ECE 536 – Integrated Optics and Optoelectronics
Lecture 10 – February 16, 2022

Spring 2022

Tu-Th 11:00am-12:20pm

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Lecture 10 Outline

- Optical Absorption-Gain in reduced dimensions (quantum wells, wires and dots)

Interband Absorption and Gain in Quantum Well

2-band model – we need to add the valence band

$$\psi_b(\mathbf{r}) = u_c(\mathbf{r}) \frac{e^{i\mathbf{k}_t' \cdot \rho}}{\sqrt{A}} \phi_n(z)$$

$$|\rho| = \sqrt{x^2 + y^2}$$

$\mathbf{k}_t'$= transverse momentum for electrons

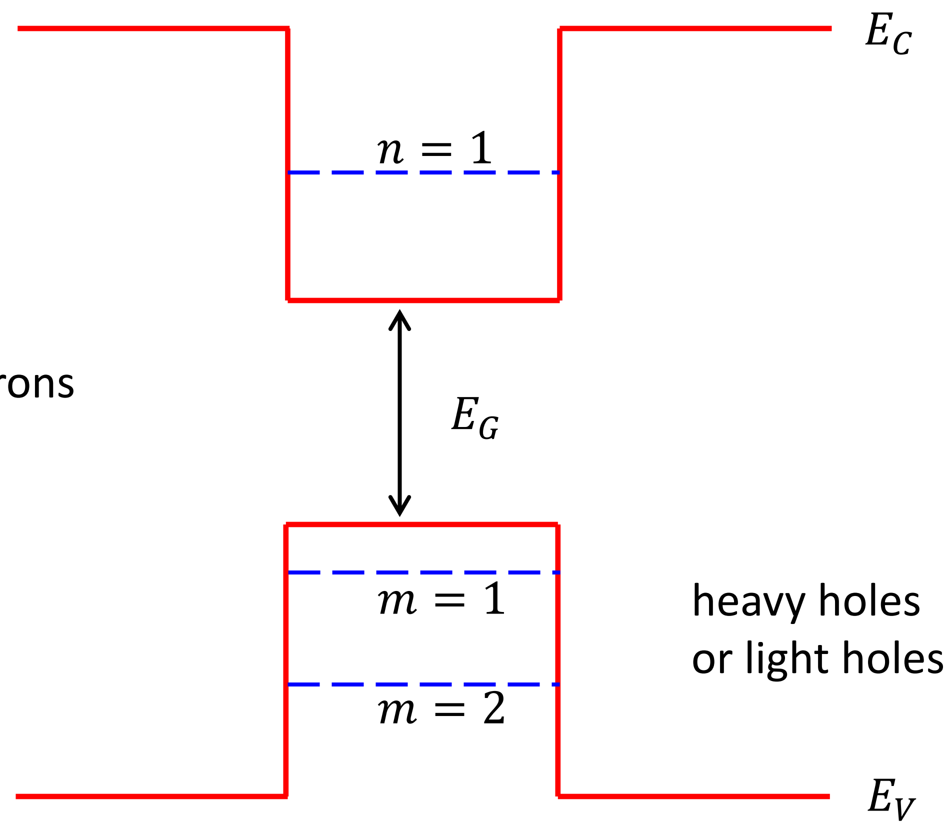
$\mathbf{k}_t$= transverse momentum for holes

$$\psi_a(\mathbf{r}) = u_v(\mathbf{r}) \frac{e^{i\mathbf{k}_t \cdot \rho}}{\sqrt{A}} g_m(z)$$

quantized envelope wavefunction

plane wave on (*x*,*y*) plane

periodic Bloch wave



Interband Absorption and Gain in Quantum Well

(We neglect here exciton interactions between electrons and holes)

Starting with the general expression for the absorption coefficient

$$\alpha(\hbar\omega) = C_o \frac{2}{V} \sum_{\mathbf{k}_a} \sum_{\mathbf{k}_b} \left| \hat{e} \cdot \mathbf{p}_{ba} \right|^2 \delta\left(E_b - E_a - \hbar\omega\right)\left(f_a - f_b\right)$$

$$\mathbf{p}_{ba} = \langle \psi_b | \mathbf{p} | \psi_a \rangle = \int \psi_b^* \frac{\hbar}{i} \nabla \psi_a \, d^3\mathbf{r}$$

$$= \int u_c^*(\mathbf{r}) \frac{e^{-i\mathbf{k}_t' \cdot \mathbf{r}}}{\sqrt{A}} \phi_n^*(z) \frac{\hbar}{i} \nabla \left(u_v(\mathbf{r}) \frac{e^{i\mathbf{k}_t \cdot \mathbf{r}}}{\sqrt{A}} g_m(z) \right) d^3\mathbf{r}$$

$$\mathbf{p}_{ba} \approx \langle u_c|\mathbf{p}|u_v\rangle \; \delta_{\mathbf{k}_t,\mathbf{k}_t'} \; I_{hm}^{en}$$

electron state n

hole state m

$\langle u_c|\mathbf{p}|u_v\rangle$: $\mathbf{p}_{cv}$

$\delta_{\mathbf{k}_t,\mathbf{k}_t'}$: k-selection rule $\rightarrow$ $\mathbf{k}_t = \mathbf{k}_t'$
conservation of transverse momentum from exponentials

I_{hm}^{en}: overlap integral of quantized wavefunctions in well

$$I_{hm}^{en} = \int_{-\infty}^{\infty} \phi_n^*(z)\, g_m(z)\, dz$$

overlap integral

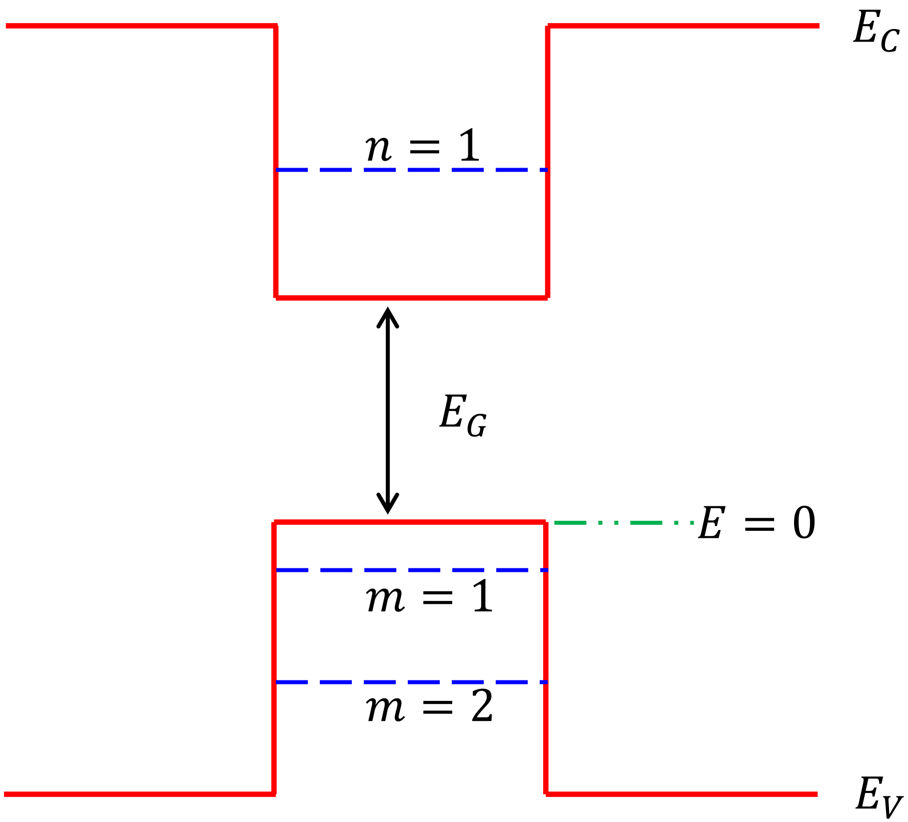
Transition energies (all are relative to top of valence band)

$$E_b = E_g + E_{en} + \frac{\hbar^2 k_t^2}{2m_e^*}$$

Energy level of electron state

$$E_a = E_{hm} - \frac{\hbar^2 k_t^2}{2m_h^*}$$

Energy level of hole state
(NOTE: it's negative)

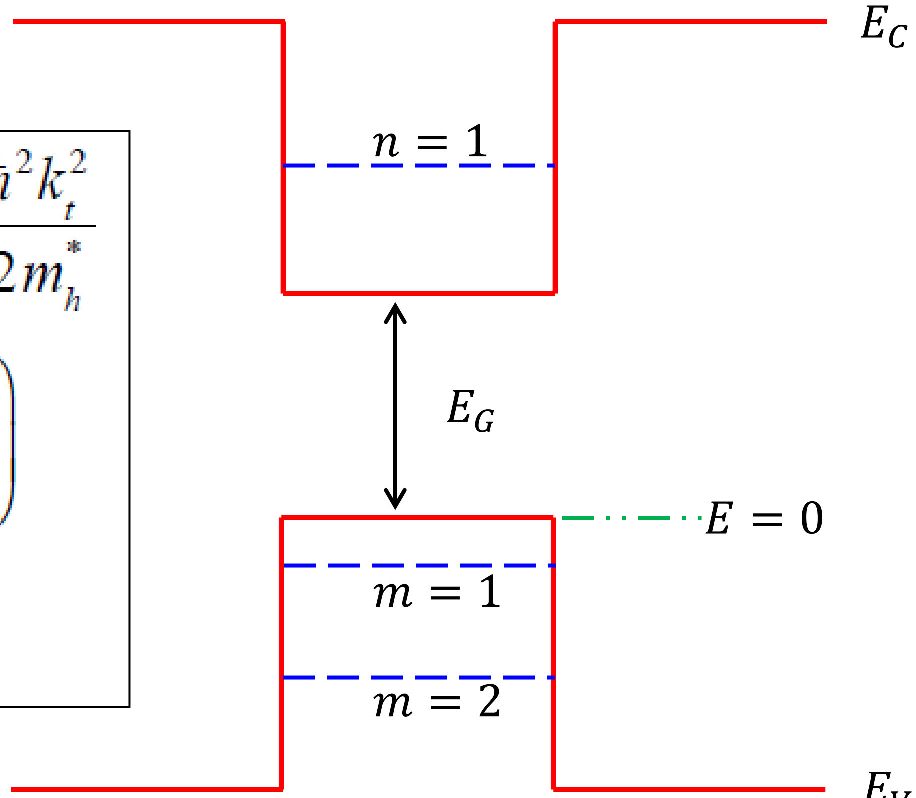


Transition energies (all are relative to top of valence band)

$$E_b - E_a = E_g + E_{en} + \frac{\hbar^2 k_t^2}{2m_e^*} - E_{hm} + \frac{\hbar^2 k_t^2}{2m_h^*}$$

$$= \left(E_g + E_{en} - E_{hm}\right) + \left(\frac{\hbar^2 k_t^2}{2m_h^*} + \frac{\hbar^2 k_t^2}{2m_e^*}\right)$$

$$= E_{hm}^{en} + E_t$$



$$E_{hm}^{en} = E_g + E_{en} - E_{hm}$$

Band edge transition energy

$$E_t = \frac{\hbar^2 k_t^2}{2m_r^*}$$

$$\frac{1}{m_r^*} = \frac{1}{m_e^*} + \frac{1}{m_h^*}$$

Transverse kinetic energy due to in-plane k-vector

Absorption expression for quantum well case

$$\alpha(\hbar\omega) = C_o \frac{2}{V} \sum_{\mathbf{k}_a} \sum_{\mathbf{k}_b} \left|\hat{e}\cdot\mathbf{p}_{ba}\right|^2 \delta\left(E_b - E_a - \hbar\omega\right)\left(f_a - f_b\right)$$

- Summations over k_a and k_b become summations over k_t and k_t'
- The k-selection rule establishes $\mathrm{k}_t = \mathrm{k}_t'$ so the sum becomes a sum over k_t
- We need to sum also over m and n

$$\alpha(\hbar\omega) = C_0 \sum_{n,m} \left|I_{hm}^{en}\right|^2 \frac{2}{V} \sum_{\mathbf{k}_t} \left|\hat{e}\cdot\mathbf{p}_{cv}\right|^2 \delta\left(E_{hm}^{en} + E_t - \hbar\omega\right)\left(f_v^m - f_c^n\right)$$

Joint Density of States

$$\rho_r^{2D} = \frac{m_r^*}{\pi \hbar^2 L_z}$$

For an unpumped semiconductor in thermal equilibrium $f_v^m = 1$ $f_c^n = 0$

$$\alpha(\hbar\omega) = C_0 \sum_{m,n} \left| I_{hm}^{en} \right|^2 \left| \hat{e} \cdot \mathbf{p}_{cv} \right|^2 \int_0^\infty \rho_r^{2D} \delta\left(E_{hm}^{en} + E_t - \hbar\omega\right) \cancelto{1}{\left(f_v^m - f_c^n\right)} dE_t$$

$$\alpha_0(\hbar\omega) = C_0 \sum_{m,n} \left| I_{hm}^{en} \right|^2 \left| \hat{e} \cdot \mathbf{p_{cv}} \right|^2 \rho_r^{2D} H\left(\hbar\omega - E_{hm}^{en}\right)$$

Optical Absorption Spectrum

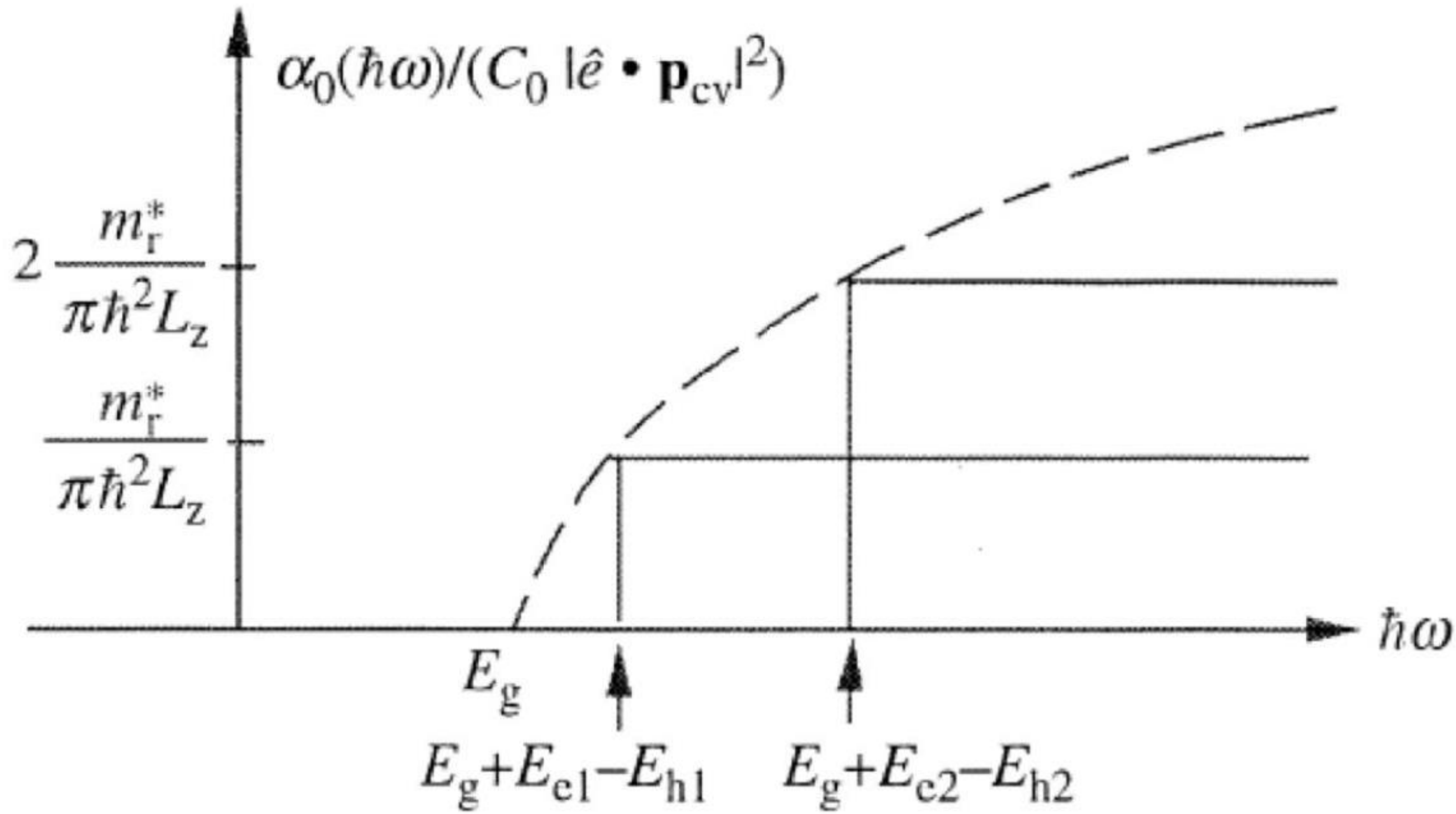
$$\alpha_0(\hbar\omega) = C_0 \left|\hat{e}\cdot\mathbf{p}_{\mathbf{cv}}\right|^2 \begin{cases} \dfrac{m_r^*}{\pi\hbar^2 L_z} & \text{for } E_{h1}^{e1} < \hbar\omega < E_{h2}^{e2} \\ 2\dfrac{m_r^*}{\pi\hbar^2 L_z} & \text{for } E_{h2}^{e2} < \hbar\omega < E_{h3}^{e3} \\ 3\dfrac{m_r^*}{\pi\hbar^2 L_z} & \text{for } E_{h3}^{e3} < \hbar\omega < E_{h4}^{e4} \\ etc \end{cases}$$

For the case of carrier injection

$$\alpha(\hbar\omega) = \alpha_0(\hbar\omega)\left[f_v^m\left(\hbar\omega - E_{hm}^{en}\right) - f_c^n\left(\hbar\omega - E_{hm}^{en}\right)\right]$$

$$= \alpha_0(\hbar\omega)\left[f_v^m\left(E_t\right) - f_c^n\left(E_t\right)\right]$$

Optical Absorption Spectrum



Electron Quasi Fermi Levels (when "*n*" electrons have been injected)

$$n = \sum_{\substack{n-occupied \\ subbands}} N_n = \sum_n \int_0^{\infty} \rho_e^{2D}(E) f_c^n(E)\, dE$$

$$N_n = \int_0^{\infty} \rho_e^{2D}(E) f_c^n(E)\, dE = \frac{m_e^*}{\pi \hbar^2 L_z} \int_{E_{en}}^{\infty} f_c^n(E)\, dE$$

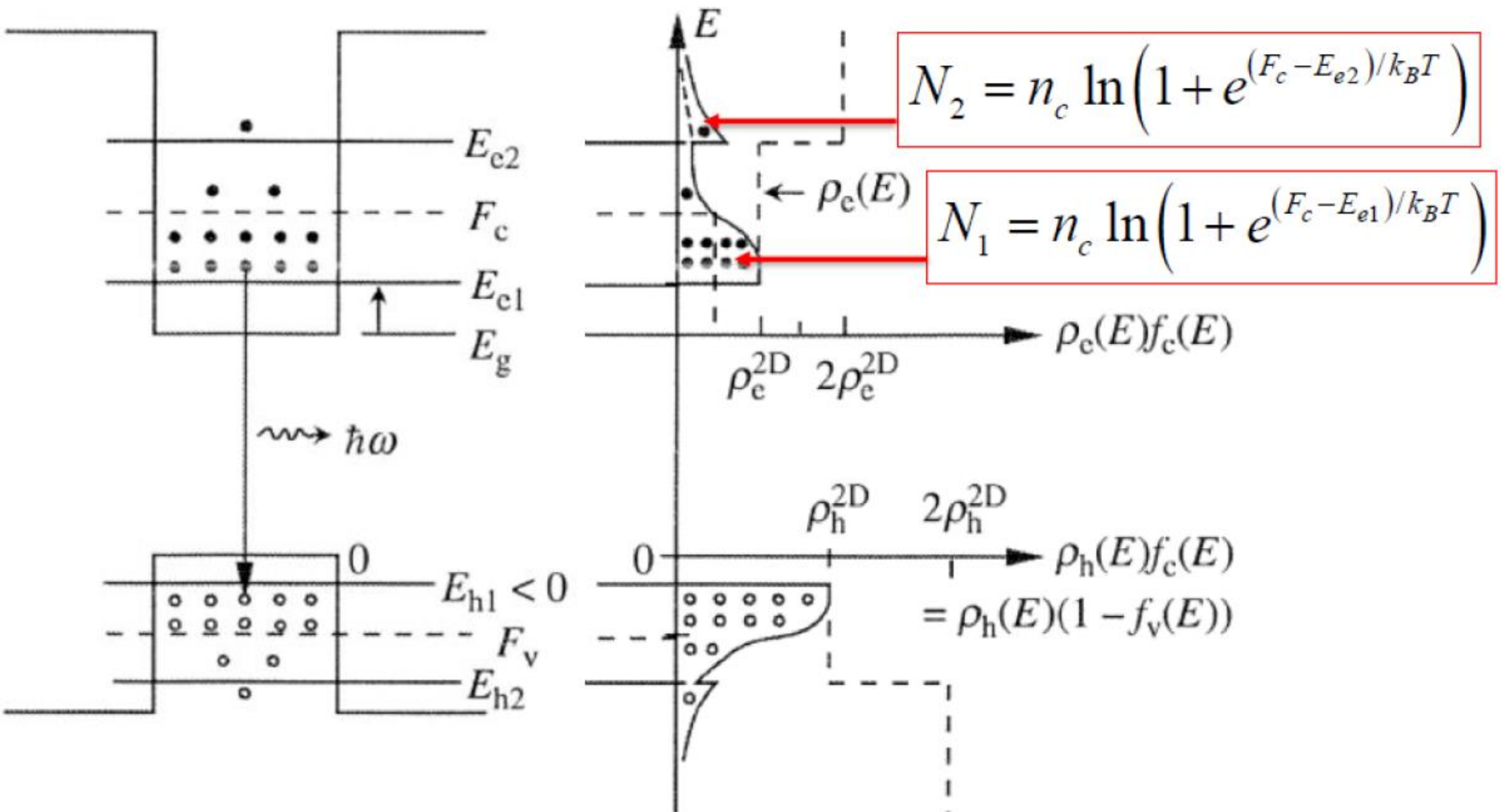
$$\int_{E_{en}}^{\infty} f_c^n(E)\, dE = \int_{E_{en}}^{\infty} \frac{1}{1+e^{(E-F_c)/k_B T}}\, dE = -k_B T \ln\left(1+e^{(F_c-E)/k_B T}\right)\Bigg|_{E_{en}}^{\infty}$$

$$\boxed{N_n = \frac{m_e^* k_B T}{\pi \hbar^2 L_z} \ln\left(1+e^{(F_c-E_{en})/k_B T}\right) = n_c \ln\left(1+e^{(F_c-E_{en})/k_B T}\right)}$$

Electron Quasi Fermi Levels (when "*n*" electrons have been injected)

$$N_n = n_c \ln\left(1 + e^{(F_c - E_{en})/k_B T}\right)$$

$$n_c = \frac{m_e^* k_B T}{\pi \hbar^2 L_z}$$



Holes Quasi Fermi level (when "*p*" holes have been injected)

$$p = n + N_A^- - N_D^+ = \sum_{\substack{occupied\\subbands}} P_m = \sum_{\substack{occupied\\subbands}} \int_{-\infty}^{0} \rho_h^{2D}(E)\left[1 - f_v^m(E)\right] dE$$

$$P_m = \int_{-\infty}^{0} \rho_h^{2D}(E) f_h^m(E)\, dE = \frac{m_h^*}{\pi \hbar^2 L_z} \int_{-\infty}^{E_{hm}} f_h^m(E)\, dE$$

$$\int_{-\infty}^{E_{hm}} f_h^m(E)\, dE = \int_{-\infty}^{E_{hm}} \frac{1}{1 + e^{-(E-F_v)/k_B T}} dE = k_B T \ln\left(1 + e^{(E-F_v)/k_B T}\right)\Big|_{-\infty}^{E_{hm}}$$

$$\boxed{P_m = \frac{m_h^* k_B T}{\pi \hbar^2 L_z} \ln\left(1 + e^{(E_{hm}-F_v)/k_B T}\right) = n_v \ln\left(1 + e^{(E_{hm}-F_v)/k_B T}\right)}$$

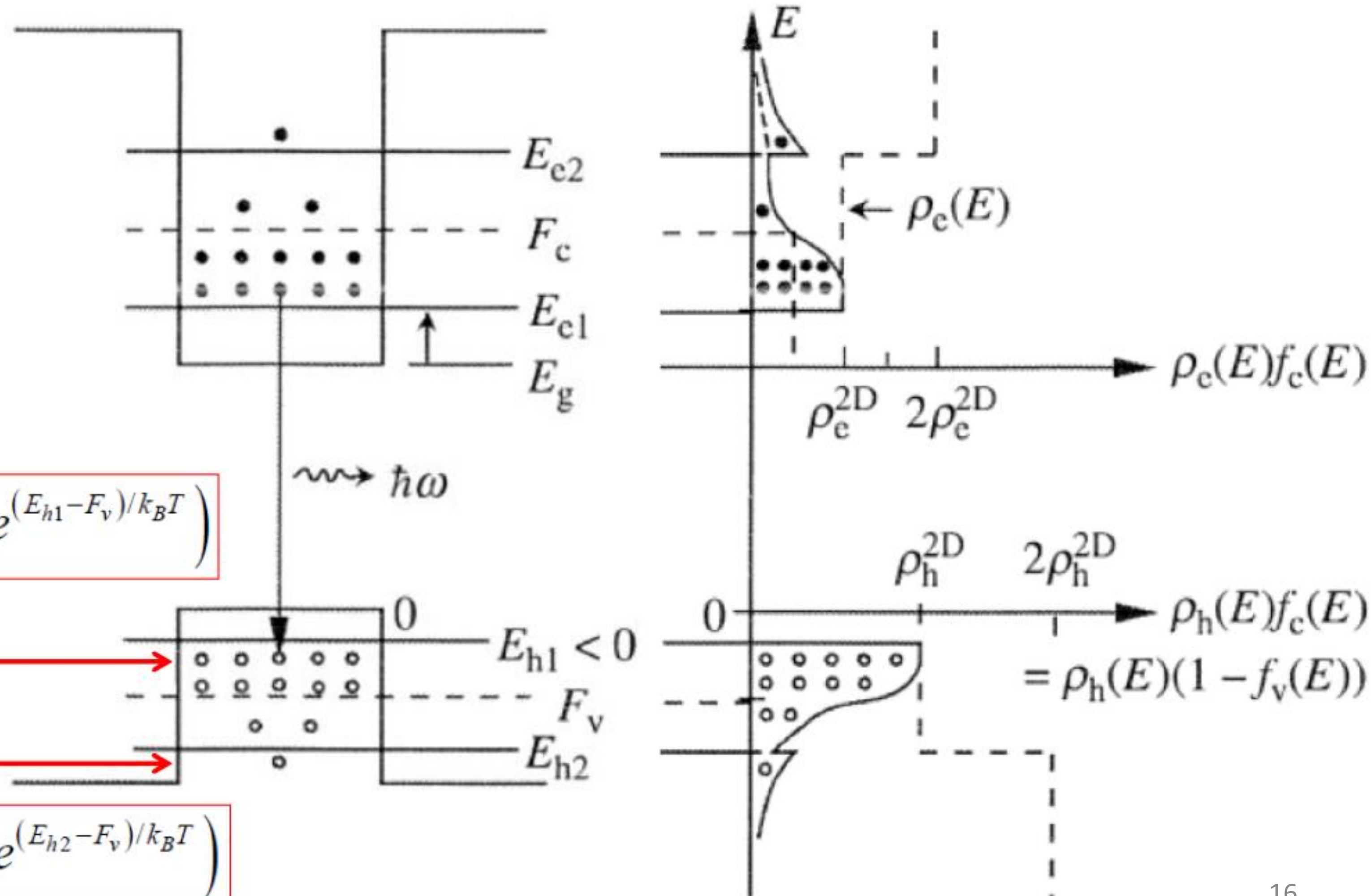
Holes Quasi Fermi level (when “*p*” holes have been injected)

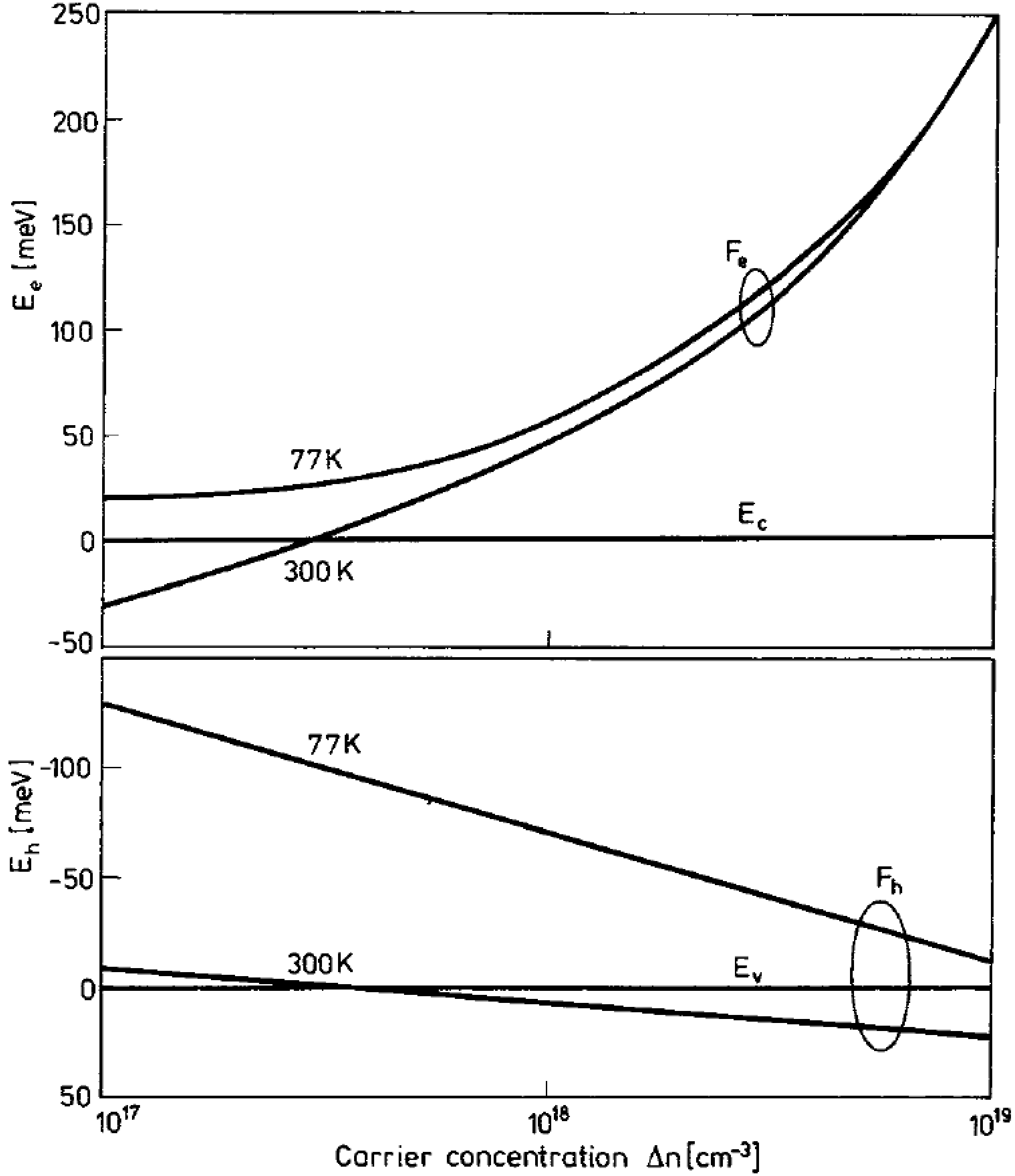
$$P_m = n_v \ln\left(1 + e^{(E_{hm} - F_v)/k_B T}\right)$$

$$n_v = \frac{m_h^* k_B T}{\pi \hbar^2 L_z}$$

$$P_1 = n_v \ln\left(1 + e^{(E_{h1} - F_v)/k_B T}\right)$$

$$P_2 = n_v \ln\left(1 + e^{(E_{h2} - F_v)/k_B T}\right)$$





GaAs bulk example

Gain Spectrum

Zero Linewidth Case:

$$g(\hbar\omega) = C_0 \sum_{m,n} \left| I_{hm}^{en} \right|^2 \left| \hat{e} \cdot \mathbf{p}_{\mathbf{cv}} \right|^2 \left[f_c^n(E_t) - f_v^m(E_t) \right] \rho_r^{2D} H\left(\hbar\omega - E_{hm}^{en}\right)$$

$$C_o = \frac{\pi e^2}{n_r c \varepsilon_0 m_0^2 \omega}$$

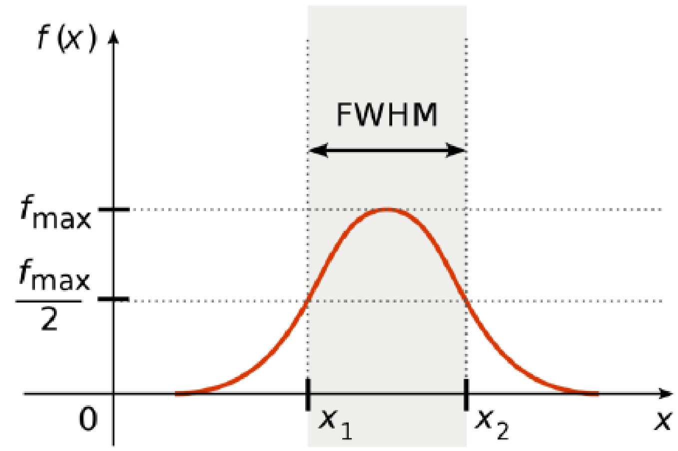
$$I_{hm}^{en} = \int_{-\infty}^{\infty} \phi_n^*(z)\, g_m(z)\, dz$$

$$\rho_r^{2D} = \frac{m_r^*}{\pi \hbar^2 L_z}$$

$$f_c^n(E_t) = \frac{1}{1 + \exp\left[\left(E_g + E_{en} + \frac{m_r^*}{m_e^*} E_t - F_c\right) / k_B T\right]}$$

$$f_v^m(E_t) = \frac{1}{1 + \exp\left[\left(E_{hm} - \frac{m_r^*}{m_h^*} E_t - F_v\right) / k_B T\right]}$$

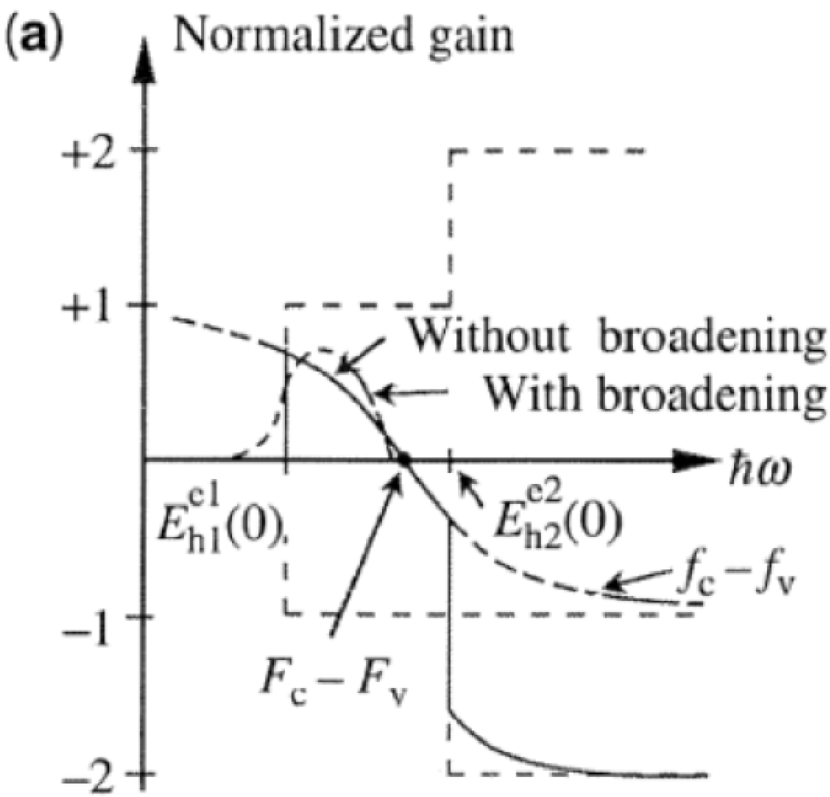
Gain Spectrum

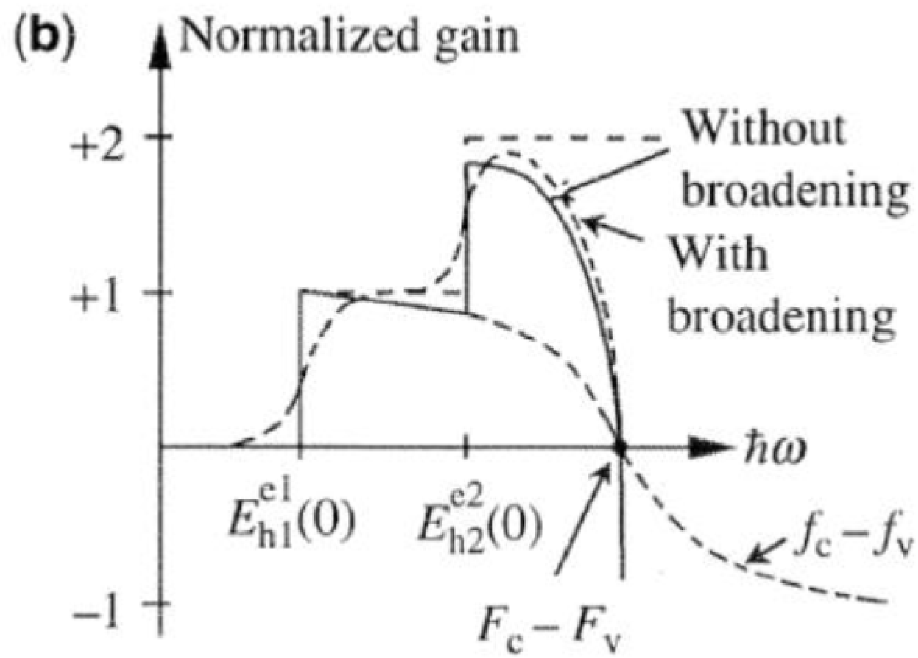


Linewidth Broadening Case
Full Width at Half Maximum (FWHM = 2γ)

$$g(\hbar\omega) = C_0 \sum_{m,n} \left| I_{hm}^{en} \right|^2 \int_0^\infty dE_t \rho_r^{2D} \left| \hat{e} \bullet \mathbf{p}_{\mathbf{cv}} \right|^2 \frac{\gamma / \pi}{\left[E_{hm}^{en} + E_t - \hbar\omega \right]^2 + \gamma^2} \left[f_c^n(E_t) - f_v^m(E_t) \right]$$

Gain Spectra with and without broadening

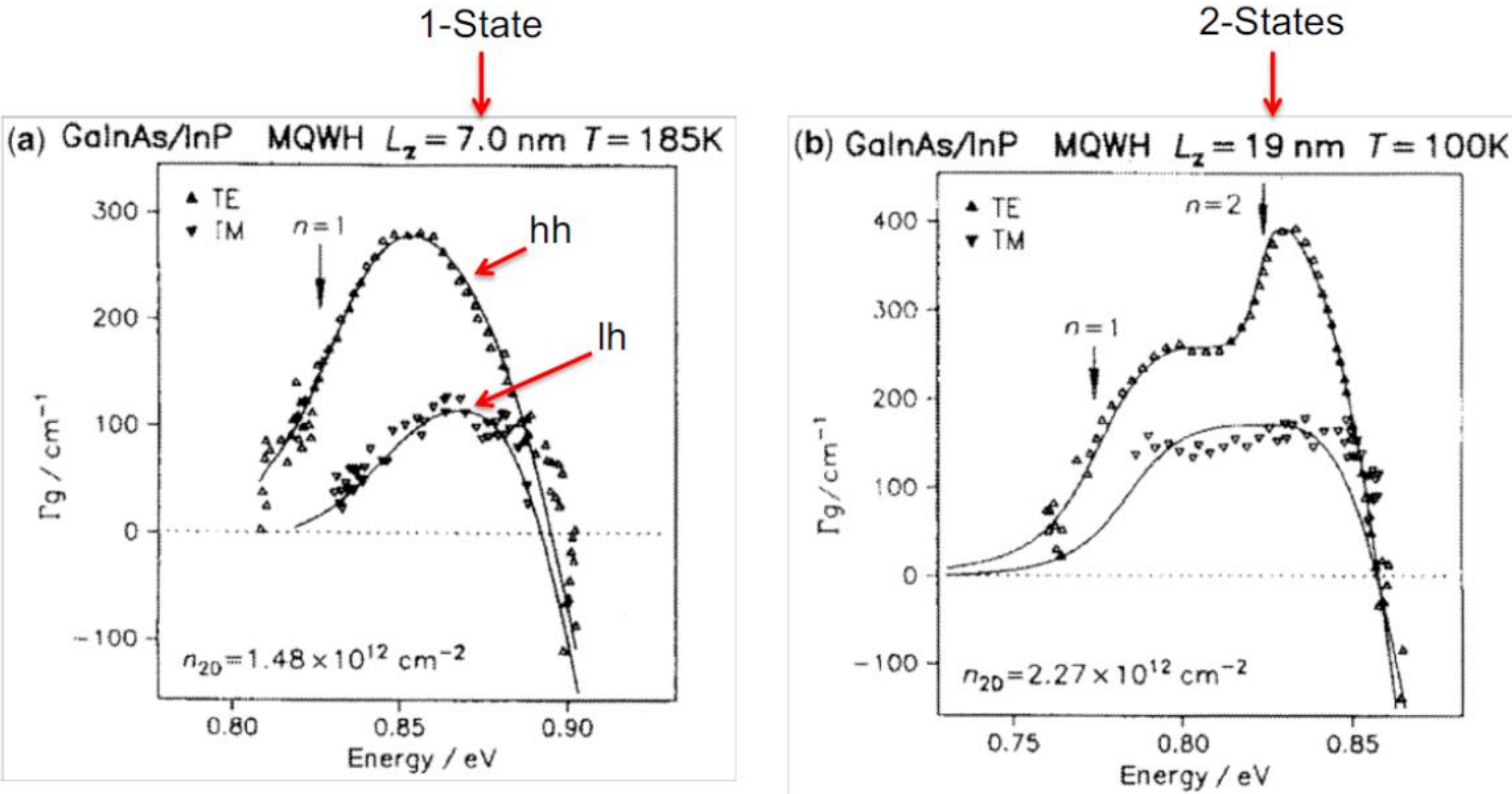




Transition involving one single electron and hole subband pair

Transition involving two electron and two hole subbands

Comparison with experimental data

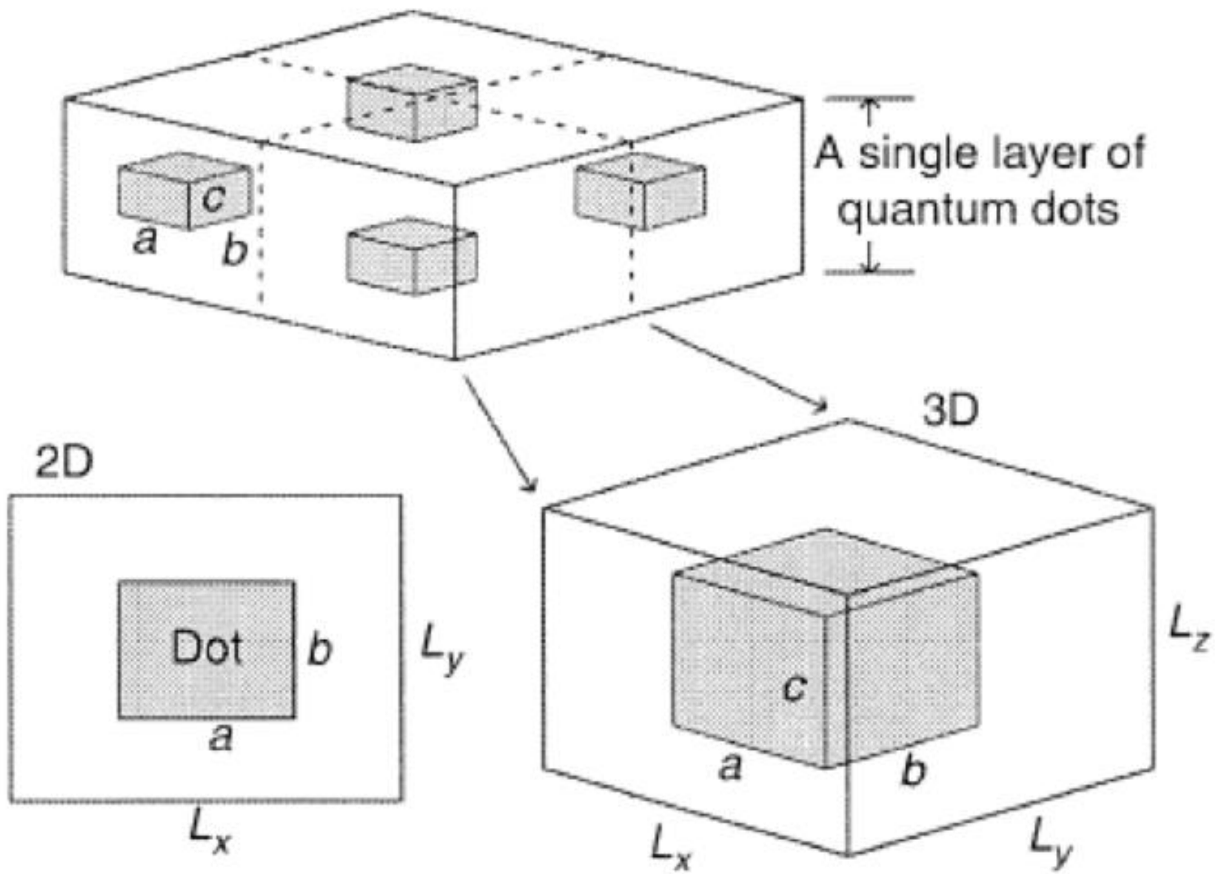


Zielinski *et al*, IEEE J. Quantum Electronics, QE-23, p.969 (1987).

Quantum Dots

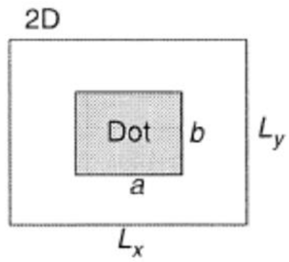
Ideal quantum dot assumptions

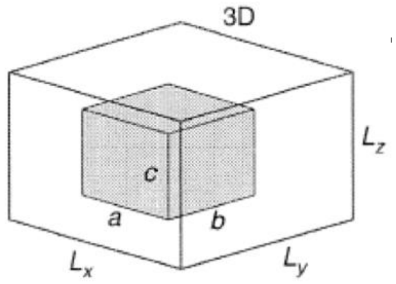
- **Uniform dot size**
- **Uniform distribution**



Ideal Quantum Dot – Wave functions and Energies

Dot Size $a \times b \times c$ **Total Volume of Single Dot** $V = L_x L_y L_z$





Dot Density

$$N_{dot}^{2D} = \frac{1}{L_x L_y}$$

$$N_{dot}^{3D} = \frac{1}{V} = \frac{1}{L_x L_y L_z}$$

Fill Factor

$$F^{2D} = \frac{ab}{L_x L_y}$$

$$F^{3D} = \frac{abc}{L_x L_y L_z}$$

Conduction Band – Wave functions and Energies

Wave Functions (Assume infinite barrier for simplicity)

$$\psi_c(x,y,z) = \frac{\sqrt{8}}{\sqrt{abc}} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sin\left(\frac{l\pi}{c}z\right) u_c(\mathbf{r})$$

Energy eigenvalues

$$E_c^{mnl} = E_{c0} + \frac{\hbar^2}{2m_e^*}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2\right]$$

Electron Density

$$n = \frac{2}{V}\sum_{m,n,l} f_c(E) = N_{dot}^{3D} 2 \sum_{m,n,l} f_c\left(E_c^{nml}\right) = 2\frac{N_{dot}^{2D}}{L_z}\sum_{m,n,l}\frac{1}{1+e^{\left(E_c^{nml}-F_c\right)/k_BT}}$$

Valence Band – Wave functions and Energies

Wave Functions (Assume infinite barrier for simplicity)

$$\psi_v(x,y,z) = \frac{\sqrt{8}}{\sqrt{abc}} \sin\left(\frac{m'\pi}{a}x\right) \sin\left(\frac{n'\pi}{b}y\right) \sin\left(\frac{l'\pi}{c}z\right) u_v(\mathbf{r})$$

Energy eigenvalues

$$E_v^{m'n'l'} = E_{v0} - \frac{\hbar^2}{2m_h^*}\left[\left(\frac{m'\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2 + \left(\frac{l'\pi}{c}\right)^2\right]$$

Hole Density

$$p = \frac{2}{V}\sum_{m',n',l'}\left(1 - f_v(E)\right) = N_{dot}^{3D}\, 2\sum_{m',n',l'}\left(1 - f_v\left(E_v^{n'm'l'}\right)\right) = 2\frac{N_{dot}^{2D}}{L_z}\sum_{m',n',l'} 1 - \frac{1}{1 + e^{\left(E_v^{m'n'l'} - F_v\right)/k_B T}}$$

$$p = 2\frac{N_{dot}^{2D}}{L_z}\sum_{m',n',l'}\frac{1}{1 + e^{\left(F_v - E_v^{m'n'l'}\right)/k_B T}}$$

Interband Absorption Spectrum

General expression

$$\alpha(\hbar\omega) = C_o \frac{2}{V} \sum_{\mathbf{k}} \left|\hat{e}\bullet\mathbf{p}_{cv}\right|^2 \delta\left(E_c - E_v - \hbar\omega\right)\left(f_v - f_c\right)$$

Quantization in all dimensions → sum over m, n, l

$$\alpha(\hbar\omega) = C_o \frac{2}{V} \sum_{m,n,l} \sum_{m',n',l'} \left|\langle\psi_c|\hat{e}\bullet\mathbf{p}|\psi_v\rangle\right|^2 \delta\left(E_c^{mnl} - E_v^{m'n'l'} - \hbar\omega\right)\left(f_v - f_c\right)$$

$$\langle\psi_c|\hat{e}\bullet\mathbf{p}|\psi_v\rangle \simeq \langle u_c|\hat{e}\bullet\mathbf{p}|u_v\rangle \delta_{mm'}\delta_{nn'}\delta_{ll'} = \hat{e}\bullet\mathbf{p}_{cv}\delta_{mm'}\delta_{nn'}\delta_{ll'}$$

$$\rightarrow \alpha(\hbar\omega) = C_o \frac{2N_{dot}^{2D}}{L_z} \sum_{m,n,l} \left|\hat{e}\bullet\mathbf{p}_{cv}\right|^2 \delta\left(E_c^{mnl} - E_v^{mnl} - \hbar\omega\right)\left(f_v - f_c\right)$$

with Interband Transition Energies

$$E_{cv}^{mnl} = E_c^{mnl} - E_v^{mnl} = E_g + \frac{\hbar^2}{2m_r^*}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2\right]$$

Absorption – Homogeneous Broadening

In case of no carrier injection, we can approximate $f_v = 1 \quad f_c = 0$

For homogeneous broadening, the delta function can be replaced by a Lorentzian

$$\alpha_0(\hbar\omega) = C_0 \frac{2N_{dot}^{2D}}{L_z} \sum_{m,n,l} \left|\hat{e} \bullet \mathbf{p}_{cv}\right|^2 \underbrace{L\left(E_{cv}^{mnl} - \hbar\omega\right)}$$

$$L(E - \hbar\omega) = \frac{\gamma / \pi}{(E - \hbar\omega)^2 + \gamma^2}$$

With carrier injection

$$\alpha(\hbar\omega) = C_0 \frac{2N_{dot}^{2D}}{L_z} \sum_{m,n,l} \left|\hat{e} \bullet \mathbf{p}_{cv}\right|^2 L\left(E_{cv}^{mnl} - \hbar\omega\right)\left(f_v - f_c\right)$$

Absorption – Inhomogeneous Broadening

For uniform dots we had: $$n = 2\frac{N_{dot}^{2D}}{L_z}\sum_{m,n,l} f_c\left(E_c^{mnl}\right)$$

In realistic case we will have variations of quantum dot size. Dot energy level become a Gaussian distribution with

$$G(E) = \frac{1}{\sqrt{2\pi\sigma_c}} e^{-\left(E-E_c^{mnl}\right)^2/2\sigma_c^2}$$

$$\begin{cases} \text{mean} = E_c^{mnl} \\ \text{FWHM} = 2\sqrt{2\ln 2}\,\sigma_c \simeq 2.35\sigma_c \end{cases}$$

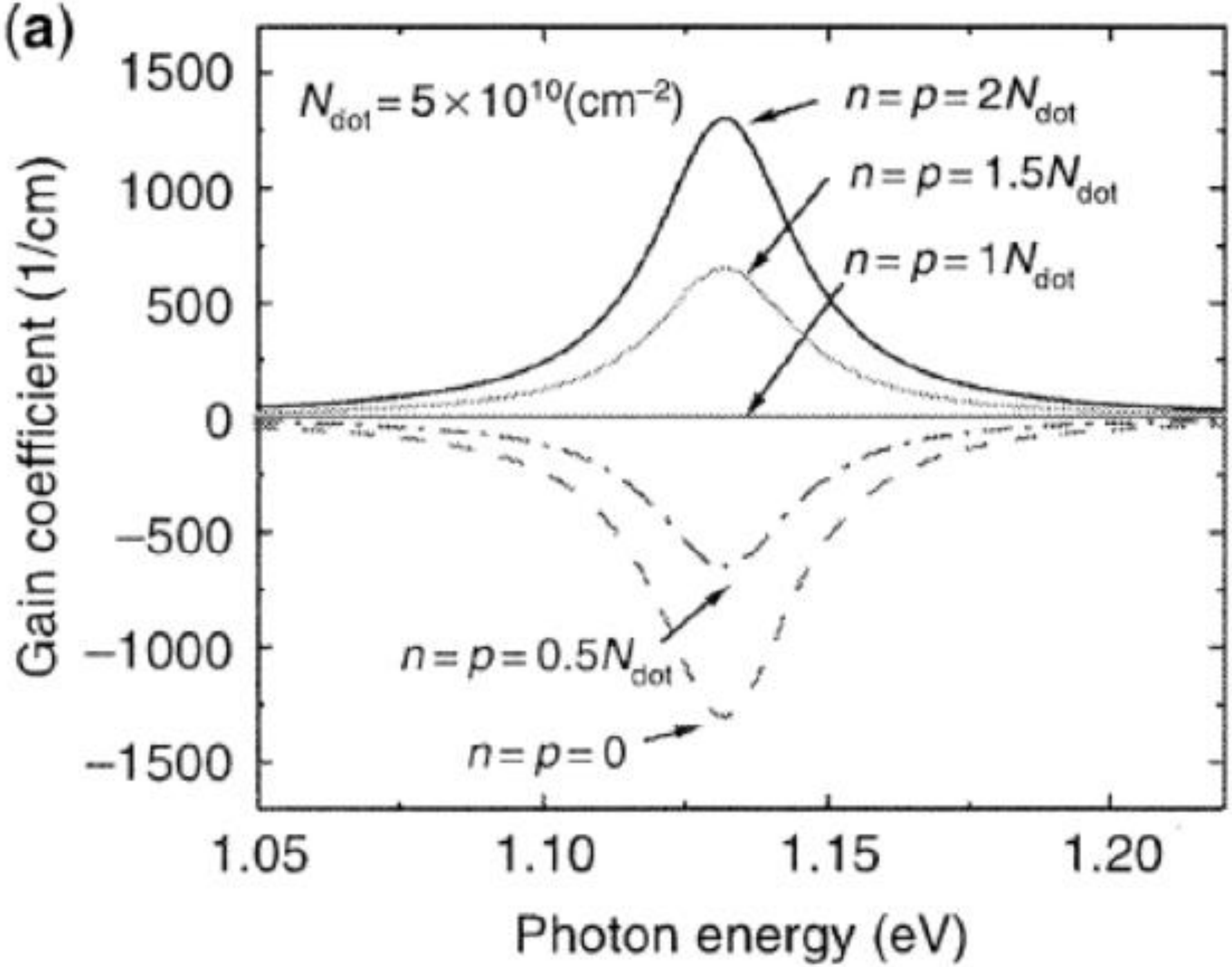
carrier density $$n = 2\frac{N_{dot}^{2D}}{L_z}\sum_{m,n,l}\int_0^\infty dE\; G(E) f_c(E)$$

$$\alpha(\hbar\omega) = C_0 \sum_{m,n,l}\int_0^\infty dE \left|\hat{e}\bullet\mathbf{p}_{\mathbf{cv}}\right|^2 D(E) L(E-\hbar\omega)(f_v - f_c)$$

$$D(E) = 2\frac{N_{dot}^{2D}}{L_z}\frac{1}{\sqrt{2\pi\sigma}} e^{-\left(E-E_{cv}^{mnl}\right)^2/2\sigma^2} \qquad \sigma^2 = \sigma_c{}^2 + \sigma_v{}^2$$

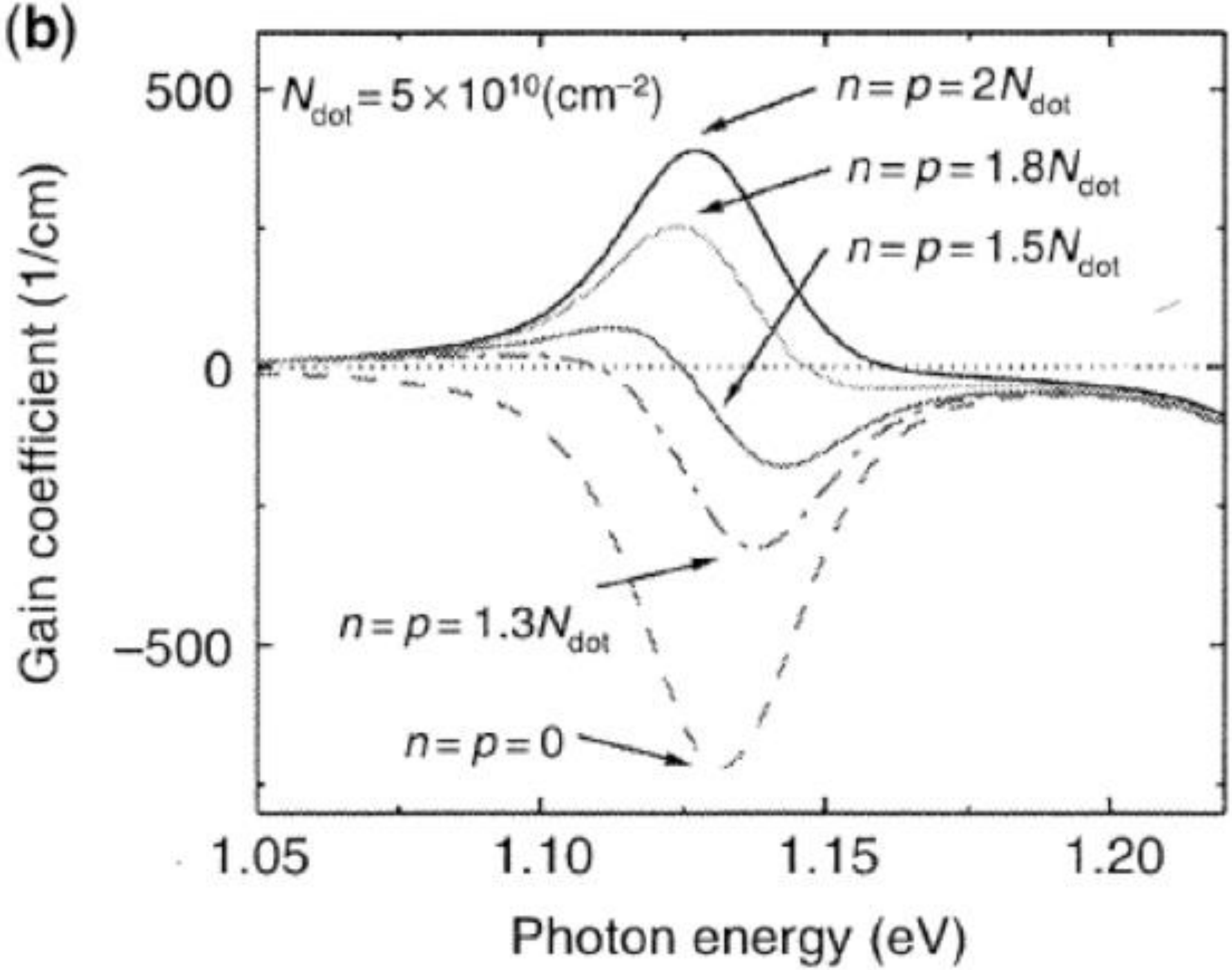
Examples = Homogeneous Broadening

FWHM = 30 meV



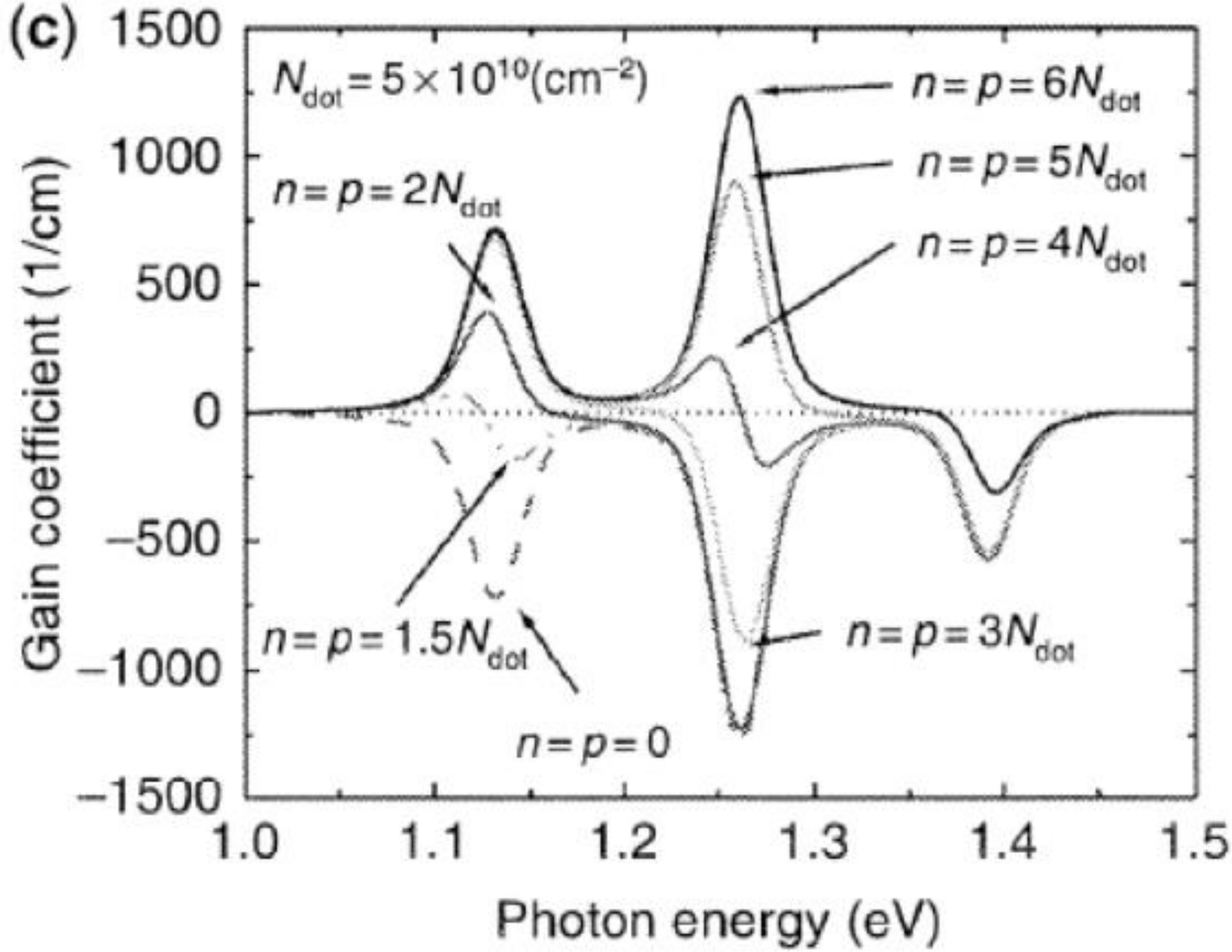
Examples = Homogeneous +Inhomogeneous Broadening

FWHM = 30 meV   FWHM = 50 meV

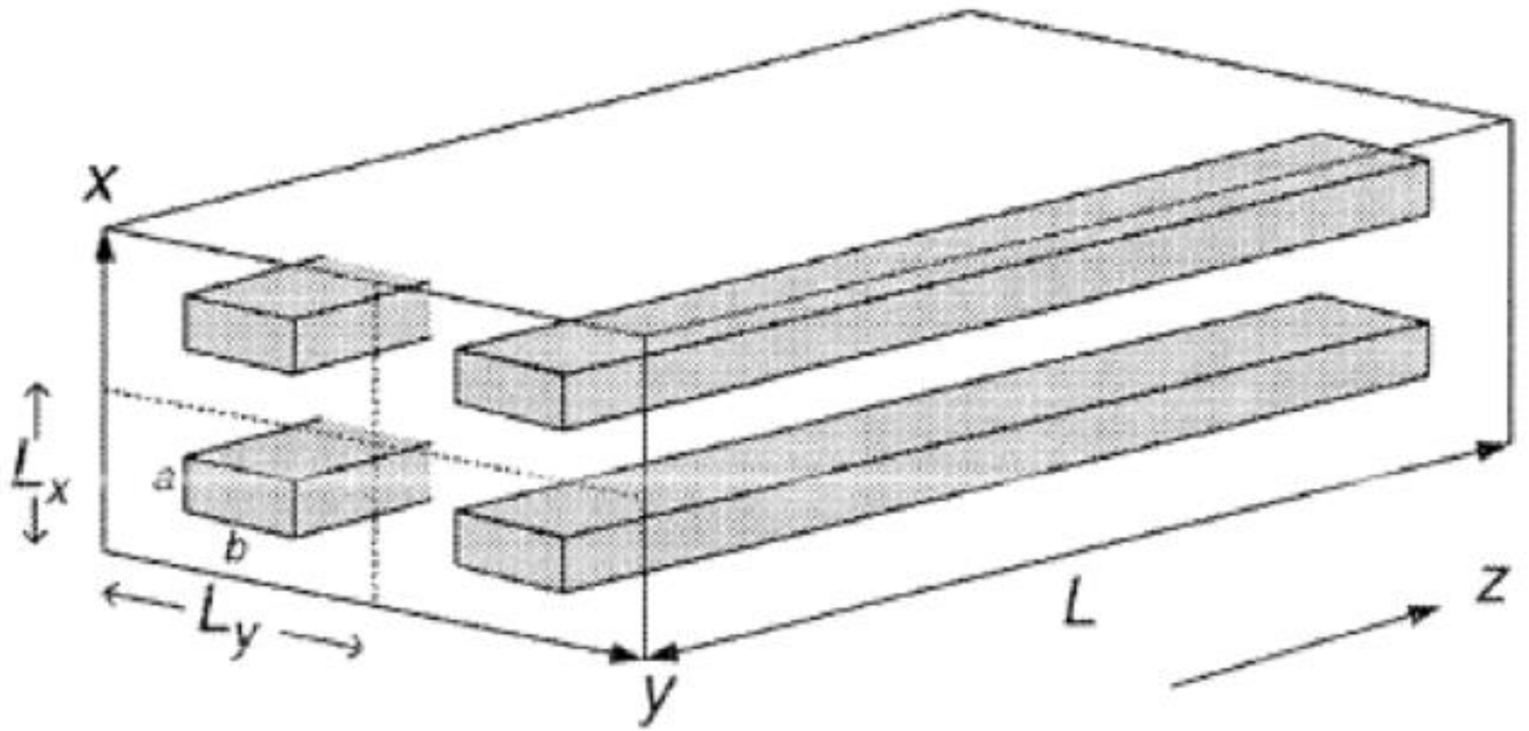


Examples = Homogeneous +Inhomogeneous Broadening

FWHM = 30 meV **FWHM = 50 meV**

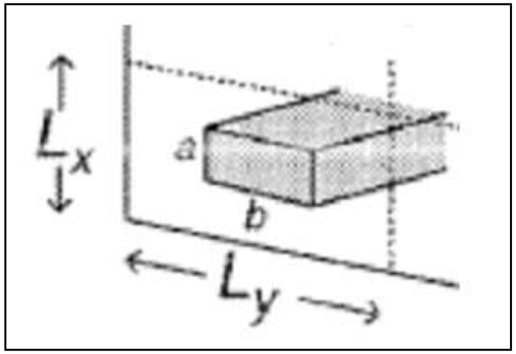


Quantum Wires



Ideal Quantum Wire – Wave functions and Energies

Areal Density in $x - y$ cross-section $N_{wr} = \frac{1}{L_x L_y}$



Fill Factor $F_{wr} = \frac{ab}{L_x L_y}$ **Wire Length** $L \gg a, b$

Wave Functions

$$\psi_c(x,y,z) = \frac{2}{\sqrt{ab}} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \frac{1}{\sqrt{L}} e^{ik_z z} u_c(\mathbf{r})$$

$$\psi_v(x,y,z) = \frac{2}{\sqrt{ab}} \sin\left(\frac{m'\pi}{a}x\right) \sin\left(\frac{n'\pi}{b}y\right) \frac{1}{\sqrt{L}} e^{ik'_z z} u_v(\mathbf{r})$$

Energy eigenvalues

$$E_c^{mn}(k_z) = E_{c0} + \frac{\hbar^2}{2m_e^*}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + k_z^2\right]$$

$$E_v^{m'n'}(k'_z) = E_{v0} - \frac{\hbar^2}{2m_h^*}\left[\left(\frac{m'\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2 + k_z'^2\right]$$

Density of States

Electron Density

$$n = \frac{2}{V}\sum_{m,n,k_z} f_c(E) = N_{wr}\frac{2}{L}\sum_{m,n}\int_{-\infty}^{\infty}\frac{dk_z}{\left(\frac{2\pi}{L}\right)} f_c(E)$$

$$= N_{wr}\int_0^{\infty} dE\,\rho_e^{1D}(E)\,f_c(E)$$

Density of state in a 1-D wire quantized along the x and y directions

$$\rho_e^{1D}(E) = \sum_{m,n}\frac{1}{\pi}\sqrt{\frac{2m_e^*}{\hbar^2}}\frac{1}{\sqrt{E-E_c^{mn}}} \quad \text{for } E > E_c^{mn}$$

Absorption coefficient

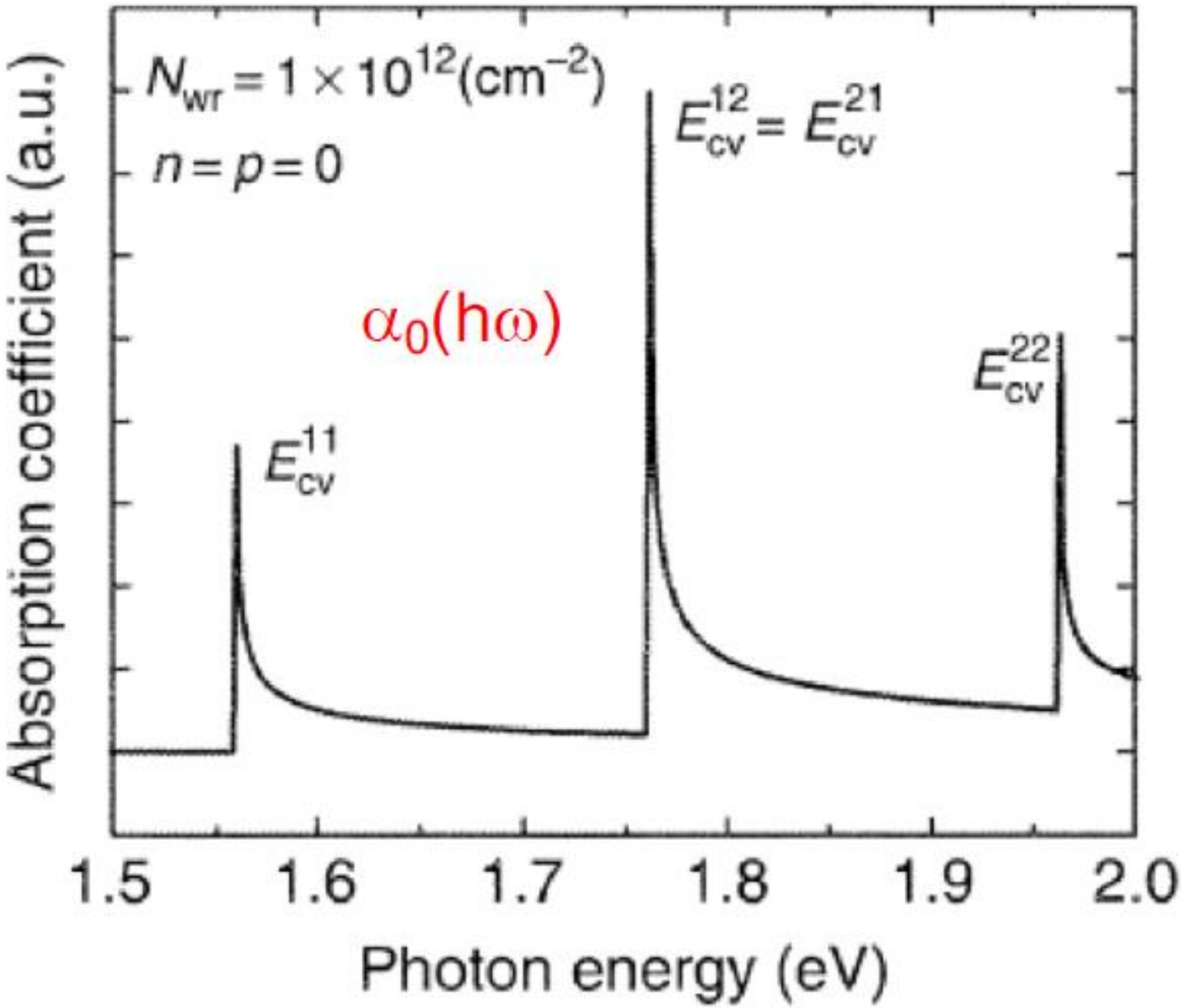
$$\alpha(\hbar\omega) = C_0 N_{wr} \sum_{m,n} \left|\hat{e} \bullet \mathbf{p}_{\mathbf{cv}}\right|^2 \rho_r^{1D}\left(\hbar\omega - E_c^{mn}\right)\left(f_v - f_c\right)$$

joint density of states

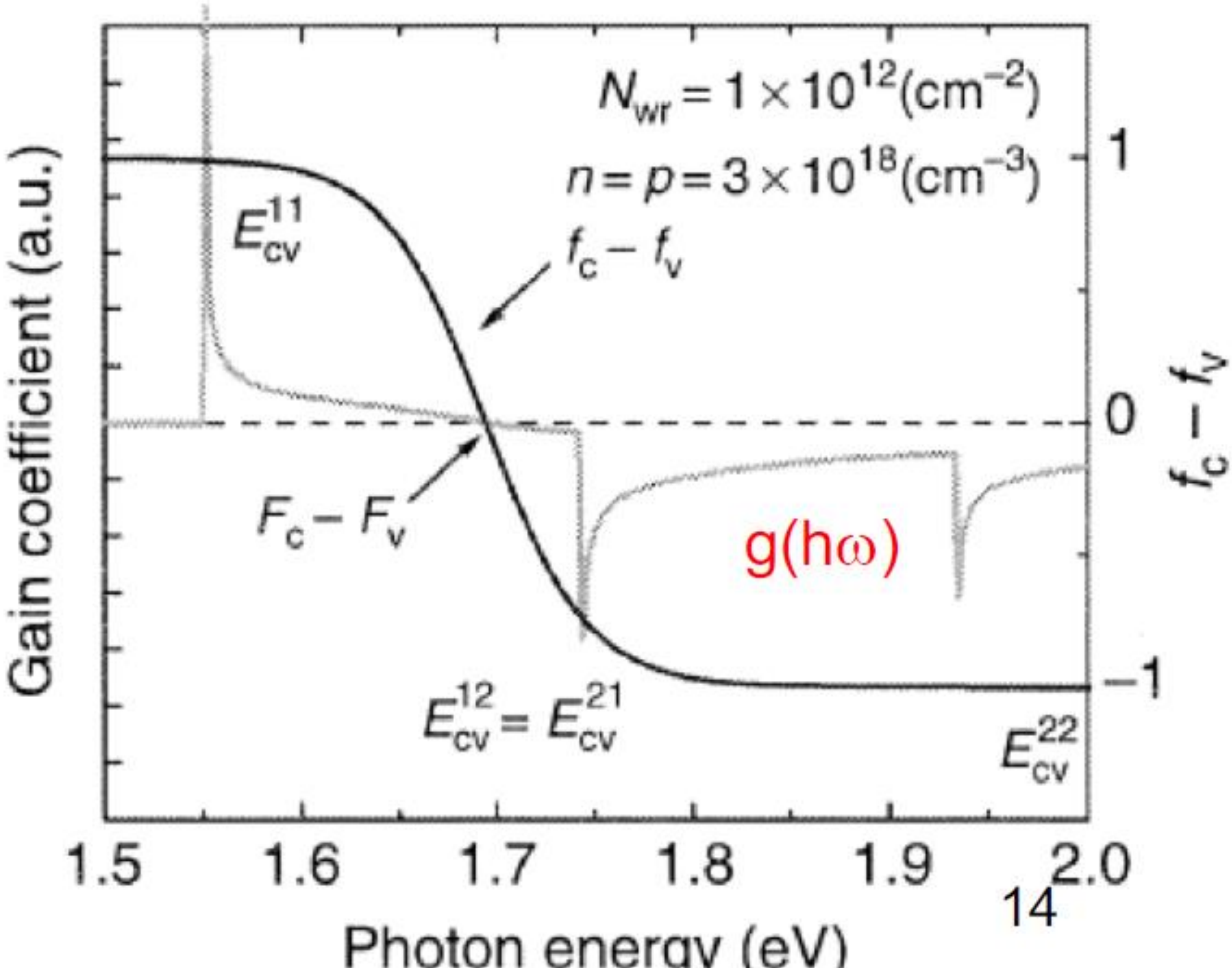
$$\rho_r^{1D}\left(\hbar\omega - E_c^{mn}\right) = \frac{1}{\pi}\sqrt{\frac{2m_r^*}{\hbar^2}}\frac{1}{\sqrt{\hbar\omega - E_c^{mn}}}$$

$$E_{cv}^{mn} = E_g + \frac{\hbar^2}{2m_r^*}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

Absorption coefficient



Gain coefficient



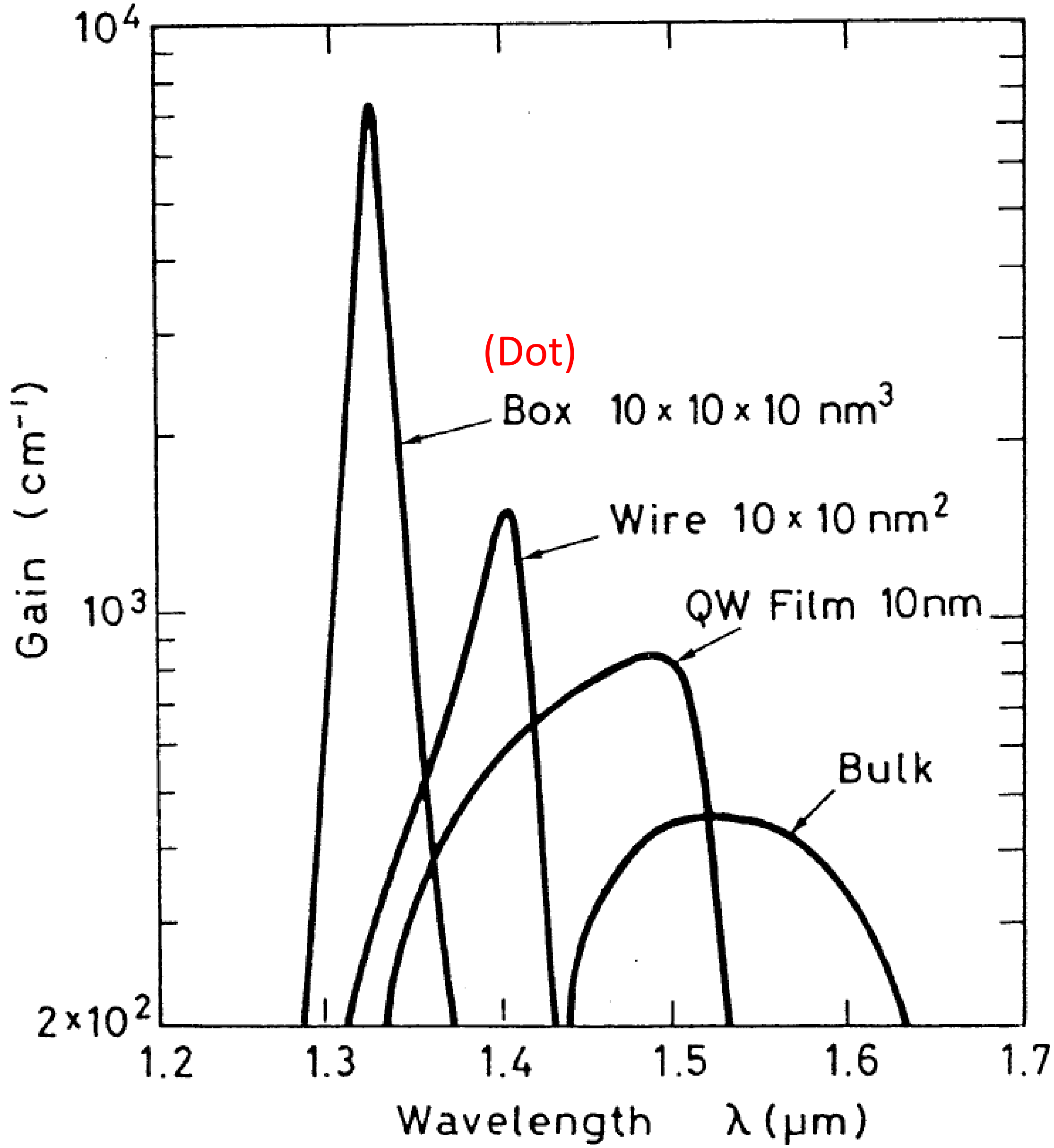
Two-dimensional quantum confinement in quantum-wire (QWR) semiconductor lasers is expected. in principle, to yield improved static and dynamic performance compared to conventional quantum-well (QW) lasers.

The improved features could include very low threshold currents (in the microampere regime), reduced temperature sensitivity, higher modulation bandwidth, and narrower spectral linewidth.

Gain in Semiconductors

Calculated gain coefficient

$Ga_{047}In_{053}As/InP$ $T = 300$ K $\tau_{in} = 1 \times 10^{-13}$ s $N = 3 \times 10^{18}$ cm^{-3}

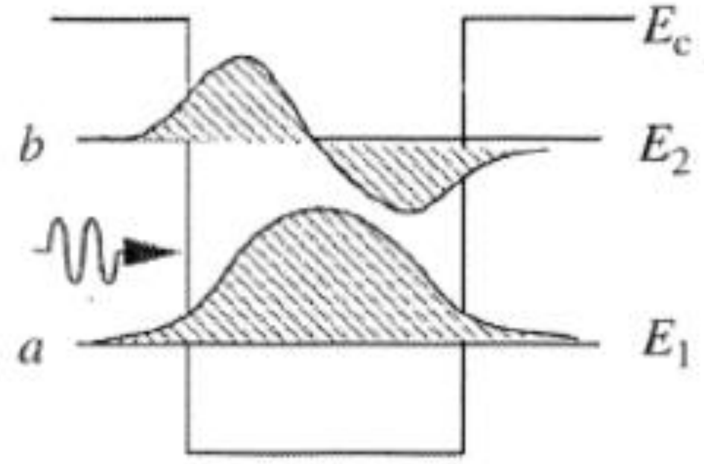


M. Asada, Y. Miyamoto, and Y. Suematsu, Gain and Threshold of Three-Dimensional Quantum-Box Lasers, *IEEE J. Quantum Electron* **QE–22**, 1915–1921 (1986).

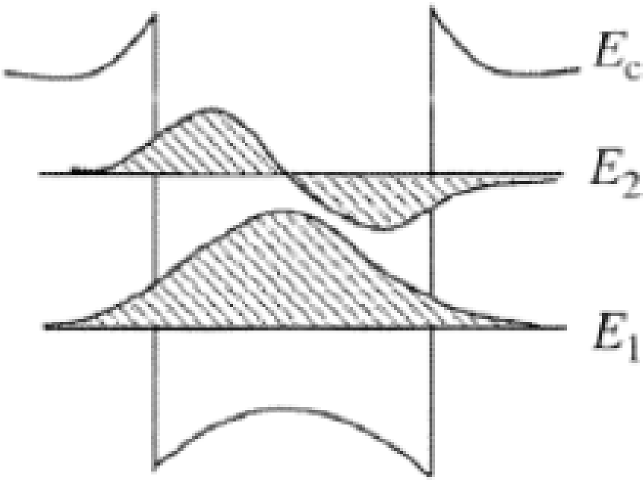
Intersubband Absorption

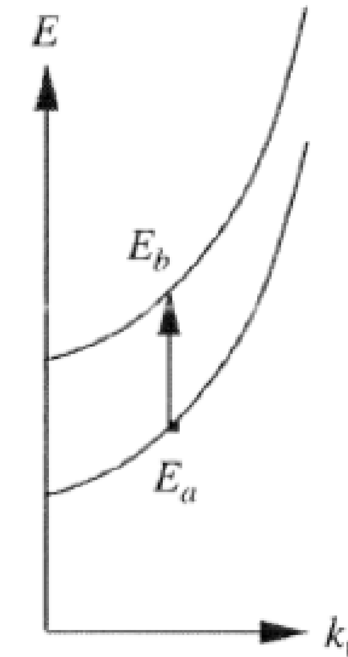
Transition between ground state and first excited state

Single QW = Low doping



Modulation doped QW





$$\psi_a(\mathbf{r}) = u_c(\mathbf{r}) \frac{e^{i\mathbf{k}_t \cdot \rho}}{\sqrt{A}} \phi_1(z)$$

$$\psi_b(\mathbf{r}) = u_{c'}(\mathbf{r}) \frac{e^{i\mathbf{k}'_t \cdot \rho}}{\sqrt{A}} \phi_2(z)$$

transverse momenta

$$\mathbf{k}_t = k_x \hat{x} + k_y \hat{y}$$

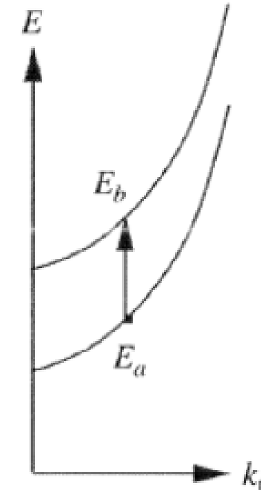
$$\mathbf{k}'_t = k'_x \hat{x} + k'_y \hat{y}$$

position vector

$$\boldsymbol{\rho} = x\hat{x} + y\hat{y}$$

Intersubband Absorption Spectrum

initial state $E_a = E_1 + \frac{\hbar^2 k_t^2}{2m_e^*}$ **final state** $E_b = E_1 + \frac{\hbar^2 k_t^2}{2m_e^*}$



$$\alpha(\hbar\omega) = \left(\frac{\omega}{n_r c \varepsilon_0}\right) \frac{2}{V} \sum_{\mathbf{k}_t} \sum_{\mathbf{k}'_t} \frac{\left|\hat{e} \cdot \boldsymbol{\mu}_{ba}\right|^2 \gamma}{\left(E_b - E_a - \hbar\omega\right)^2 + \gamma^2} \left(f_a - f_b\right)$$

$$= \left(\frac{\omega}{n_r c \varepsilon_0}\right) \frac{\left|\boldsymbol{\mu}_{21}\right|^2 \gamma}{\left(E_2 - E_1 - \hbar\omega\right)^2 + \gamma^2} \frac{2}{V} \sum_{\mathbf{k}_t} \left(f_a - f_b\right) = \left(\frac{\omega}{n_r c \varepsilon_0}\right) \frac{\left|\boldsymbol{\mu}_{21}\right|^2 \gamma}{\left(E_2 - E_1 - \hbar\omega\right)^2 + \gamma^2} \left(N_1 - N_2\right)$$

intersubband dipole moment

$$\boldsymbol{\mu}_{21} = \langle \phi_2 | ez | \phi_1 \rangle = \int \phi^*{}_2(z)\, ez\, \phi_1(z)\, dz$$

electrons per unit volume in n^th subband

$$N_n = \frac{m_e^* k_B T}{\pi \hbar^2 L_z} \ln\left[1 + e^{(E_F - E_n)/k_B T}\right]$$

low temperature $(E_F - E_n) \gg k_B T$

$$N_n = \frac{m_e^*}{\pi \hbar^2 L_z} \left(E_F - E_n\right)$$

Intersubband Absorption Spectrum

Using the two previous results

$$\alpha(\hbar\omega)=\left(\frac{\omega}{n_r c\varepsilon_0}\right)\frac{|\mu_{21}|^2\gamma}{(E_2-E_1-\hbar\omega)^2+\gamma^2}(N_1-N_2)$$

$$N_n=\frac{m_e^*}{\pi\hbar^2 L_z}(E_F-E_n)$$

considering two levels occupied

$$\alpha(\hbar\omega)=\left(\frac{\omega}{n_r c\varepsilon_0}\right)\frac{|\mu_{21}|^2\gamma}{(E_2-E_1-\hbar\omega)^2+\gamma^2}\left(\frac{m_e^*}{\pi\hbar^2 L_z}\right)(E_2-E_1)$$

INTEGRATED ABSORBANCE

$$A=\int_0^\infty \alpha(\hbar\omega)\,d(\hbar\omega)\simeq\left(\frac{\omega_{21}}{n_r c\varepsilon_0}\right)|\mu_{21}|^2\pi(N_1-N_2)$$

Example – Peak absorption coefficient in QW

Consider a GaAs quantum well of width $L_z = 100\text{Å}$, doped *n*-type corresponding to a 3D carrier concentration $N = 10^{18}\ \text{cm}^{-3}$. The electron effective mass is $m_e^* = 0.0665 m_0$. Let's assume a simple infinite barrier model and $T = 300\text{K}$.

$$E_1 = \frac{\hbar^2}{2m_e^*}\left(\frac{\pi}{L_z}\right)^2 = 56.6\ \text{meV} \qquad \varphi_1(z) = \sqrt{\frac{2}{L_z}}\sin\left(\frac{\pi}{L_z}z\right)$$

$$E_2 = \frac{2\hbar^2}{m_e^*}\left(\frac{\pi}{L_z}\right)^2 = 226\ \text{meV} \qquad \varphi_1(z) = \sqrt{\frac{2}{L_z}}\sin\left(\frac{2\pi}{L_z}z\right)$$

Assume the number of electrons per unit volume (3D density) occupy only the first subband and the 2nd subband is empty

$$N_{2D} = \frac{m_e^* k_B T}{\pi\hbar^2} = 7.19\times 10^{11} cm^{-2} \qquad L_z = 100\ \text{Å} = 10^{-6}\ \text{cm}$$

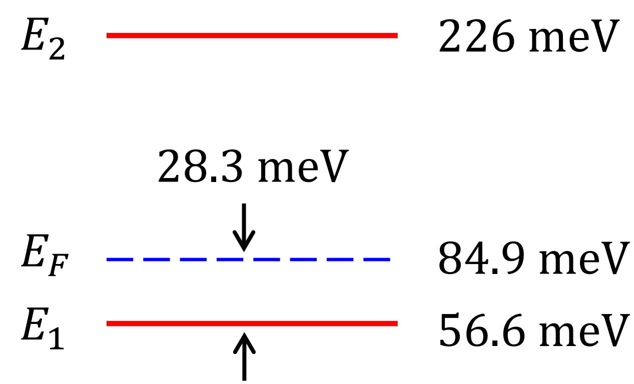
$$N_1 = \frac{m_e^* k_B T}{\pi\hbar^2 L_z}\ln\left(1+\exp\left(\frac{E_F - E_1}{k_B T}\right)\right) = \frac{N_{2D}}{L_z}\ln\left(1+\exp\left(\frac{E_F - E_1}{k_B T}\right)\right) \approx 10^{18}\text{cm}^{-3}$$

This is satisfied for

$$(E_F - E_1) \approx 0.0283\ \text{eV} = 28.3\ \text{meV}$$

Now, calculate the density N_2 with this Fermi level, to verify that $N_2 \ll N$

$$N_2 = \frac{N_{2D}}{L_z} \ln\left(1 + \exp\left(\frac{E_F - E_2}{k_B T}\right)\right) = 2.9 \times 10^{15} \mathrm{cm}^{-3}$$



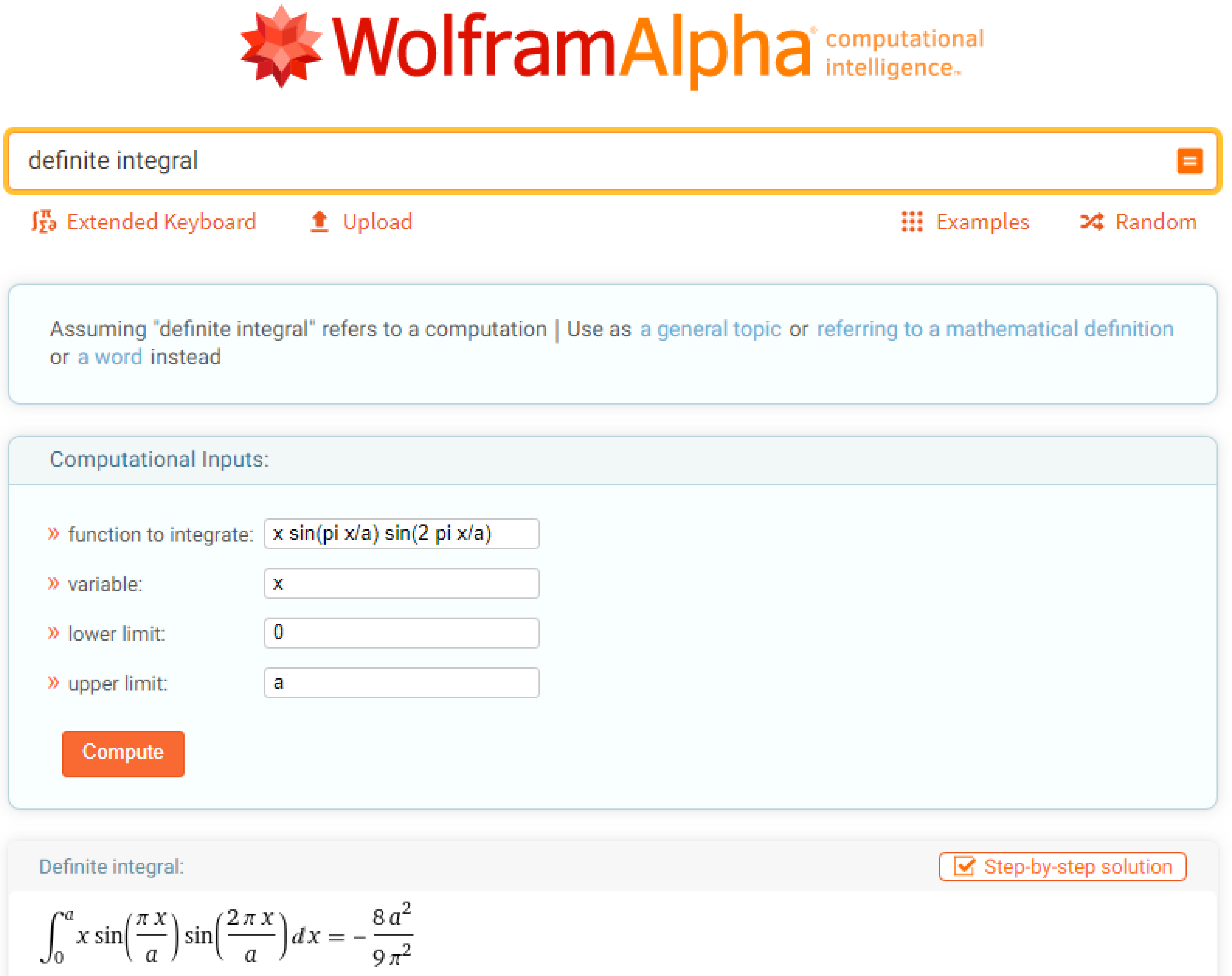
It is indeed $N_2 \ll N$. Otherwise, we would need to use the expression for $N_1 + N_2 = N$ to determine numerically the Fermi level.

We can calculate now the dipole moment

$$\mu_{21} = e \int_0^{L_z} \varphi_2(z)\, z\, \varphi_1(z)\, dz = e \frac{2}{L_z} \int_0^{L_z} z\ \sin\left(\frac{\pi}{L_z} z\right) \sin\left(\frac{2\pi}{L_z} z\right) dz =$$

$$= e \frac{2}{L_z} \left[-\frac{8L_z^2}{9\pi^2}\right] = -\frac{16}{9\,\pi^2}\, e\, L_z = -18.013 e\ \text{Å} = -2.882 \times 10^{-28} \mathrm{C \cdot m}$$

Resources for symbolic manipulation can be very handy to evaluate integrals with complicated trigonometric functions



The peak absorption occurs for a photon energy

$$\hbar\omega \approx E_2 - E_1 = 169.4 \text{ meV}$$

with corresponding peak wavelength

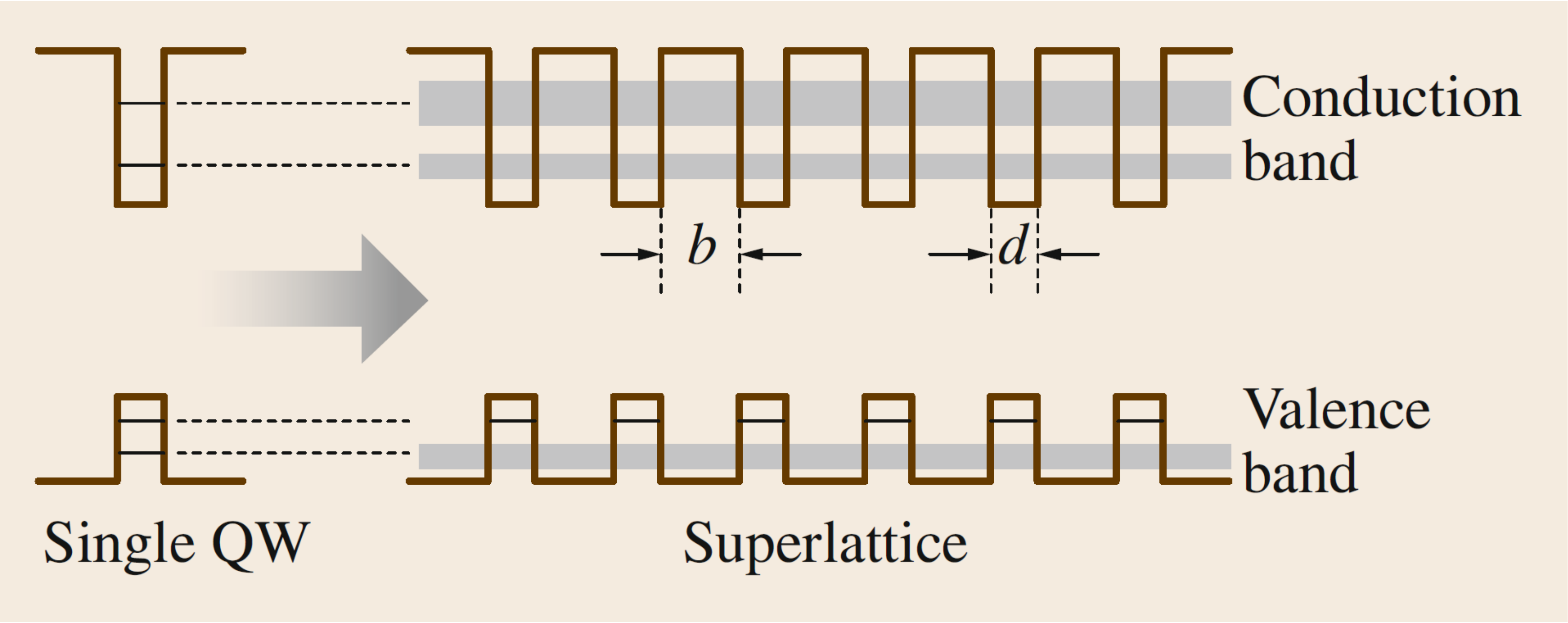
$$\lambda \approx \frac{1.24}{0.170} = 7.3 \ \mu\text{m}$$

Assume refractive index $n_r = 3.3$ and a linewidth $2\gamma = 30 \text{ meV}$ we obtain the peak absorption coefficient

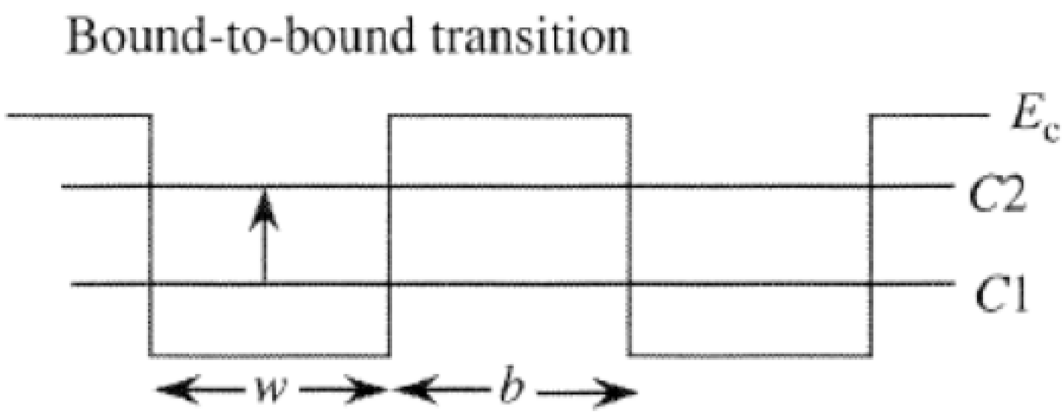
$$\alpha = \frac{\omega}{n_r c \varepsilon_0} \frac{|\mu_{21}|^2}{\gamma} (N_1 - N_2) \approx 1.015 \times 10^4 \text{ cm}^{-1}$$

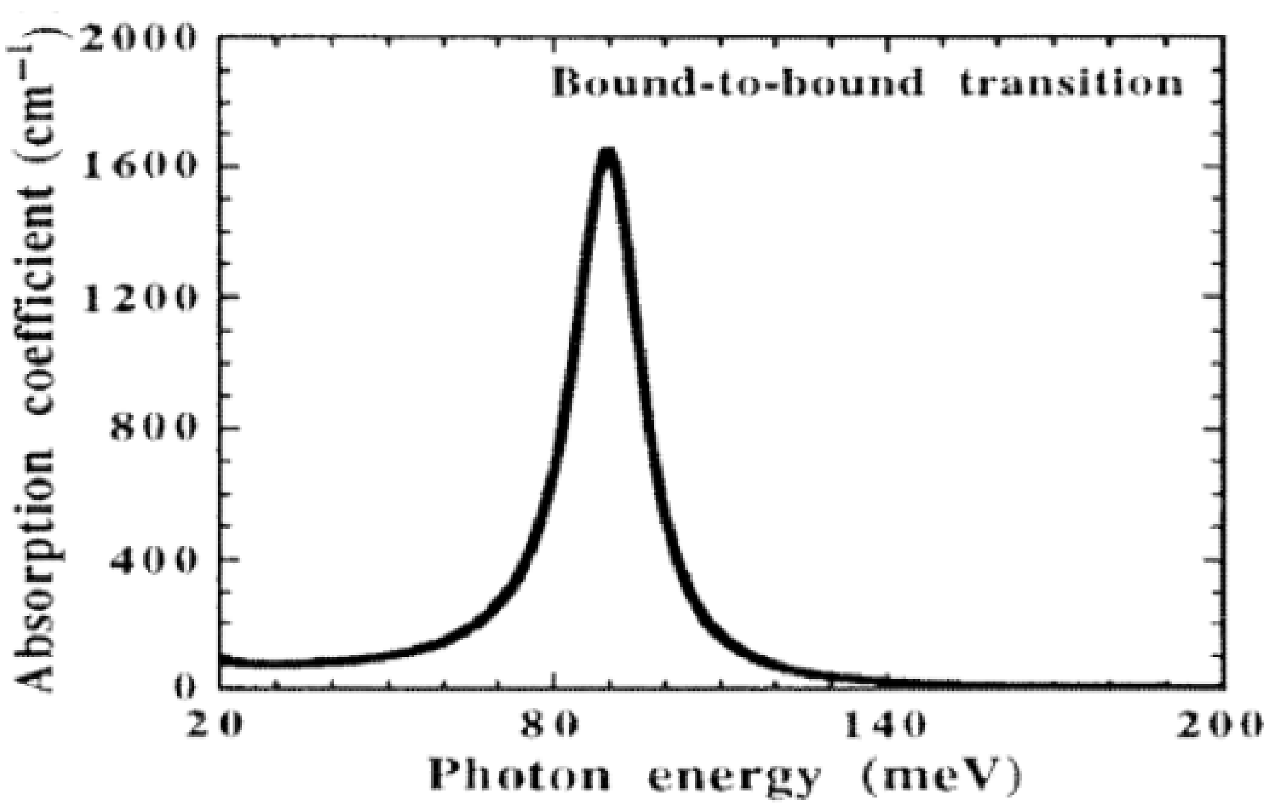
Absorption in Superlattices

Superlattice



Absorption in a Superlattice ($GaAs/Al_{0.3}Ga_{0.7}As$)





$T = 77\text{K}$

$N_s = 2 \times 10^{11}\text{cm}^{-2}$

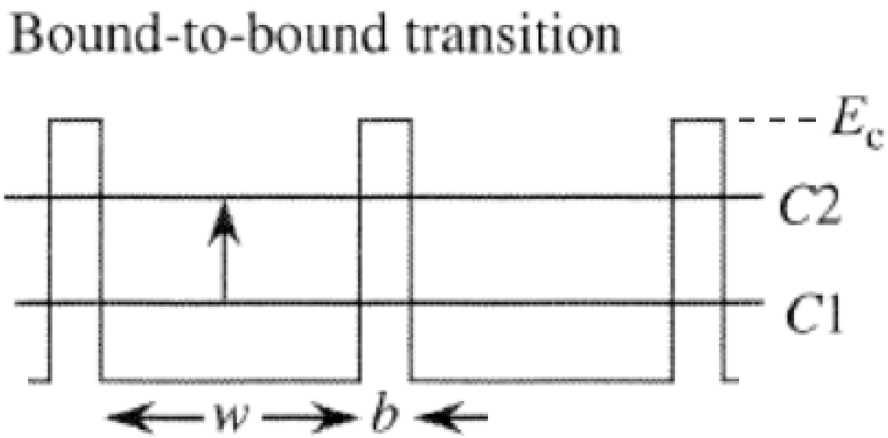
$2\gamma = 15\text{ meV}$

$w = b = 100\text{Å}$

Thick barriers *b*

Small miniband widths

Absorption in a Superlattice



$T = 77\text{K}$

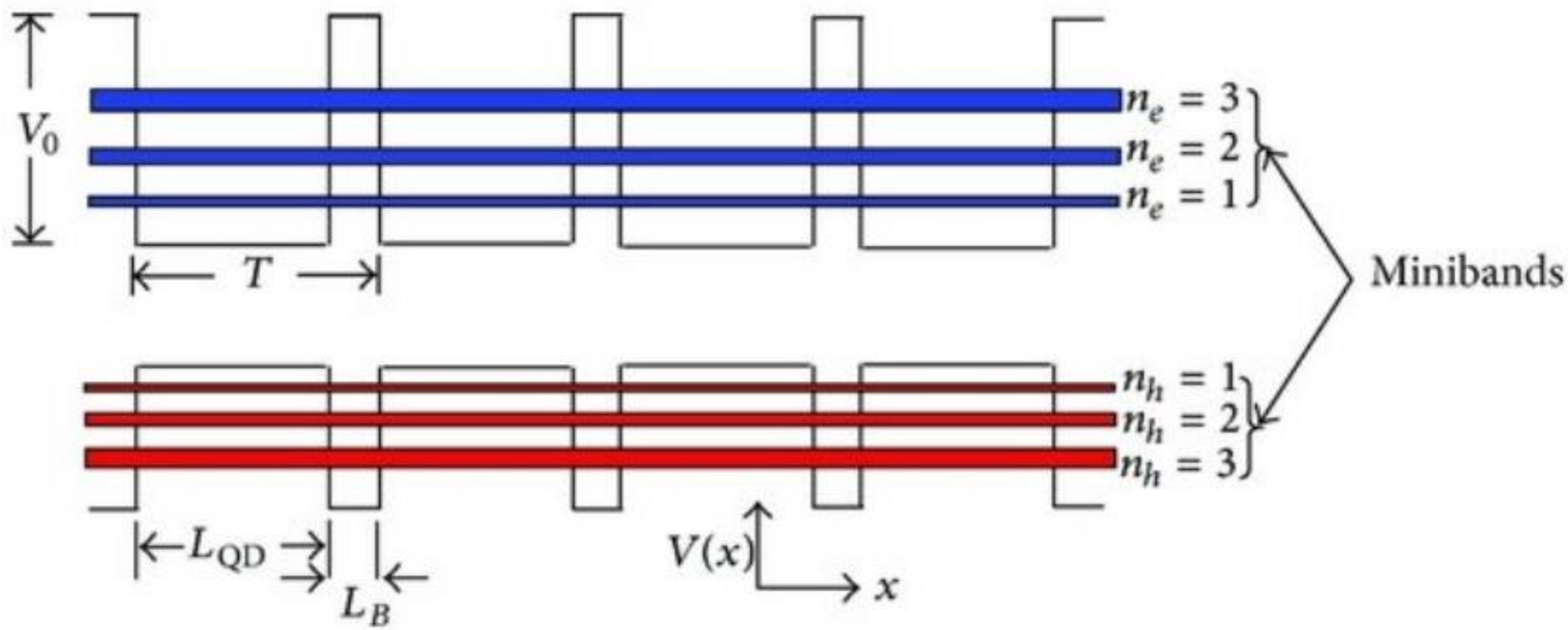
$N_s = 2 \times 10^{11}\text{cm}^{-2}$

$2\gamma = 15\ \text{meV}$

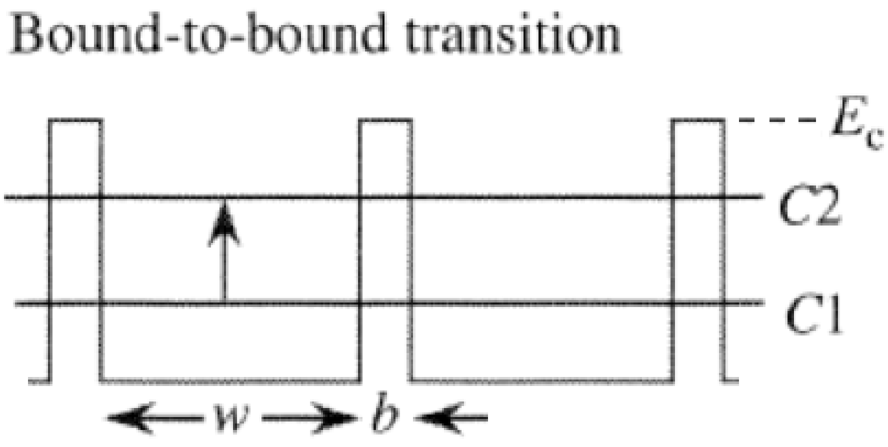
$w = 100\text{Å}$
$b = 20\text{Å}$

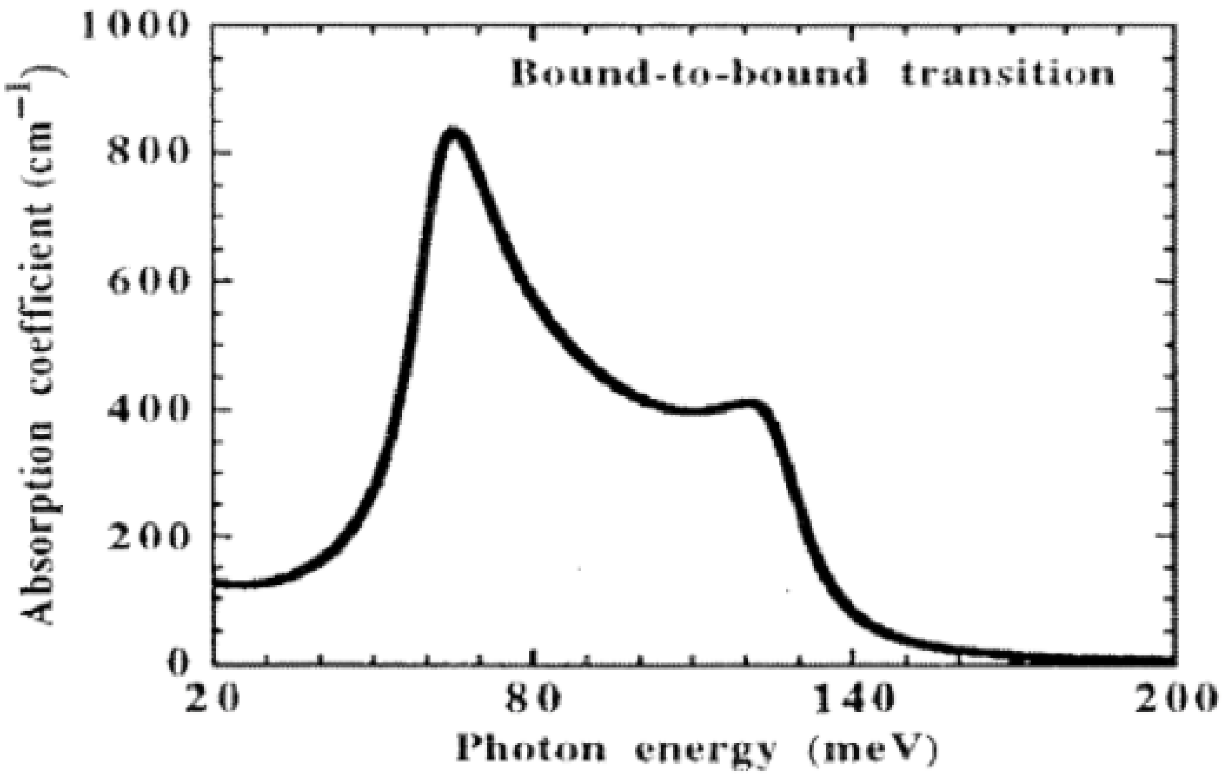
Thin barriers *b*

Large miniband widths



Absorption in a Superlattice





$T = 77\text{K}$

$N_s = 2 \times 10^{11}\text{cm}^{-2}$

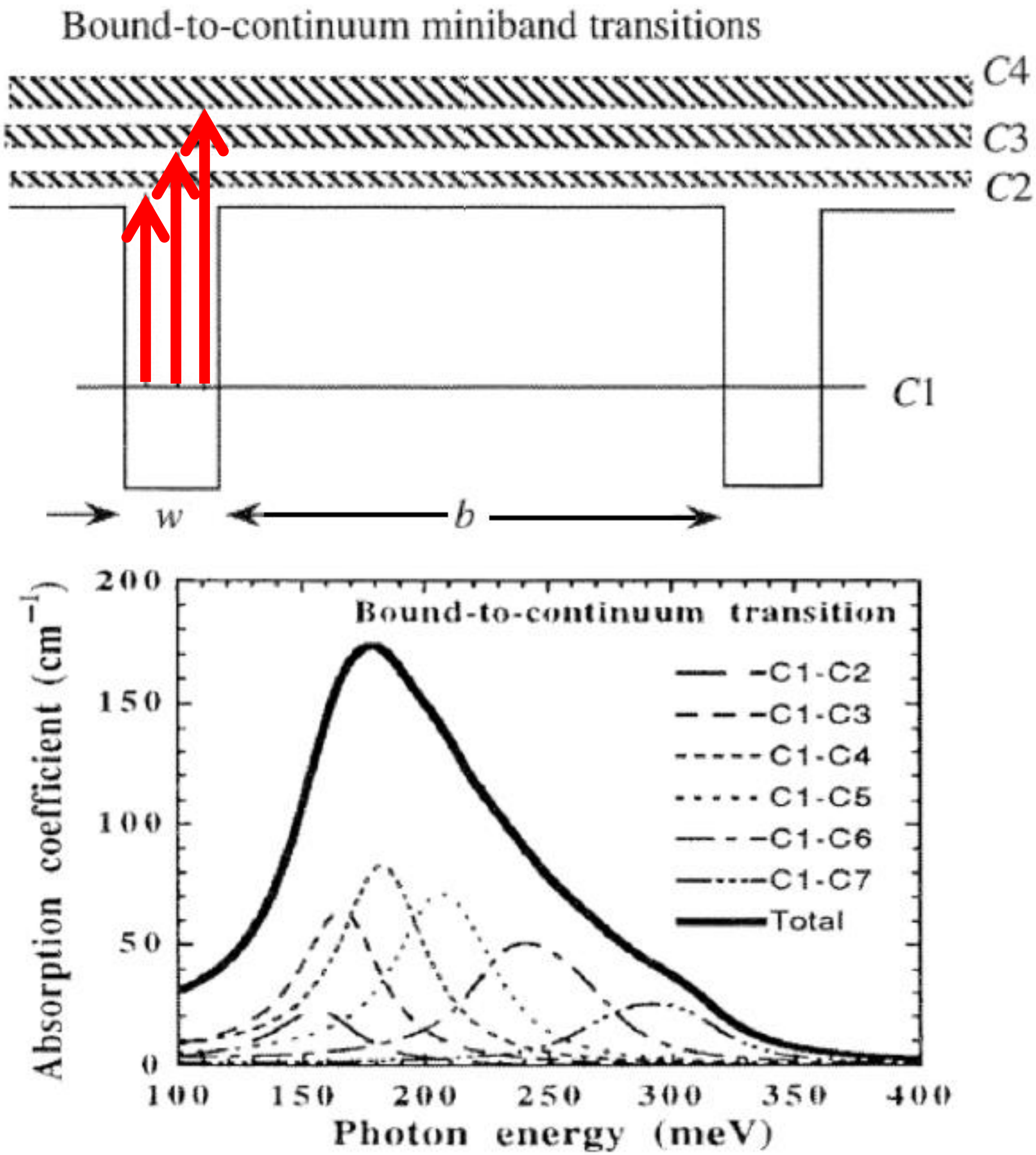
$2\gamma = 15 \text{ meV}$

$w = 100\text{Å}$
$b = 20\text{Å}$

Thin barriers *b*

Large miniband widths

Absorption in a Superlattice



$T = 77\text{K}$

$N_s = 2 \times 10^{11} \text{cm}^{-2}$

$2\gamma = 15 \text{ meV}$

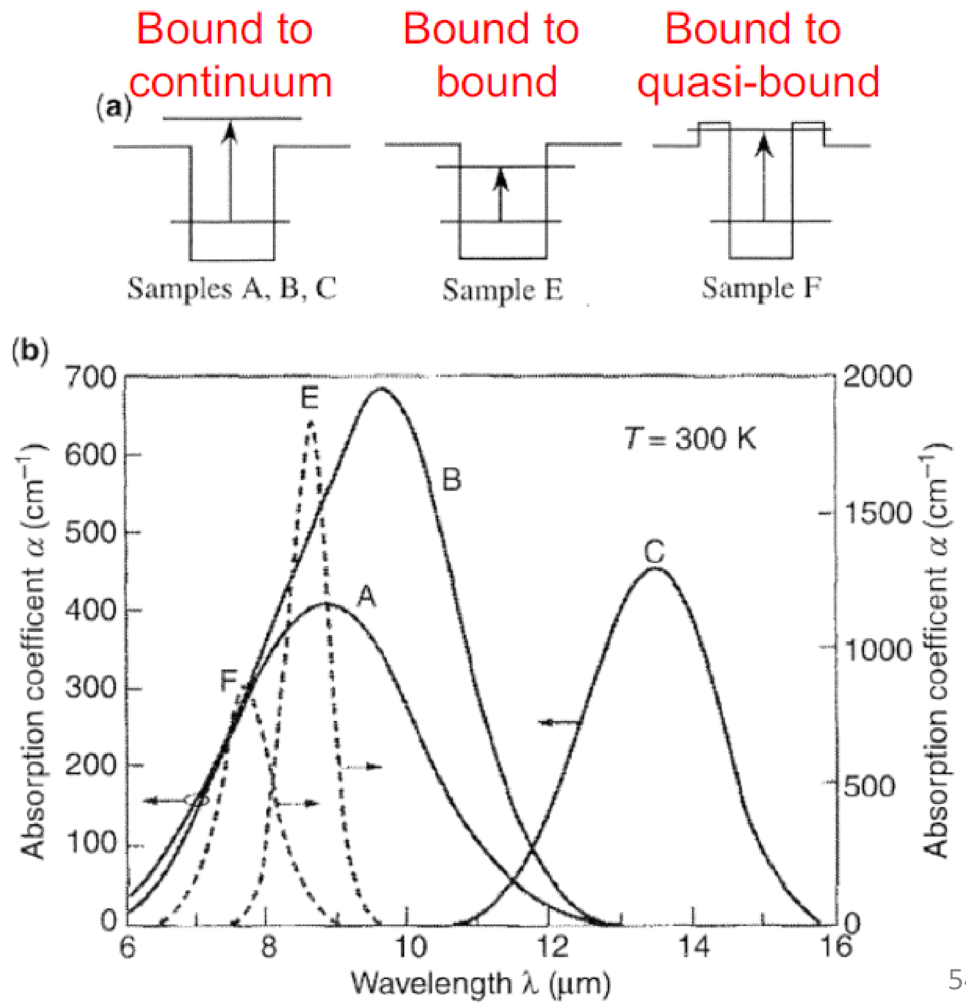
$w = 40\text{Å}$
$b = 300\text{Å}$

Experimental Subband Transition Examples

GaAs QW
$Al_xGa_{1-x}As$ Barrier

[from Levine, *J. Appl. Phys.*, 1993]

Sample	Well Width (Å)	Barrier Width (Å)	x	N_D ($10^{18}/cm^3$)
A	40	500	0.26	1
B	40	500	0.25	1.6
C	60	500	0.15	0.5
E	50	500	0.26	0.42
F	50	50	0.30	0.42
		500	0.26	



A, B, C have weak absorption but good carrier collection

E has the best absorption but doesn't have good carrier collection

F has good absorption and carrier collection

Quantum Cascade Laser (1994)

QCL is an **intersubband** laser where electrons are injected by tunneling through the barrier into E_3 (t_3=0.2ps).

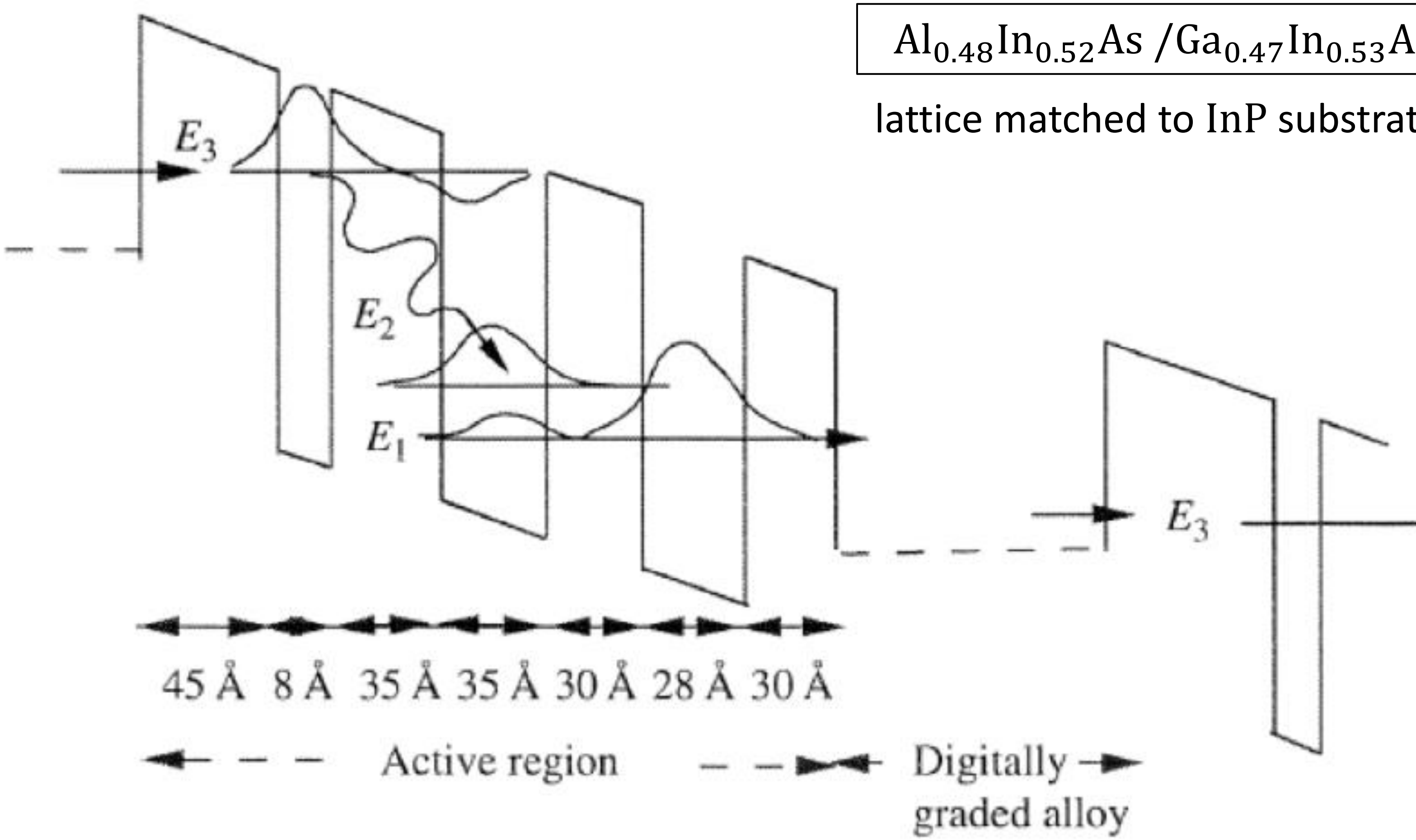
Small overlap between E_3 and E_2 wavefunctions creates long decay time (t_{32}=4.3ps) and thus a population inversion between states E_3 and E_2 for lasing action.

Quick decay from E_2 to E_1 (t_{21}=0.6ps).

barriers	**wells**

$Al_{0.48}In_{0.52}As\ /Ga_{0.47}In_{0.53}As$

lattice matched to InP substrate



Reading Assignments:

Section 9.6 of Chuang's book