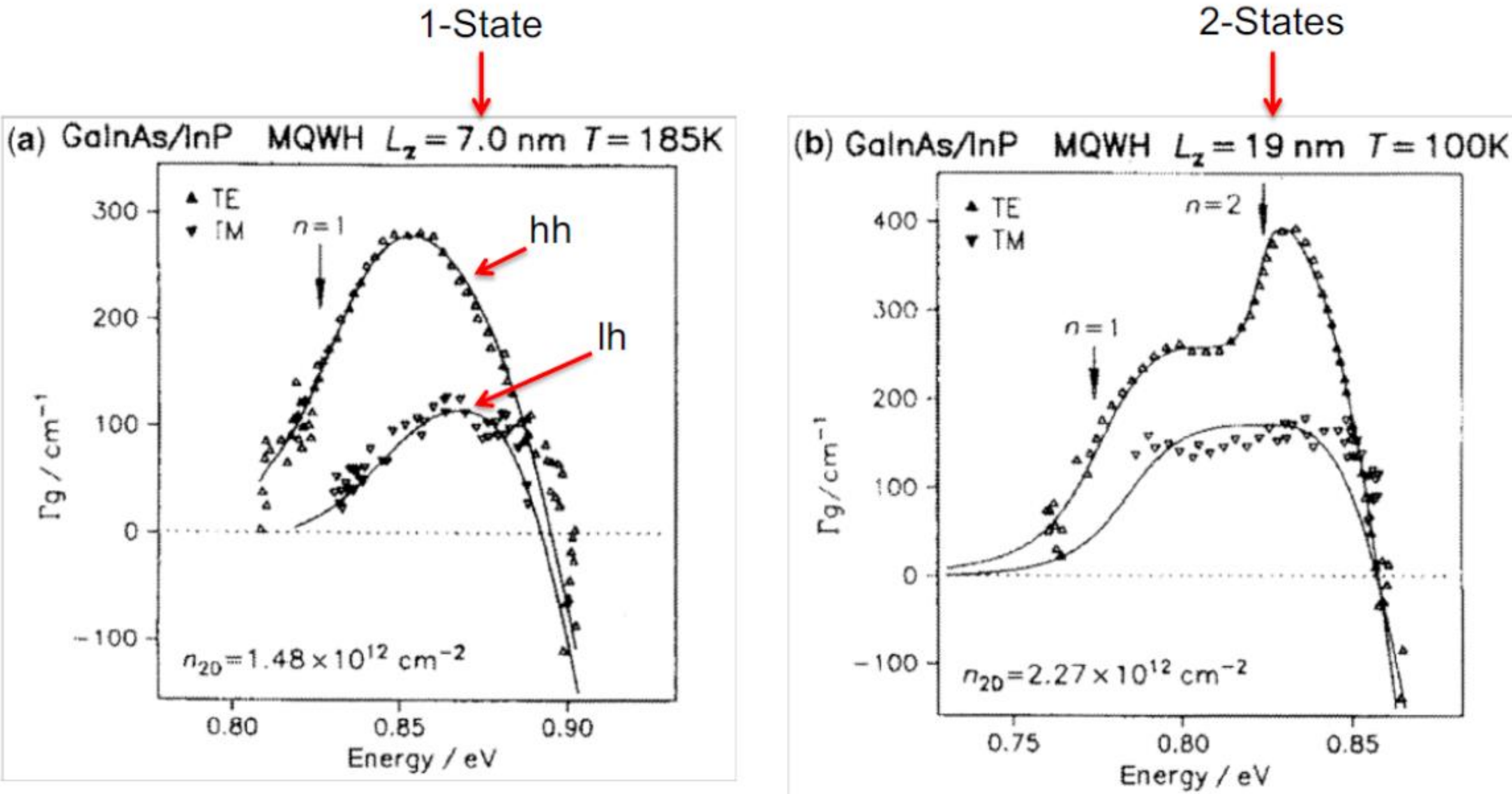




Transition involving one single electron and hole subband pair

Transition involving two electron and two hole subbands

Comparison with experimental data

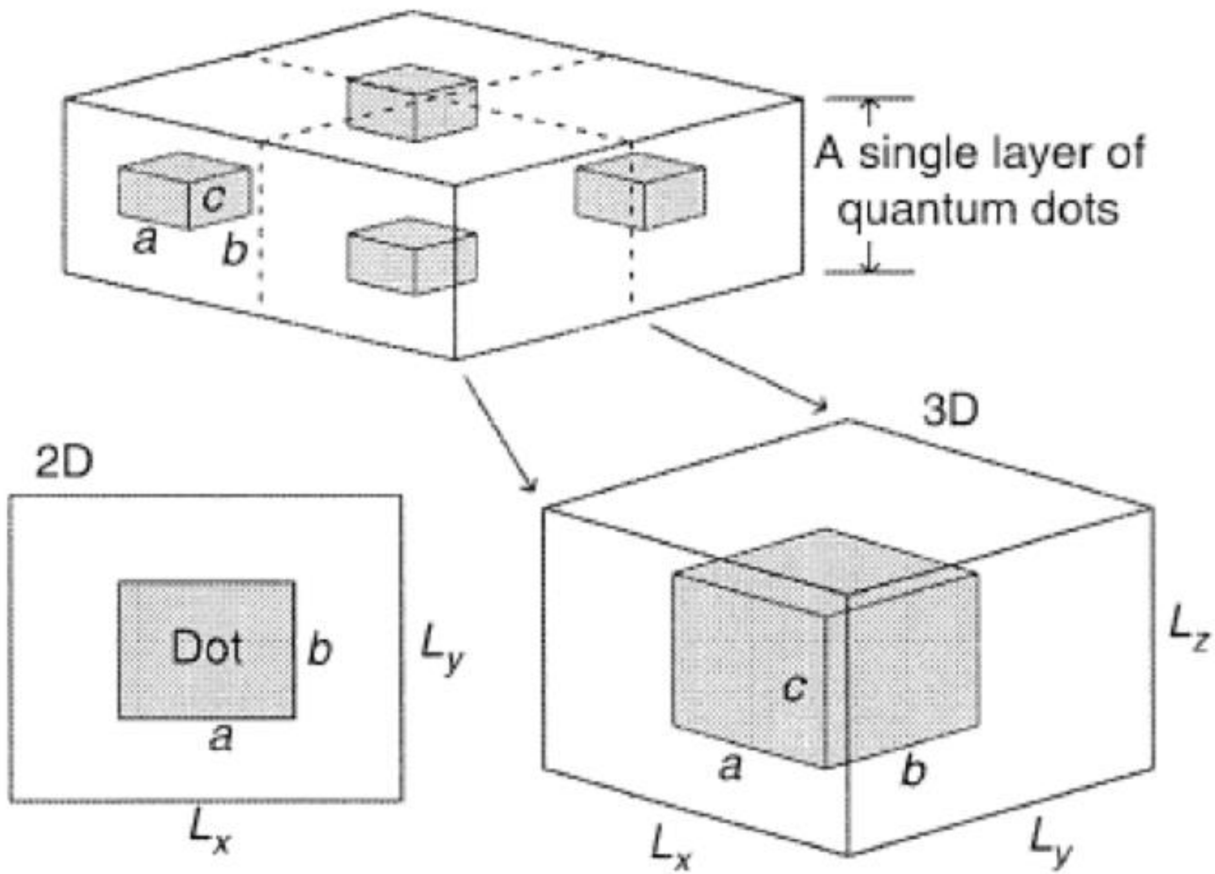


Zielinski *et al*, IEEE J. Quantum Electronics, QE-23, p.969 (1987).

Quantum Dots

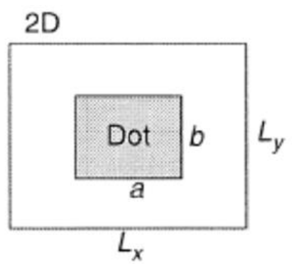
Ideal quantum dot assumptions

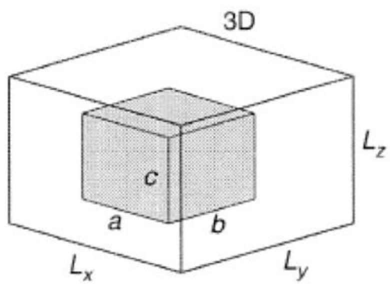
- **Uniform dot size**
- **Uniform distribution**



Ideal Quantum Dot – Wave functions and Energies

Dot Size $a \times b \times c$ **Total Volume of Single Dot** $V = L_x L_y L_z$





Dot Density

$$N_{dot}^{2D} = \frac{1}{L_x L_y}$$

$$N_{dot}^{3D} = \frac{1}{V} = \frac{1}{L_x L_y L_z}$$

Fill Factor

$$F^{2D} = \frac{ab}{L_x L_y}$$

$$F^{3D} = \frac{abc}{L_x L_y L_z}$$

Conduction Band – Wave functions and Energies

Wave Functions (Assume infinite barrier for simplicity)

$$\psi_c(x,y,z) = \frac{\sqrt{8}}{\sqrt{abc}} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sin\left(\frac{l\pi}{c}z\right) u_c(\mathbf{r})$$

Energy eigenvalues

$$E_c^{mnl} = E_{c0} + \frac{\hbar^2}{2m_e^*}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2\right]$$

Electron Density

$$n = \frac{2}{V}\sum_{m,n,l} f_c(E) = N_{dot}^{3D} 2\sum_{m,n,l} f_c\left(E_c^{nml}\right) = 2\frac{N_{dot}^{2D}}{L_z}\sum_{m,n,l}\frac{1}{1+e^{\left(E_c^{nml}-F_c\right)/k_BT}}$$

Valence Band – Wave functions and Energies

Wave Functions (Assume infinite barrier for simplicity)

$$\psi_v(x,y,z) = \frac{\sqrt{8}}{\sqrt{abc}} \sin\left(\frac{m'\pi}{a}x\right) \sin\left(\frac{n'\pi}{b}y\right) \sin\left(\frac{l'\pi}{c}z\right) u_v(\mathbf{r})$$

Energy eigenvalues

$$E_v^{m'n'l'} = E_{v0} - \frac{\hbar^2}{2m_h^*}\left[\left(\frac{m'\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2 + \left(\frac{l'\pi}{c}\right)^2\right]$$

Hole Density

$$p = \frac{2}{V}\sum_{m',n',l'}\left(1 - f_v(E)\right) = N_{dot}^{3D} 2 \sum_{m',n',l'}\left(1 - f_v\left(E_v^{n'm'l'}\right)\right) = 2\frac{N_{dot}^{2D}}{L_z}\sum_{m',n',l'} 1 - \frac{1}{1 + e^{\left(E_v^{m'n'l'} - F_v\right)/k_B T}}$$

$$p = 2\frac{N_{dot}^{2D}}{L_z}\sum_{m',n',l'} \frac{1}{1 + e^{\left(F_v - E_v^{m'n'l'}\right)/k_B T}}$$

Interband Absorption Spectrum

General expression

$$\alpha(\hbar\omega) = C_o \frac{2}{V} \sum_{\mathbf{k}} \left| \hat{e} \bullet \mathbf{p}_{cv} \right|^2 \delta(E_c - E_v - \hbar\omega)(f_v - f_c)$$

Quantization in all dimensions $\rightarrow$ sum over m, n, l

$$\alpha(\hbar\omega) = C_o \frac{2}{V} \sum_{m,n,l} \sum_{m',n',l'} \left| \langle \psi_c | \hat{e} \bullet \mathbf{p} | \psi_v \rangle \right|^2 \delta\left(E_c^{mnl} - E_v^{m'n'l'} - \hbar\omega\right)(f_v - f_c)$$

$$\boxed{\langle \psi_c | \hat{e} \bullet \mathbf{p} | \psi_v \rangle \simeq \langle u_c | \hat{e} \bullet \mathbf{p} | u_v \rangle \delta_{mm'} \delta_{nn'} \delta_{ll'} = \hat{e} \bullet \mathbf{p}_{cv} \delta_{mm'} \delta_{nn'} \delta_{ll'}}$$

$$\rightarrow \alpha(\hbar\omega) = C_o \frac{2N_{dot}^{2D}}{L_z} \sum_{m,n,l} \left| \hat{e} \bullet \mathbf{p}_{cv} \right|^2 \delta\left(E_c^{mnl} - E_v^{mnl} - \hbar\omega\right)(f_v - f_c)$$

with Interband Transition Energies

$$E_{cv}^{mnl} = E_c^{mnl} - E_v^{mnl} = E_g + \frac{\hbar^2}{2m_r^*}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2\right]$$

Absorption – Homogeneous Broadening

In case of no carrier injection, we can approximate $f_v = 1 \quad f_c = 0$

For homogeneous broadening, the delta function can be replaced by a Lorentzian

$$\alpha_0(\hbar\omega) = C_0 \frac{2N_{dot}^{2D}}{L_z} \sum_{m,n,l} \left|\hat{e} \bullet \mathbf{p}_{cv}\right|^2 \underbrace{L\left(E_{cv}^{mnl} - \hbar\omega\right)}$$

$$\boxed{L\left(E - \hbar\omega\right) = \frac{\gamma / \pi}{\left(E - \hbar\omega\right)^2 + \gamma^2}}$$

With carrier injection

$$\alpha(\hbar\omega) = C_0 \frac{2N_{dot}^{2D}}{L_z} \sum_{m,n,l} \left|\hat{e} \bullet \mathbf{p}_{cv}\right|^2 L\left(E_{cv}^{mnl} - \hbar\omega\right)\left(f_v - f_c\right)$$

Absorption – Inhomogeneous Broadening

For unifom dots we had: $$n = 2\frac{N_{dot}^{2D}}{L_z}\sum_{m,n,l} f_c\left(E_c^{mnl}\right)$$

In realistic case we will have variations of quantum dot size. Dot energy level become a Gaussian distribution with

$$G(E) = \frac{1}{\sqrt{2\pi}\sigma_c} e^{-\left(E-E_c^{mnl}\right)^2/2\sigma_c^2}$$

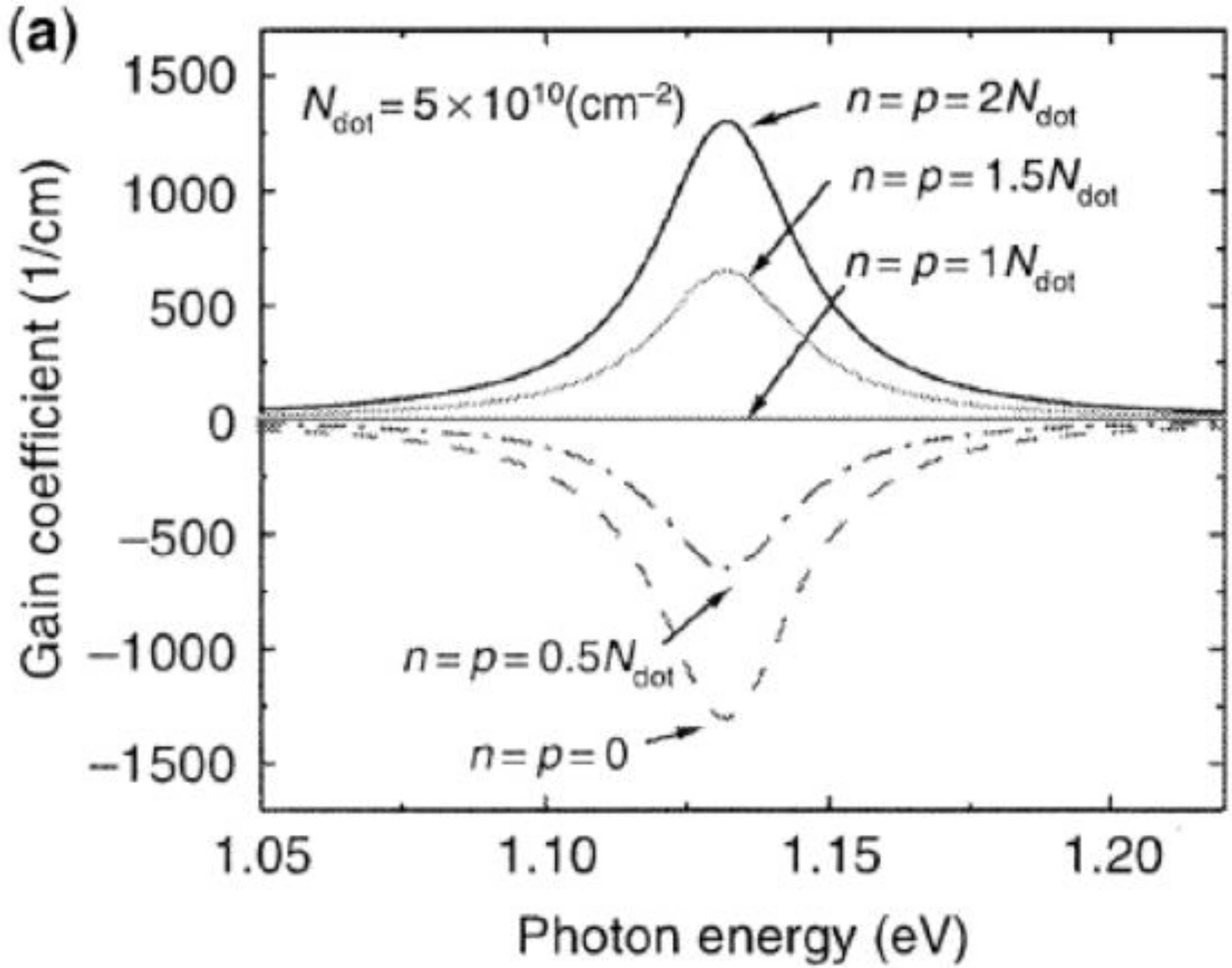
$$\begin{cases} \text{mean}=E_c^{mnl} \\ \text{FWHM}=2\sqrt{2\ln 2}\sigma_c \simeq 2.35\sigma_c \end{cases}$$

carrier density $$n = 2\frac{N_{dot}^{2D}}{L_z}\sum_{m,n,l}\int_0^\infty dE\; G(E) f_c(E)$$

$$\boxed{\begin{aligned} &\alpha(\hbar\omega) = C_0 \sum_{m,n,l}\int_0^\infty dE \left|\hat{e}\cdot\mathbf{p}_{\mathbf{cv}}\right|^2 D(E) L(E-\hbar\omega)(f_v - f_c) \\ &D(E) = 2\frac{N_{dot}^{2D}}{L_z}\frac{1}{\sqrt{2\pi}\sigma} e^{-\left(E-E_{cv}^{mnl}\right)^2/2\sigma^2} \qquad \sigma^2 = \sigma_c^{\ 2} + \sigma_v^{\ 2} \end{aligned}}$$

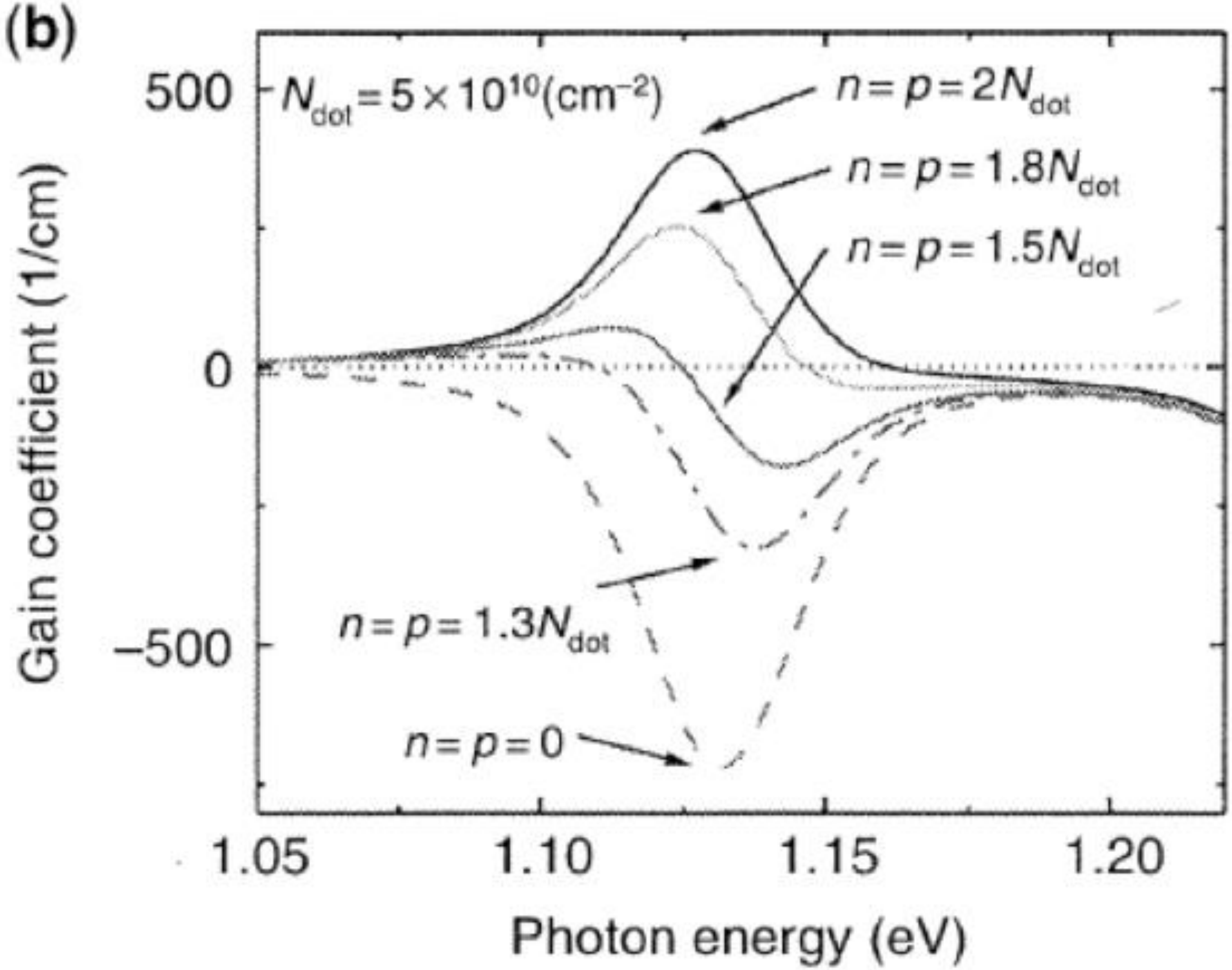
Examples = Homogeneous Broadening

FWHM = 30 meV



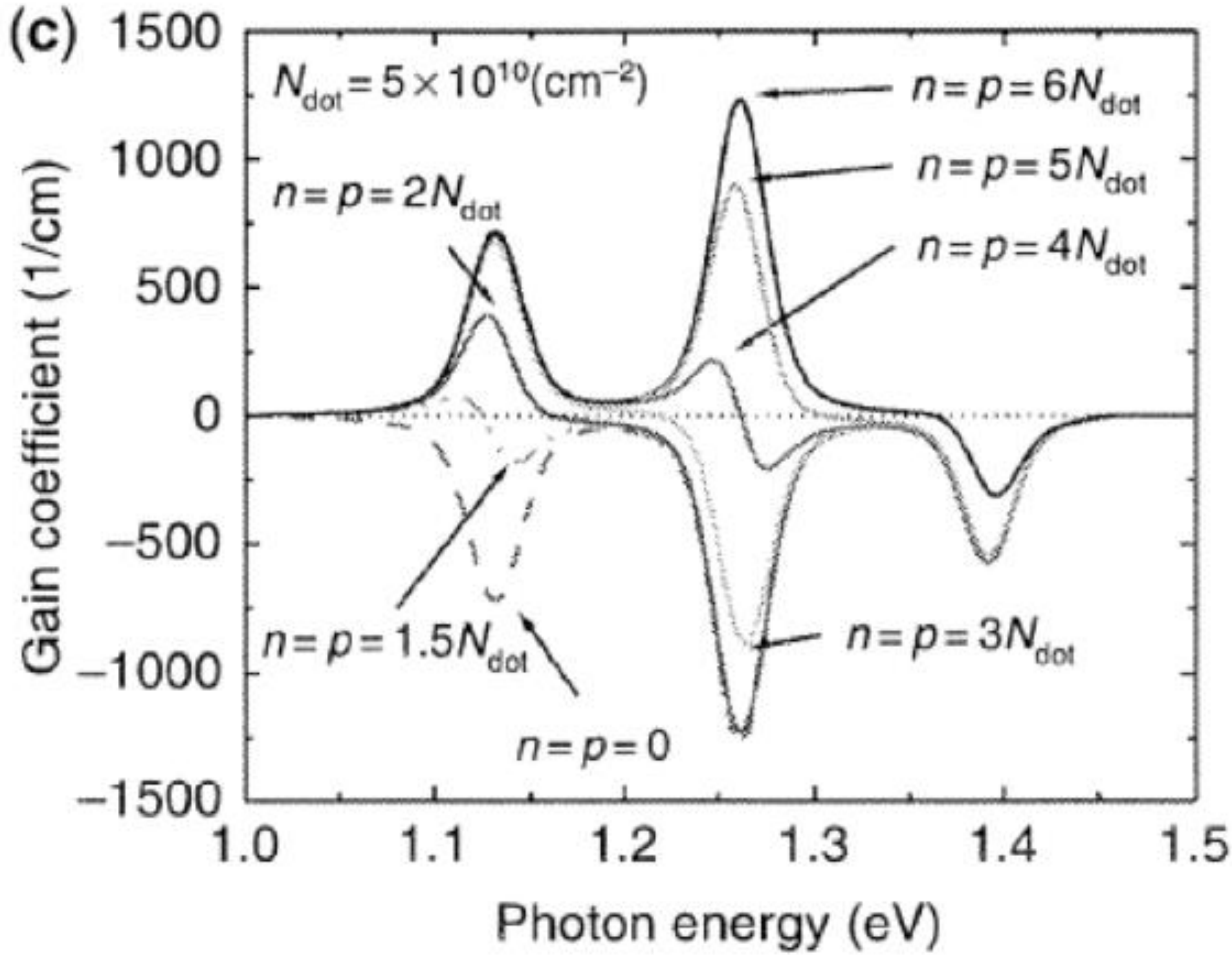
Examples = Homogeneous +Inhomogeneous Broadening

FWHM = 30 meV **FWHM = 50 meV**

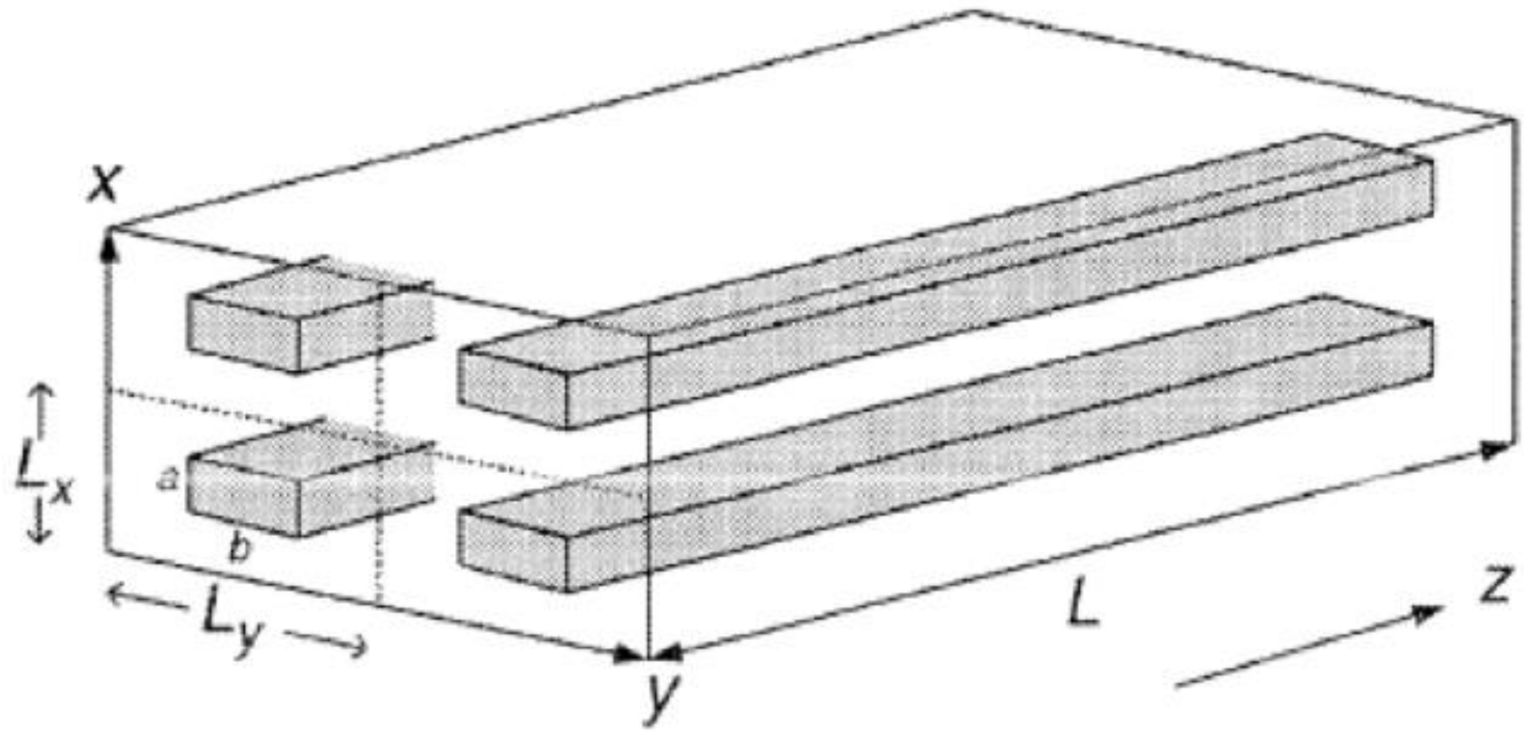


Examples = Homogeneous +Inhomogeneous Broadening

FWHM = 30 meV **FWHM = 50 meV**

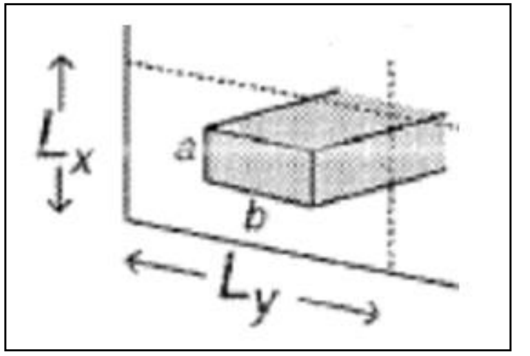


Quantum Wires



Ideal Quantum Wire – Wave functions and Energies

Areal Density in $x-y$ cross-section $$N_{wr} = \frac{1}{L_x L_y}$$



Fill Factor $$F_{wr} = \frac{ab}{L_x L_y}$$ **Wire Length** $L \gg a, b$

Wave Functions

$$\psi_c(x,y,z) = \frac{2}{\sqrt{ab}} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \frac{1}{\sqrt{L}} e^{ik_z z} u_c(\mathbf{r})$$

$$\psi_v(x,y,z) = \frac{2}{\sqrt{ab}} \sin\left(\frac{m'\pi}{a}x\right) \sin\left(\frac{n'\pi}{b}y\right) \frac{1}{\sqrt{L}} e^{ik'_z z} u_v(\mathbf{r})$$

Energy eigenvalues

$$E_c^{mn}(k_z) = E_{c0} + \frac{\hbar^2}{2m_e^*}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + k_z^2\right]$$

$$E_v^{m'n'}(k'_z) = E_{v0} - \frac{\hbar^2}{2m_h^*}\left[\left(\frac{m'\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2 + k_z'^2\right]$$

Density of States

Electron Density

$$n = \frac{2}{V}\sum_{m,n,k_z} f_c\left(E\right) = N_{wr}\frac{2}{L}\sum_{m,n}\int_{-\infty}^{\infty}\frac{dk_z}{\left(\dfrac{2\pi}{L}\right)} f_c\left(E\right)$$

$$= N_{wr}\int_{0}^{\infty} dE\,\rho_e^{1D}\left(E\right) f_c\left(E\right)$$

Density of state in a 1-D wire quantized along the x and y directions

$$\rho_e^{1D}\left(E\right) = \sum_{m,n}\frac{1}{\pi}\sqrt{\frac{2m_e^*}{\hbar^2}}\frac{1}{\sqrt{E-E_c^{mn}}} \quad \text{for } E > E_c^{mn}$$

Absorption coefficient

$$\alpha(\hbar\omega) = C_0 N_{wr} \sum_{m,n} \left|\hat{e}\bullet\mathbf{p}_{\mathbf{cv}}\right|^2 \rho_r^{1D}\left(\hbar\omega - E_c^{mn}\right)\left(f_v - f_c\right)$$

joint density of states

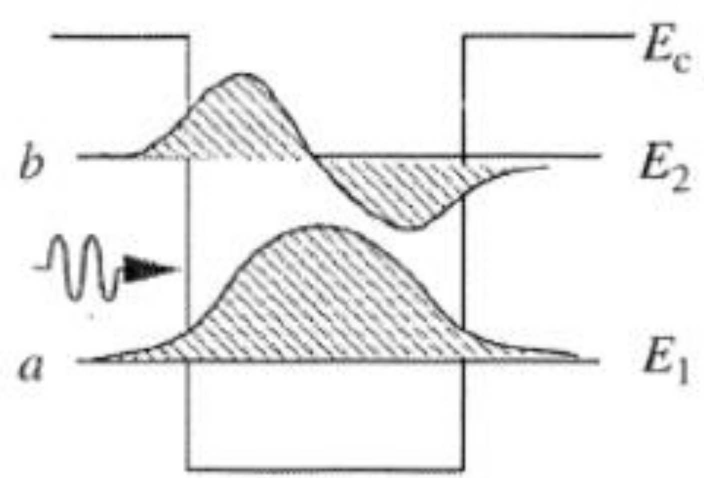
$$\rho_r^{1D}\left(\hbar\omega - E_c^{mn}\right) = \frac{1}{\pi}\sqrt{\frac{2m_r^*}{\hbar^2}}\frac{1}{\sqrt{\hbar\omega - E_c^{mn}}}$$

$$E_{cv}^{mn} = E_g + \frac{\hbar^2}{2m_r^*}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

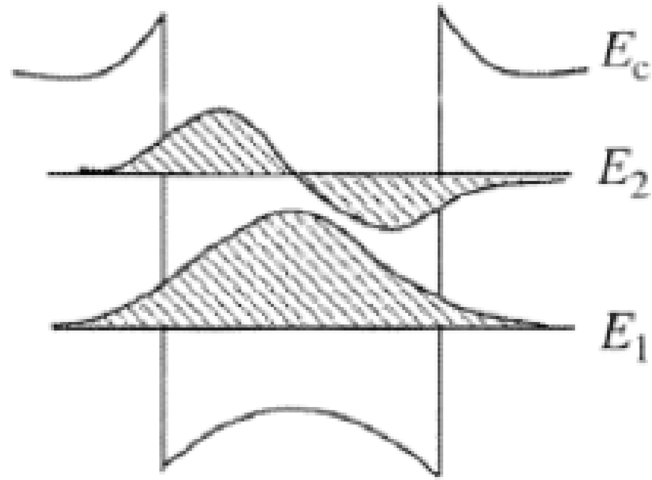
Intersubband Absorption

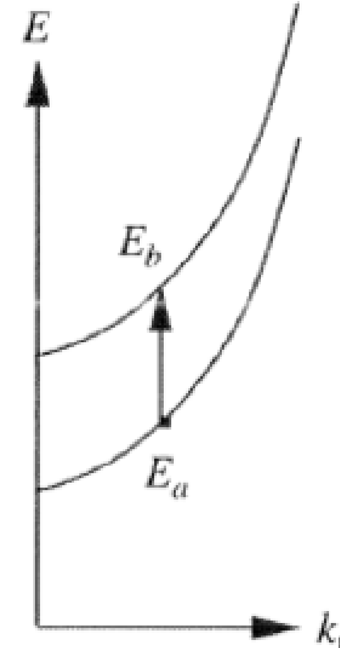
Transition between ground state and first excited state

Single QW = Low doping



Modulation doped QW





$$\psi_a(\mathbf{r}) = u_c(\mathbf{r})\frac{e^{i\mathbf{k}_t \cdot \boldsymbol{\rho}}}{\sqrt{A}}\phi_1(z)$$

$$\psi_b(\mathbf{r}) = u_{c'}(\mathbf{r})\frac{e^{i\mathbf{k}'_t \cdot \boldsymbol{\rho}}}{\sqrt{A}}\phi_2(z)$$

transverse momenta

$$\mathbf{k}_t = k_x\hat{x} + k_y\hat{y}$$

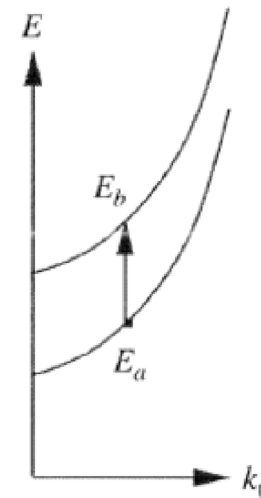
$$\mathbf{k}'_t = k'_x\hat{x} + k'_y\hat{y}$$

position vector

$$\boldsymbol{\rho} = x\hat{x} + y\hat{y}$$

Intersubband Absorption Spectrum

initial state $E_a = E_1 + \frac{\hbar^2 k_t^2}{2m_e^*}$ **final state** $E_b = E_1 + \frac{\hbar^2 k_t^2}{2m_e^*}$



$$\alpha(\hbar\omega) = \left(\frac{\omega}{n_r c\varepsilon_0}\right)\frac{2}{V}\sum_{\mathbf{k}_t}\sum_{\mathbf{k}'_t}\frac{|\hat{e}\bullet\boldsymbol{\mu}_{ba}|^2\gamma}{(E_b - E_a - \hbar\omega)^2 + \gamma^2}(f_a - f_b)$$

$$= \left(\frac{\omega}{n_r c\varepsilon_0}\right)\frac{|\boldsymbol{\mu}_{21}|^2\gamma}{(E_2 - E_1 - \hbar\omega)^2 + \gamma^2}\frac{2}{V}\sum_{\mathbf{k}_t}(f_a - f_b) = \left(\frac{\omega}{n_r c\varepsilon_0}\right)\frac{|\boldsymbol{\mu}_{21}|^2\gamma}{(E_2 - E_1 - \hbar\omega)^2 + \gamma^2}(N_1 - N_2)$$

intersubband dipole moment

$$\boxed{\boldsymbol{\mu}_{21} = \langle\phi_2|ez|\phi_1\rangle = \int \phi^*{}_2(z)\, ez\, \phi_1(z)\, dz}$$

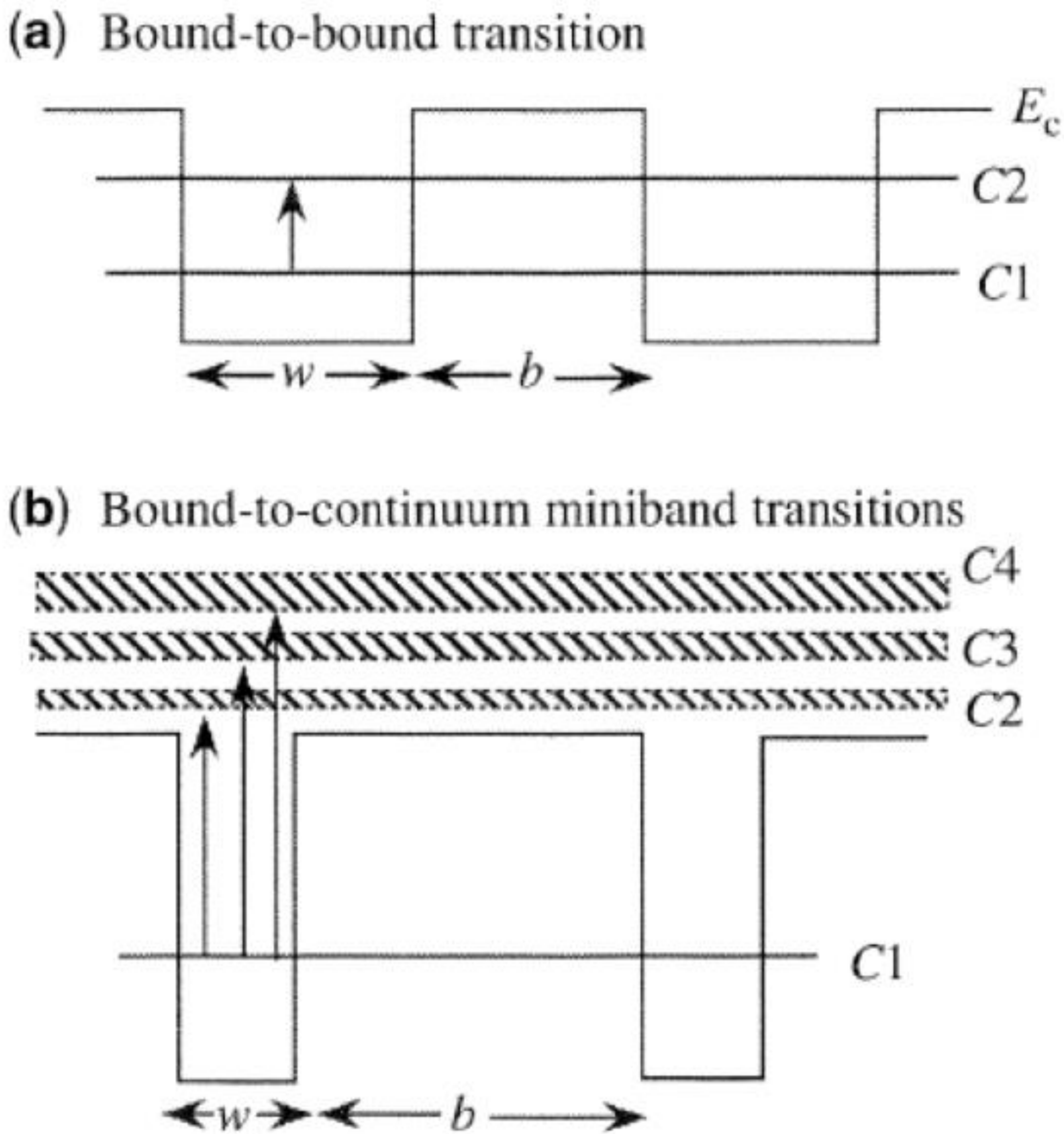
electrons per unit volume in n^th subband

$$\boxed{N_n = \frac{m_e^* k_B T}{\pi\hbar^2 L_z}\ln\left[1 + e^{(E_F - E_n)/k_B T}\right]}$$

low temperature $(E_F - E_n) \gg k_B T$

$$\boxed{N_n = \frac{m_e^*}{\pi\hbar^2 L_z}(E_F - E_n)}$$

Absorption in a Superlattice



Intersubband Absorption Spectrum

Using the se two previous results

$$\alpha(\hbar\omega) = \left(\frac{\omega}{n_r c \varepsilon_0}\right) \frac{|\mu_{21}|^2 \gamma}{(E_2 - E_1 - \hbar\omega)^2 + \gamma^2} (N_1 - N_2)$$

$$\boxed{N_n = \frac{m_e^*}{\pi \hbar^2 L_z}(E_F - E_n)}$$

considering two levels occupied

$$\alpha(\hbar\omega) = \left(\frac{\omega}{n_r c \varepsilon_0}\right) \frac{|\mu_{21}|^2 \gamma}{(E_2 - E_1 - \hbar\omega)^2 + \gamma^2} \left(\frac{m_e^*}{\pi \hbar^2 L_z}\right)(E_2 - E_1)$$

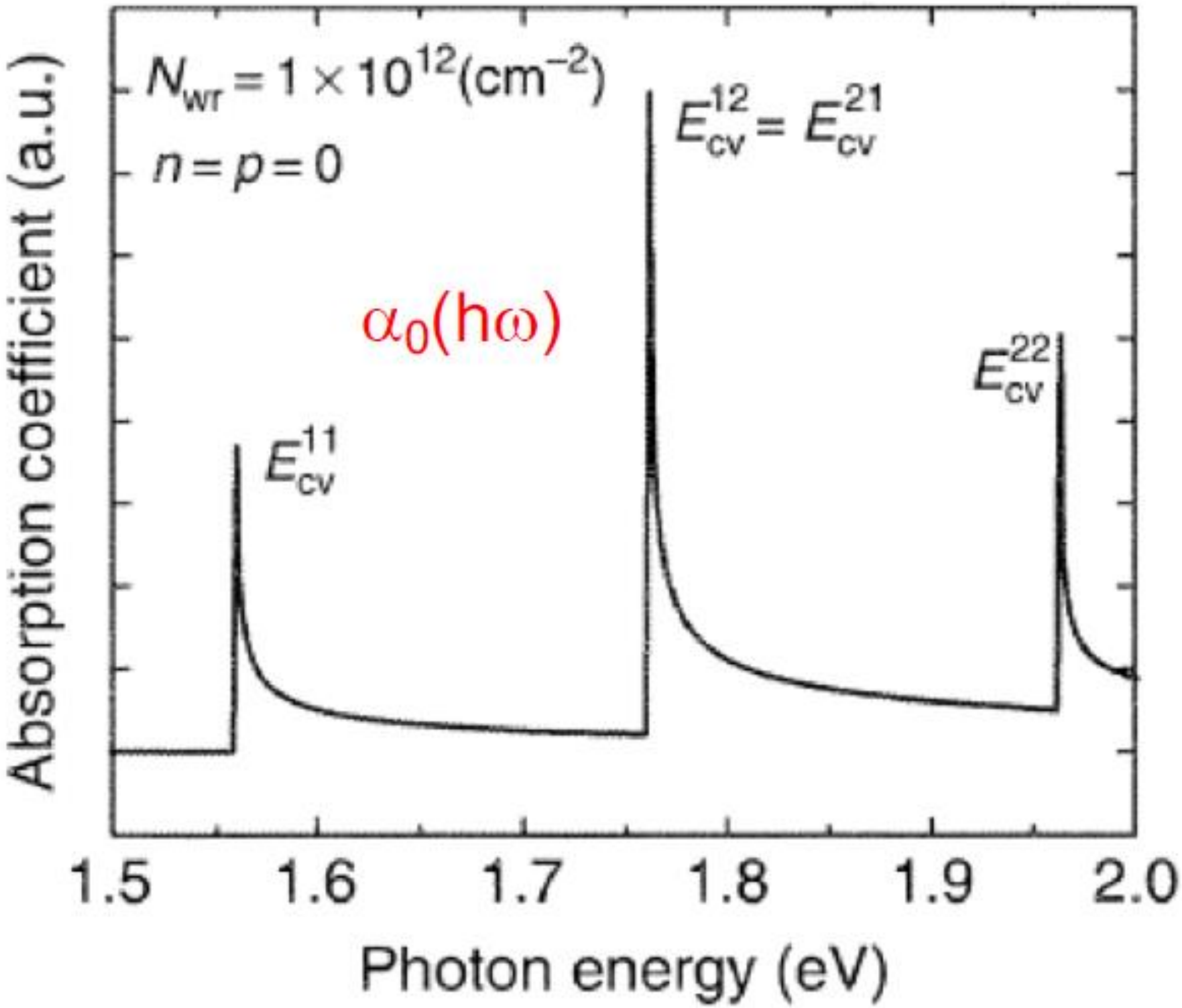
INTEGRATED ABSORBANCE

$$A = \int_0^\infty \alpha(\hbar\omega)\, d(\hbar\omega) \simeq \left(\frac{\omega_{21}}{n_r c \varepsilon_0}\right) |\mu_{21}|^2 \pi (N_1 - N_2)$$

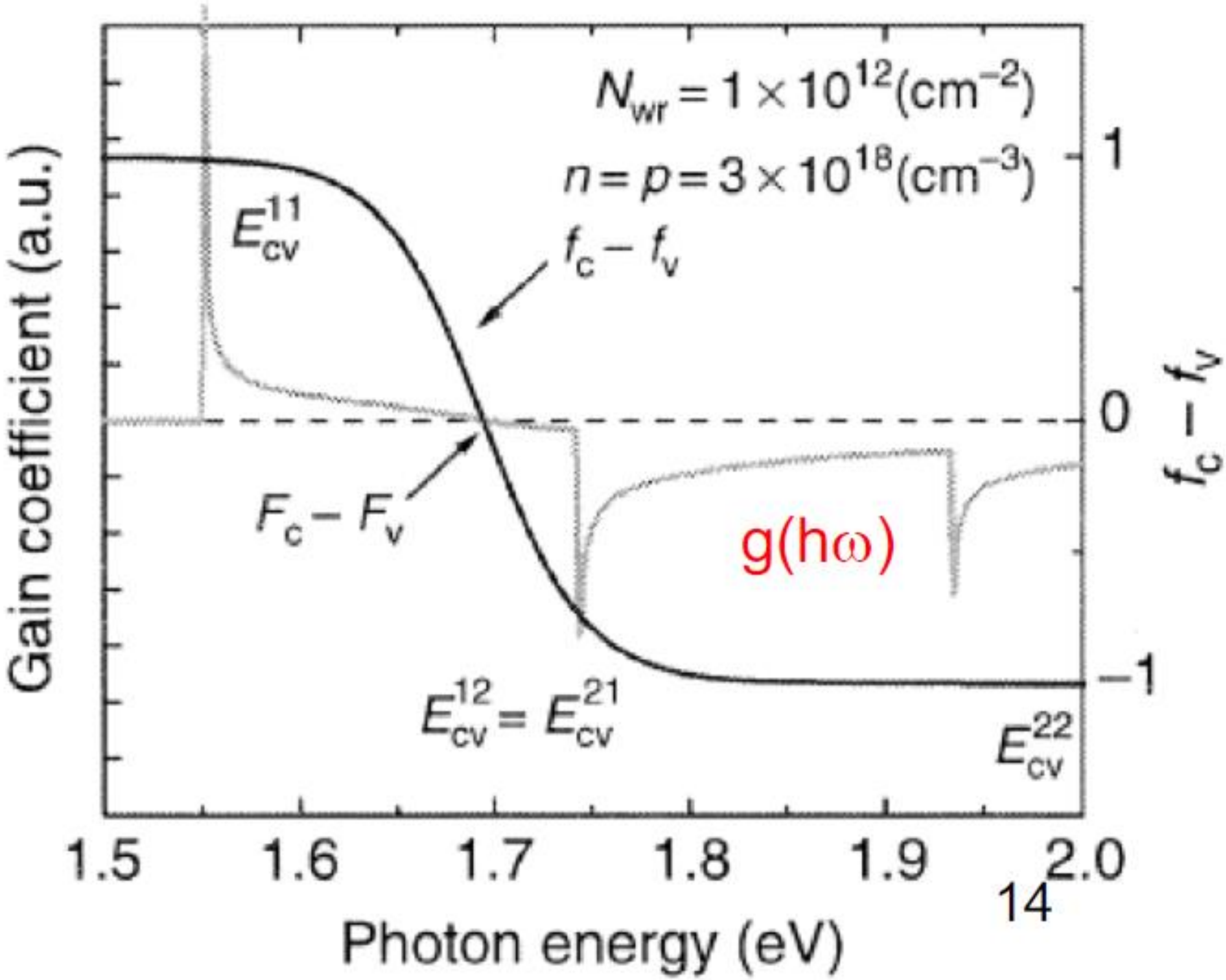
Reading Assignments:

Sections 9.3 and 9.4 of Chuang's book

Absorption coefficient



Gain coefficient



Two-dimensional quantum confinement in quantum-wire (QWR) semiconductor lasers is expected to yield improved static and dynamic performance compared to conventional quantum-well (QW) lasers.[1,2] The improved features include very low threshold currents (in the microampere regime), reduced temperature sensitivity, higher modulation bandwidth, and narrower spectral linewidth.