ECE 536 – Integrated Optics and Optoelectronics
Lecture 20 – March 31, 2022

Spring 2022

Tu-Th 11:00am-12:20pm

Prof. Umberto Ravaioli

ECE Department, University of Illinois

Lecture 20 Outline

- Integrated Optics Simulation Approaches
- Transmission Matrix Method

Most popular numerical simulation approaches for optical waveguide devices

- **Full solution of Maxwell's equations:**
  Finite-Differences Time-Domain (FDTD) Method

- **Approximate EM solution techniques for light propagation in slowly varying waveguides**
  Beam Propagation Method

- **Scattering formalism (Transmission Line)**
  Transmission Matrix Method

Maxwell's Equation

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\partial_t \mathbf{B}(\mathbf{r},t),$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \partial_t \mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t),$$

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t),$$

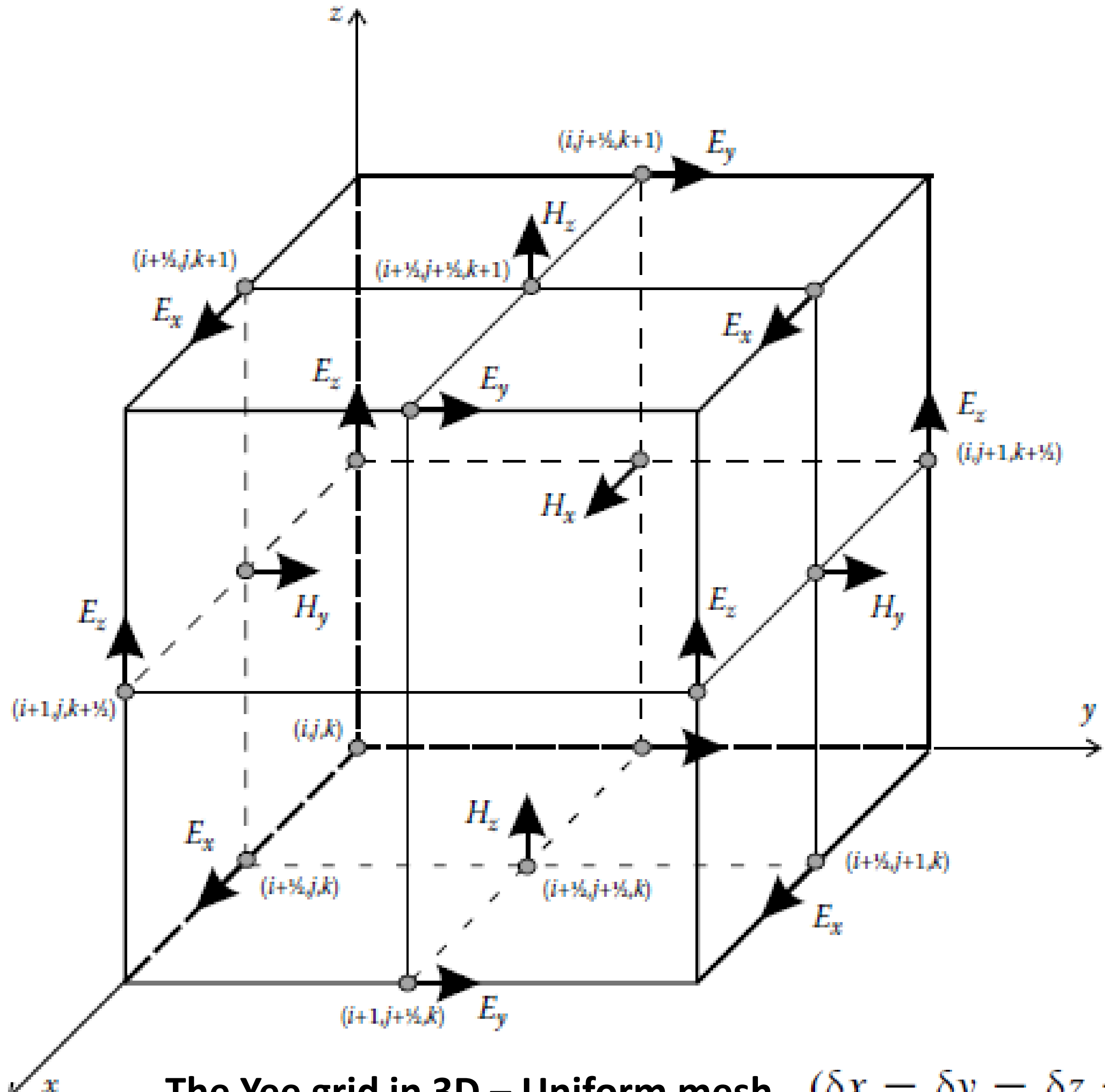
$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0,$$

Simple case of linear, isotropic, and non-dispersive medium

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \varepsilon(\mathbf{r}) \mathbf{E}(\mathbf{r},t),$$

$$\mathbf{B}(\mathbf{r},t) = \mu_0 \mu(\mathbf{r}) \mathbf{H}(\mathbf{r},t).$$

Finite Differences Time Dependent (FDTD)



The Yee grid in 3D – Uniform mesh $(\delta x = \delta y = \delta z = h)$

Finite Differences Time Dependent (FDTD)

- **Components of the electric fields are defined in the middle of the edges of the cube**

- **Components of the magnetic field are defined in the centers of the cube faces**

- **Time discretization: leapfrog scheme with time step δt**

Electric Field

$$\frac{E_i^{n+1/2} - E_i^{n-1/2}}{\delta t} = -\frac{1}{\varepsilon_0 \varepsilon_i} \frac{H_{i+1/2}^n - H_{i-1/2}^n}{h}.$$

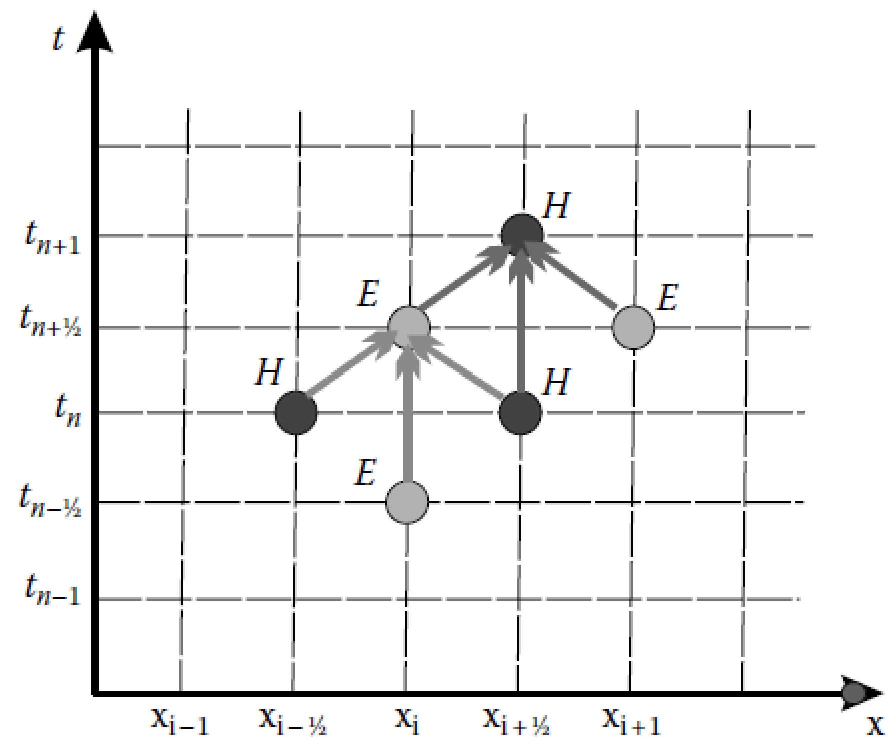
$$E_i^{n+1/2} = E_i^{n-1/2} - \frac{\delta t}{h \varepsilon_0 \varepsilon_i} (H_{i+1/2}^n - H_{i-1/2}^n).$$

Finite Differences Time Dependent (FDTD)

Magnetic Field

$$\frac{H_{i+1/2}^{n+1} - H_{i+1/2}^{n}}{\delta t} = -\frac{1}{\mu_0 \mu_{i+1/2}} \frac{E_{i+1}^{n+1/2} - E_{i}^{n+1/2}}{h},$$

$$H_{i+1/2}^{n+1} = H_{i+1/2}^{n} - \frac{\delta t}{h \mu_0 \mu_{i+1/2}} \left(E_{i+1}^{n+1/2} - E_{i}^{n+1/2}\right).$$



Finite Differences Time Dependent (FDTD)

$$E_x\big|^{n+1/2}_{i+1/2,j,k} = E_x\big|^{n-1/2}_{i+1/2,j,k} + \frac{\delta t}{\varepsilon_0 \varepsilon_{i+1/2,j,k}} \times \left(\frac{H_z\big|^{n}_{i+1/2,j+1/2,k} - H_z\big|^{n}_{i+1/2,j-1/2,k}}{\delta y} - \frac{H_y\big|^{n}_{i+1/2,j,k+1/2} - H_y\big|^{n}_{i+1/2,j,k-1/2}}{\delta z} \right),$$

Similar discretization for y and z

$$H_x\big|^{n+1}_{i,j+1/2,k+1/2} = H_x\big|^{n}_{i,j+1/2,k+1/2} + \frac{\delta t}{\mu_0 \mu_{i,j+1/2,k+1/2}} \times \left(\frac{E_y\big|^{n+1/2}_{i,j+1/2,k+1} - E_y\big|^{n+1/2}_{i,j+1/2,k}}{\delta z} - \frac{E_z\big|^{n+1/2}_{i,j+1,k+1/2} - E_z\big|^{n+1/2}_{i,j,k+1/2}}{\delta y} \right),$$

Similar discretization for y and z

Example of discretization

Finite Differences Time Dependent (FDTD)

$$E^{n+1}_{x;i,j,k} = E^{n}_{x;i,j,k} + \frac{Q}{\varepsilon_{i,j,k}} \left(H^{n}_{z;i,j,k} - H^{n}_{z;i,j-1,k} - H^{n}_{y;i,j,k} + H^{n}_{y;i,j,k-1}\right),$$

$$E^{n+1}_{y;i,j,k} = E^{n}_{y;i,j,k} + \frac{Q}{\varepsilon_{i,j,k}} \left(H^{n}_{x;i,j,k} - H^{n}_{x;i,j,k-1} - H^{n}_{z;i,j,k} + H^{n}_{z;i-1,j,k}\right),$$

$$E^{n+1}_{z;i,j,k} = E^{n}_{z;i,j,k} + \frac{Q}{\varepsilon_{i,j,k}} \left(H^{n}_{y;i,j,k} - H^{n}_{y;i-1,j,k} - H^{n}_{x;i,j,k} + H^{n}_{x;i,j-1,k}\right),$$

$$H^{n+1}_{x;i,j,k} = H^{n}_{x;i,j,k} + \frac{Q}{\mu_{i,j,k}} \left(E^{n+1}_{y;i,j,k+1} - E^{n+1}_{y;i,j,k} - E^{n+1}_{z;i,j+1,k} + E^{n+1}_{z;i,j,k}\right),$$

$$H^{n+1}_{y;i,j,k} = H^{n}_{y;i,j,k} + \frac{Q}{\mu_{i,j,k}} \left(E^{n+1}_{z;i+1,j,k} - E^{n+1}_{z;i,j,k} - E^{n+1}_{x;i,j,k+1} + E^{n+1}_{x;i,j,k}\right),$$

$$H^{n+1}_{z;i,j,k} = H^{n}_{z;i,j,k} + \frac{Q}{\mu_{i,j,k}} \left(E^{n+1}_{x;i,j+1,k} - E^{n+1}_{x;i,j,k} - E^{n+1}_{y;i+1,j,k} + E^{n+1}_{y;i,j,k}\right),$$

Scaling $\hat{E} = \frac{E}{\eta_0}$ **Uniform mesh** $(\delta x = \delta y = \delta z = h)$ $Q = \frac{c_0 \delta t}{h}$

Beam Propagation Method (BPM)

Approximate simulation technique for light propagation in slowly varying optical waveguides

Helmholtz equation

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} + k^2 n^2(x, y, z)E = 0.$$

Electric Field

slowly varying term

rapidly varying term

$$E(x, y, z) = \phi(x, y, z)\exp(-jkn_0 z).$$

cladding index

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^2\phi - j2kn_0\frac{\partial\phi}{\partial z} + k^2(n^2 - n_0^2)\phi = 0,$$

Beam Propagation Method (BPM)

Assuming weakly guiding condition

$$(n^2 - n_0^2) \cong 2\, n_0\, (n - n_0)$$

$$\boxed{\nabla^2\phi - j2kn_0\frac{\partial\phi}{\partial z} + k^2(n^2 - n_0^2)\phi = 0,}$$

↓

$$\frac{\partial\phi}{\partial z} = -j\frac{1}{2kn_0}\nabla^2\phi - jk(n - n_0)\phi.$$

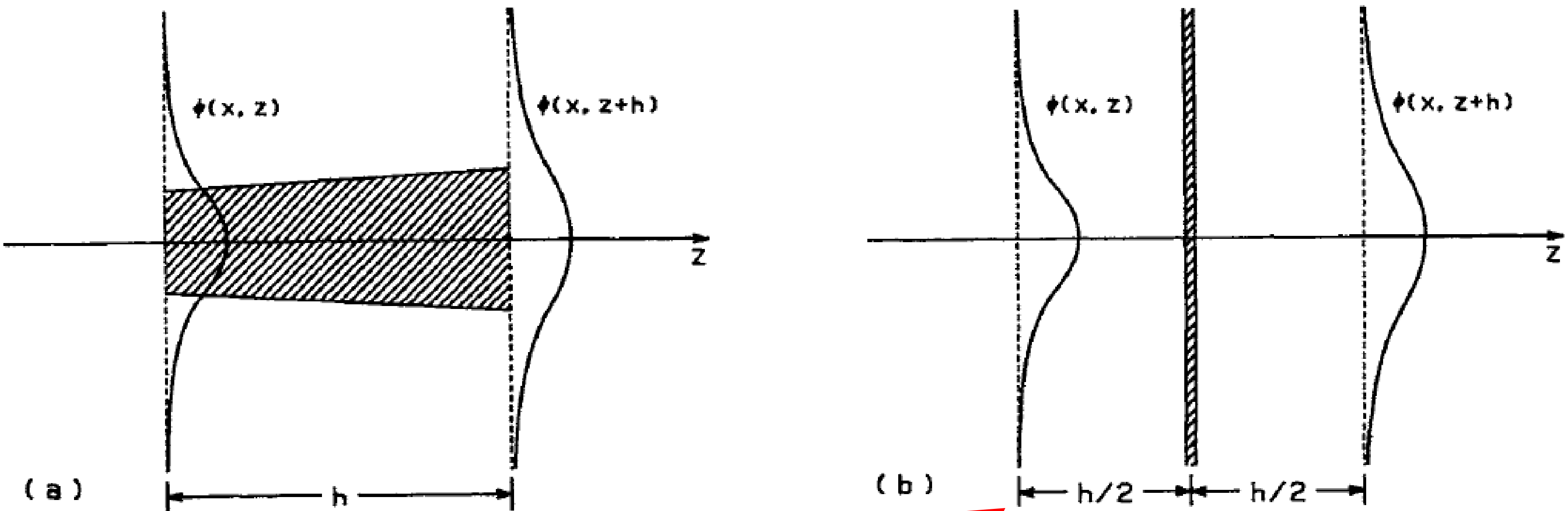
free-light propagation term

guidance term

Beam Propagation Method (BPM)

The two terms are applied separately

Approximation with BPM step



- The electric field $\varphi(x,z)$ is first propagated freely in index n_0 over a distance $h/2$.
- Phase retardation of the entire length h is taken into account at the center of the interval
- The resulting electric field is propagated again freely in index n_0 for another distance $h/2$ to obtain $\varphi(x,z+h)$

Beam Propagation Method (BPM)

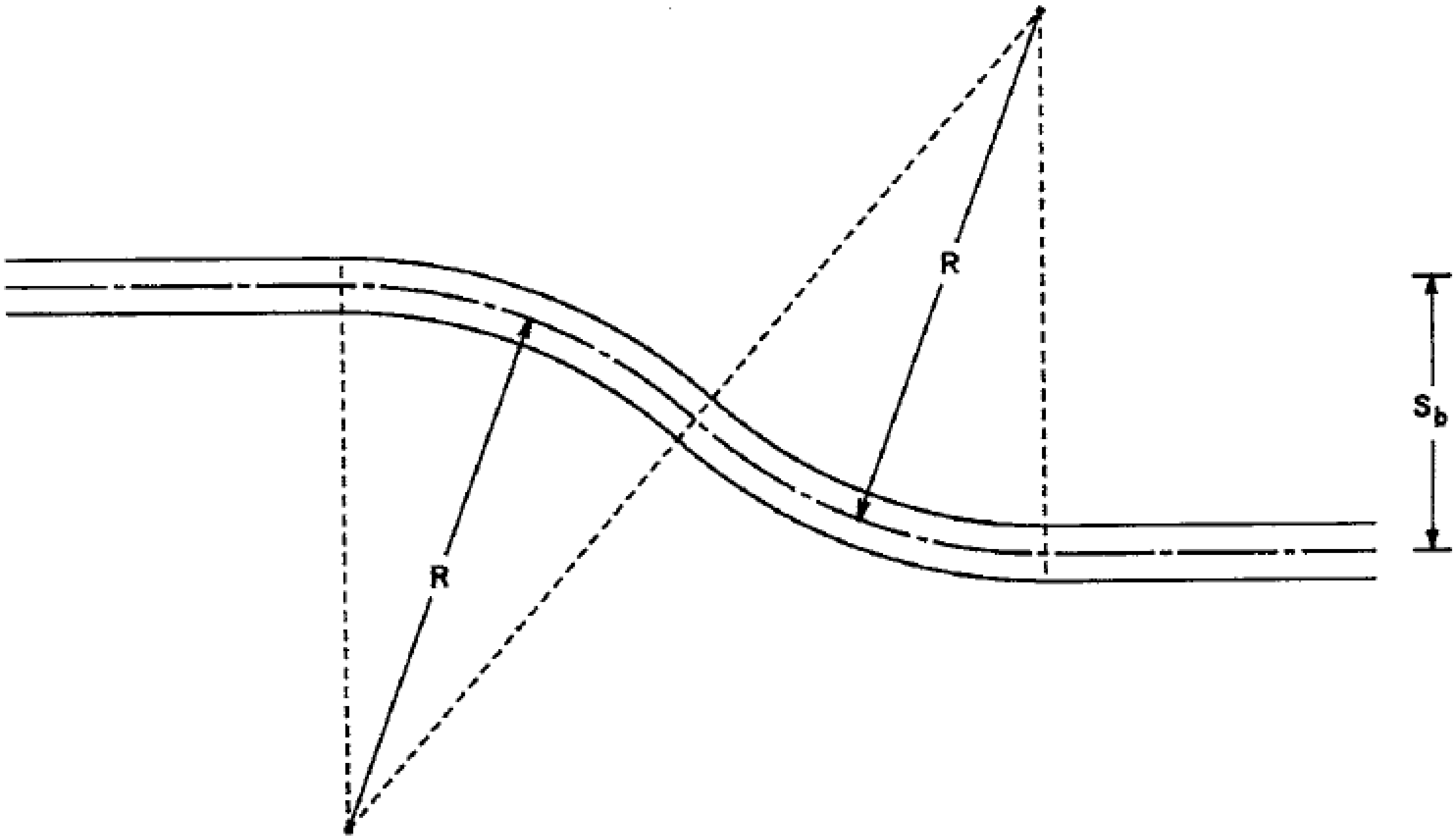
An extensive theoretical reference for BPM is Chapter 7 of

K. Okamoto, "Fundamentals of Optical Waveguides" Elsevier (2006)

(Available for download from the digital library)

Some representative examples follow

Beam Propagation Method (BPM) - Examples



S-shaped bent waveguide without offset.

Beam Propagation Method (BPM) - Examples

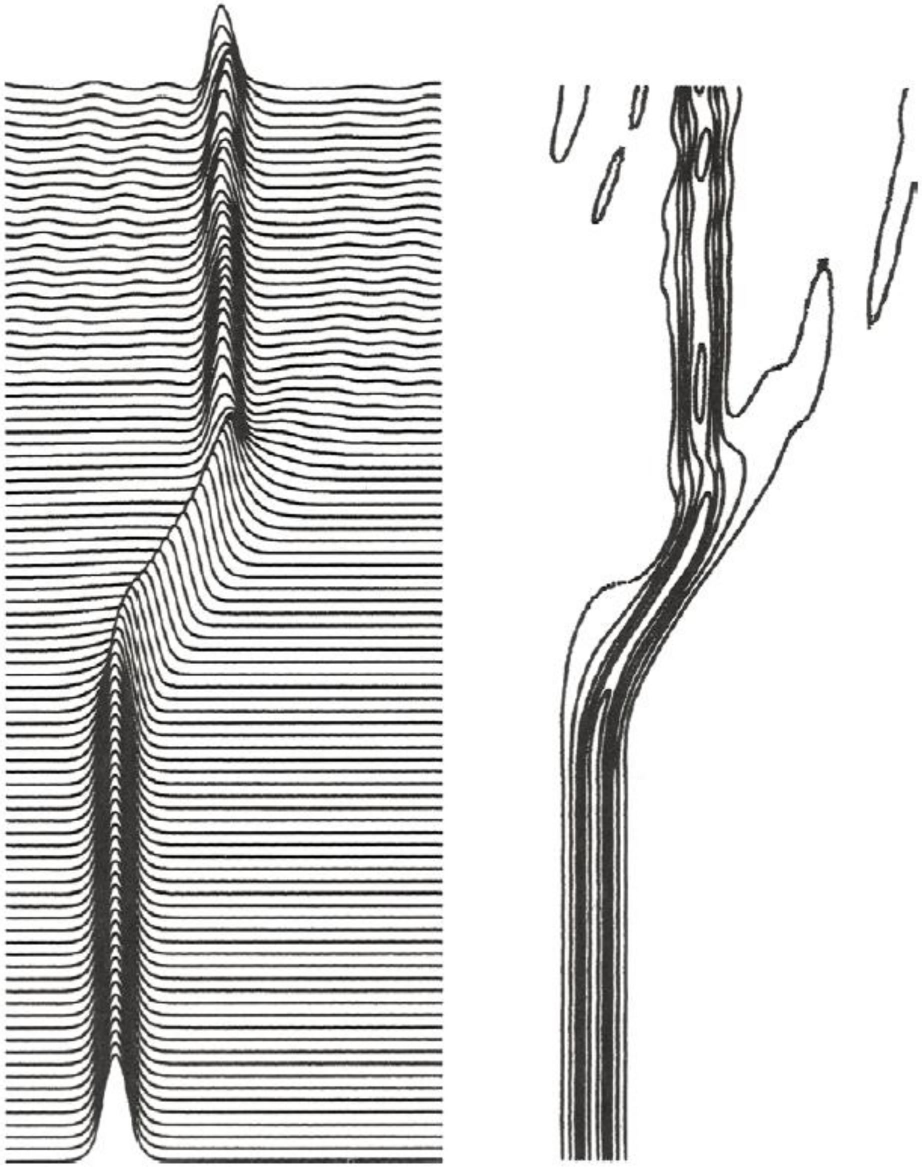


Figure 7.12 BPM simulation of the light propagation in an S-bend waveguide consisting of a fixed radius of curvature without offset.

Beam Propagation Method (BPM) - Examples

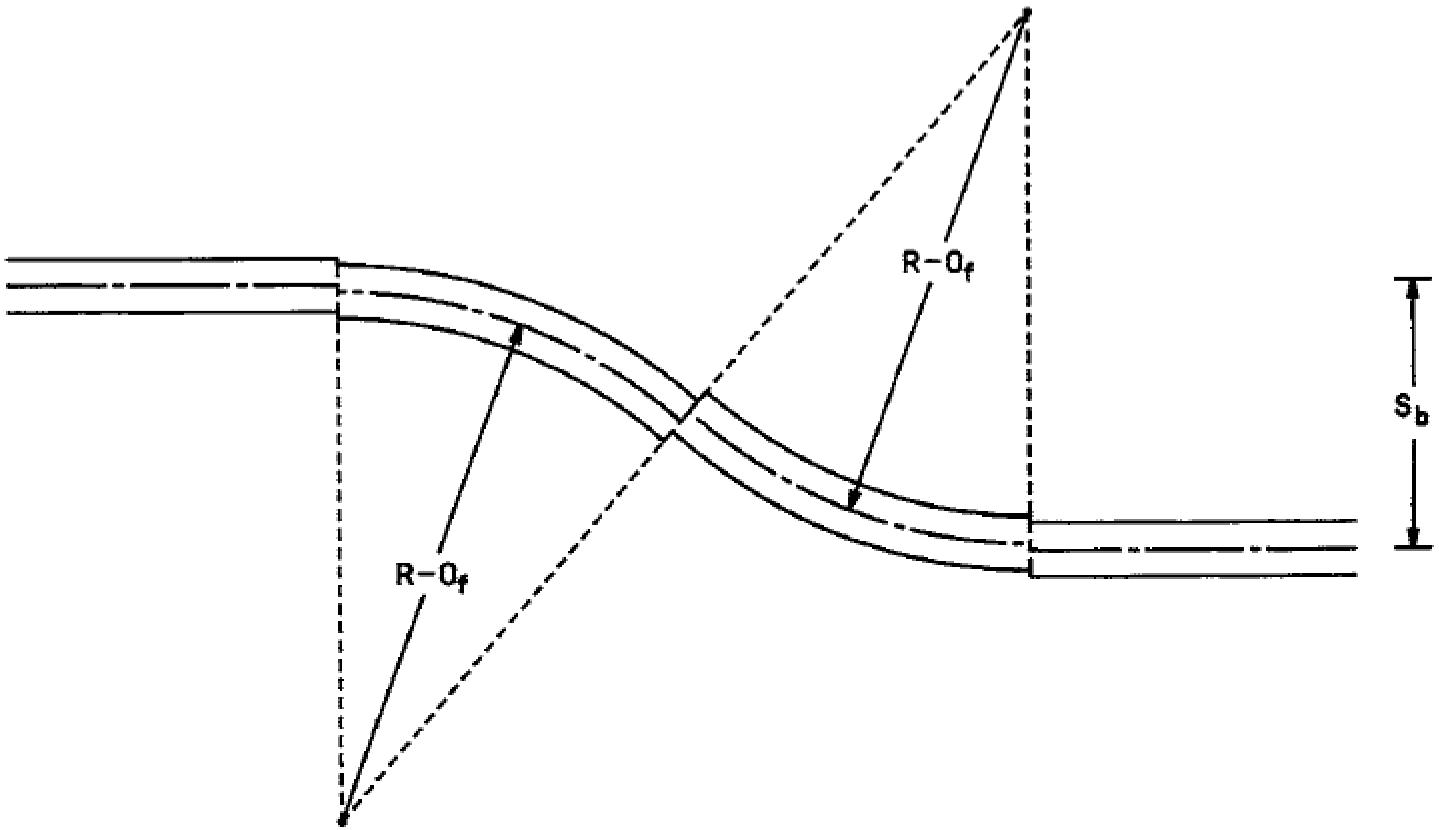


Figure 7.13 S-shaped bent waveguide with a waveguide offset.

Beam Propagation Method (BPM) - Examples

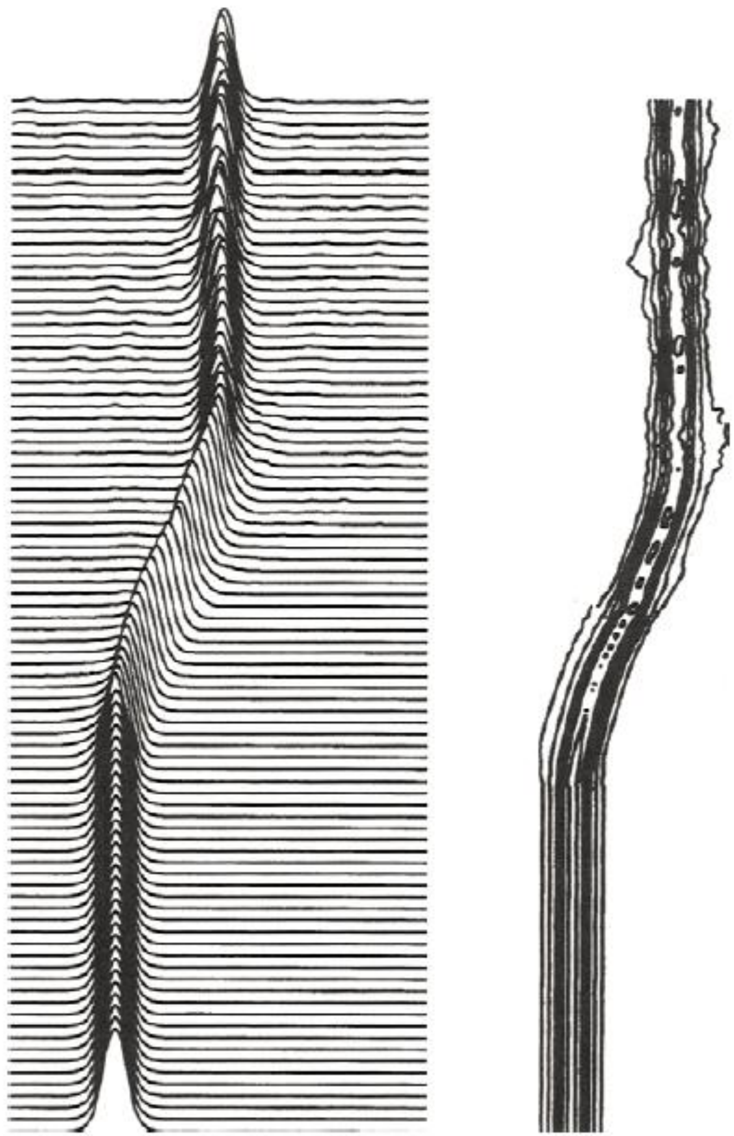


Figure 7.14 BPM simulation of the light propagation in an S-bend waveguide having an offset of $O_f = 1.4\,\mu\text{m}$.

Beam Propagation Method (BPM) - Examples

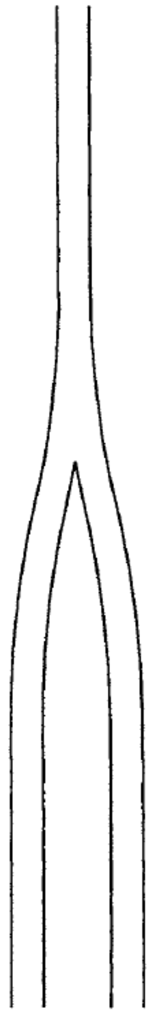


Figure 7.15 Schematic configuration of a Y-combiner consisting of single-mode waveguides.

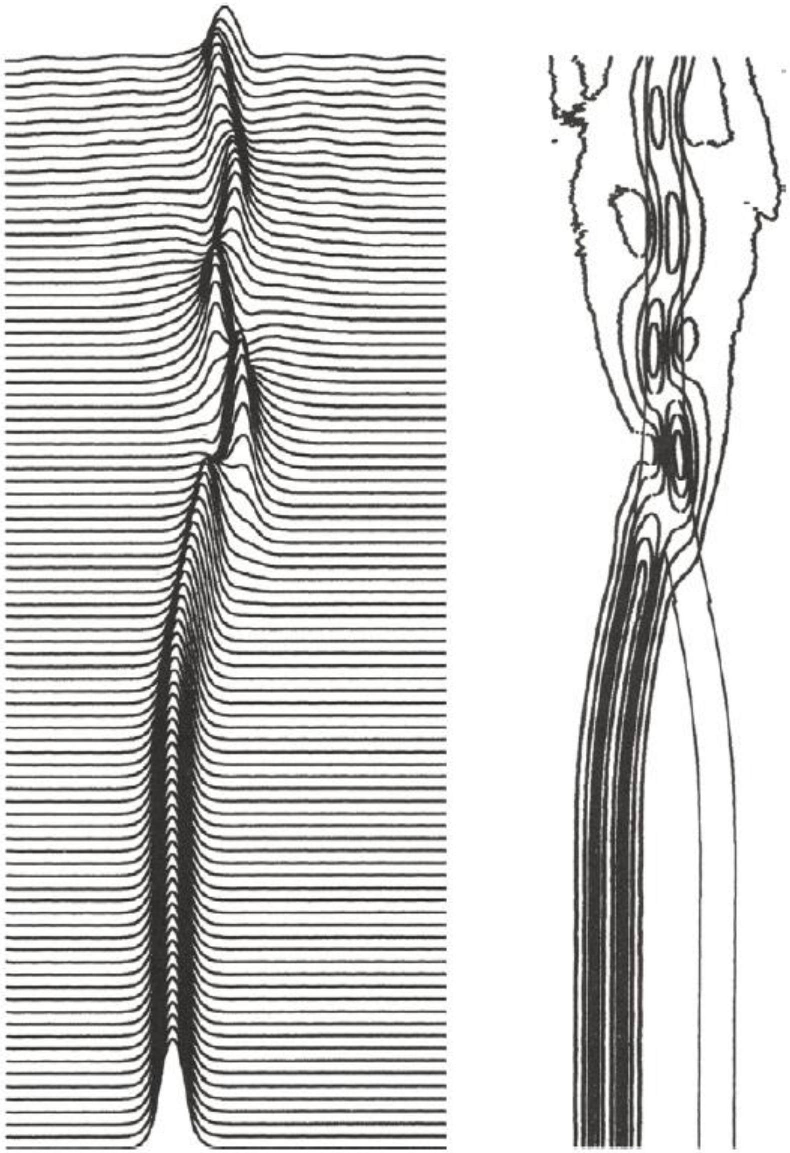


Figure 7.16 BPM analysis of single-mode Y-combiner when when light is coupled into one of the two waveguides.

Beam Propagation Method (BPM) - Examples

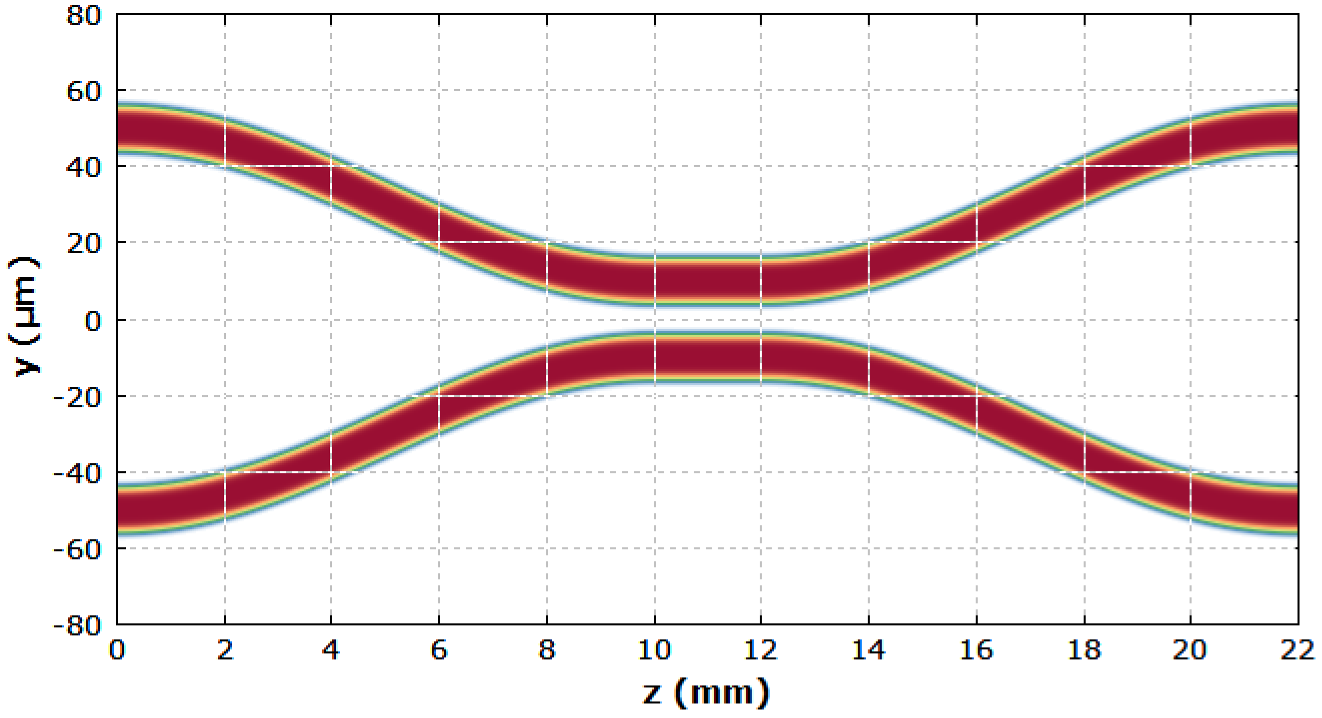


Figure 2: Refractive index profile of a fiber coupler.

https://www.rp-photonics.com/passive_fiber_optics8.html

Beam Propagation Method (BPM) - Examples

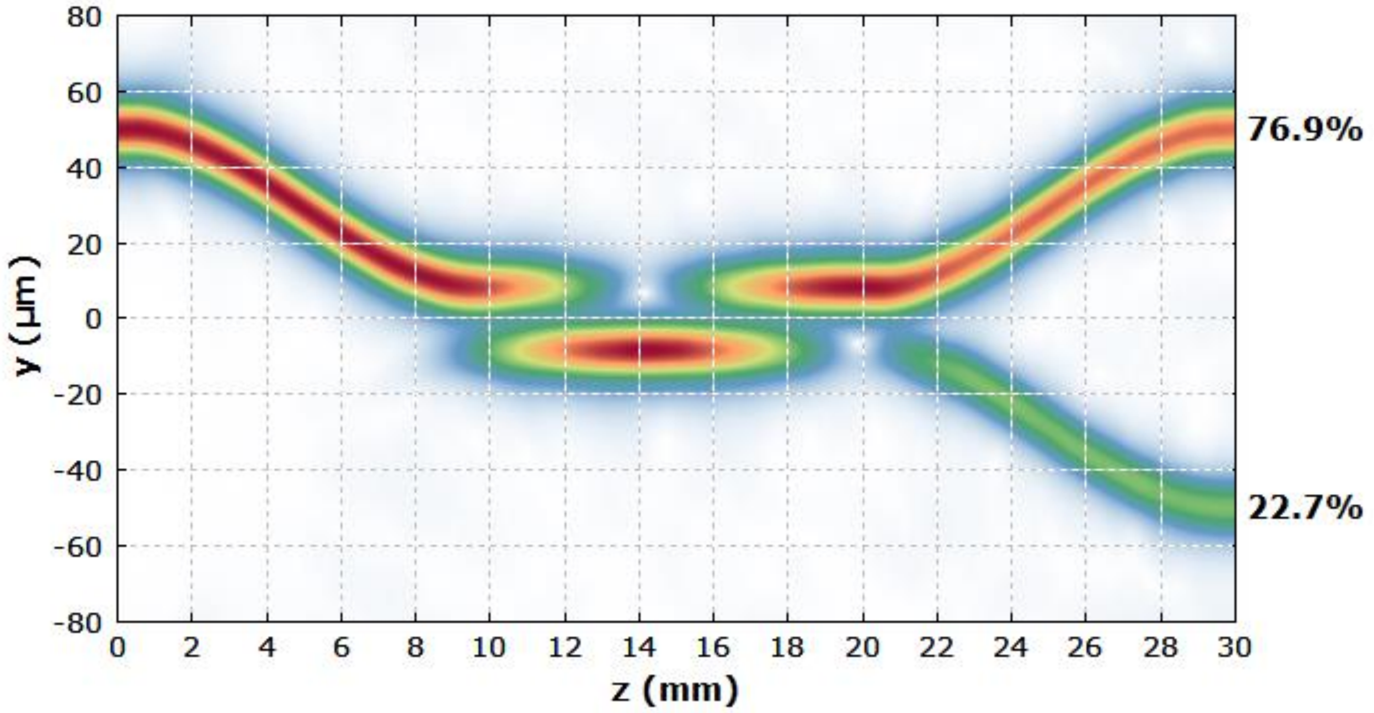


Figure 3: Amplitude distribution in a fiber coupler, obtained with a numerical simulation of beam propagation, done with the software **RP Fiber Power**.

https://www.rp-photonics.com/passive_fiber_optics8.html

Transfer Matrix Method

Sections 3.1-3.5 in Coldren, Corzine and Mašanović

Definition of the Scattering Matrix

Linear Networks **S** allows you to find outputs from inputs

Inputs: a_n

Outputs: b_n

$\mathbf{b} = \mathbf{S}\mathbf{a}$

$$b_i = \sum_j S_{ij} a_j$$

Can measure S_{ij} by setting $a_k = 0$ for $k \neq j$ and measuring b_i

$$\mathcal{E}(x,y,z,t) = \hat{\boldsymbol{e}} E_0 U(x,y) e^{j(\omega t - \tilde{\beta} z)}$$

$$a_j = \frac{E_0}{\sqrt{2\eta_j}} e^{-j\tilde{\beta} z} \quad \text{where} \quad \eta_j = \frac{377\Omega}{\tilde{n}_j}$$

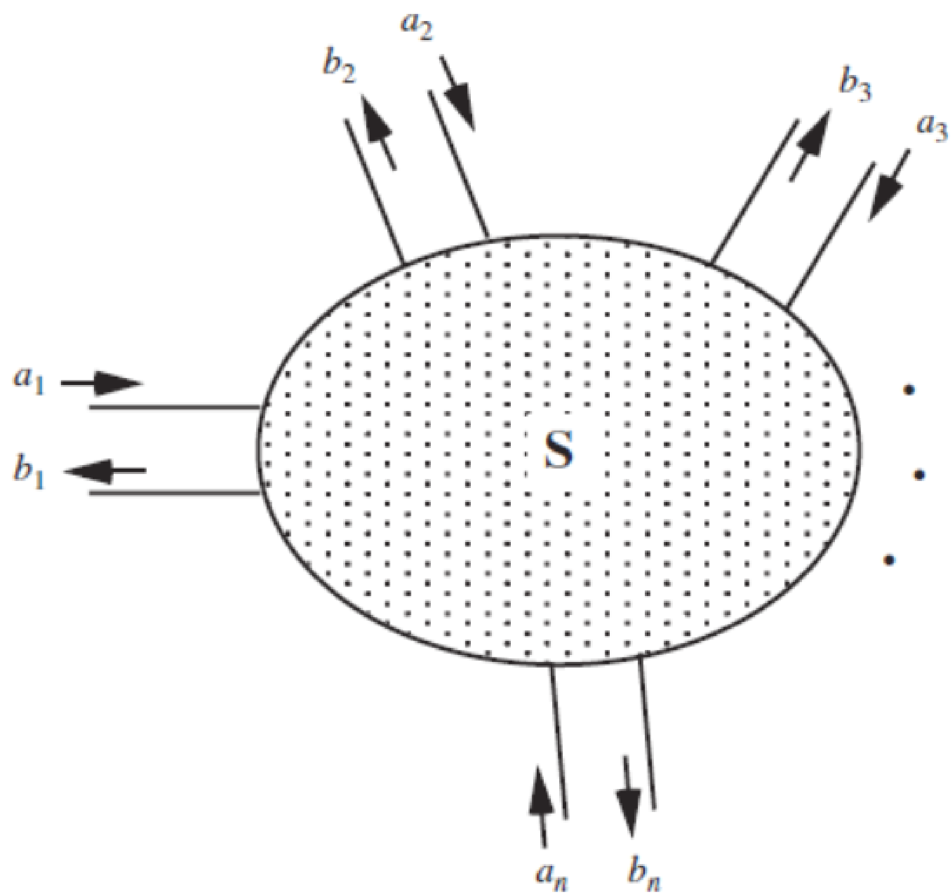
For $\int |U|^2 \, dx\, dy = 1$ we have $a_j a_j^* = P_j^+$

The net power flowing into the port is:

$$P_j = a_j a_j^* - b_j b_j^*$$

Important case (2-port junction):

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

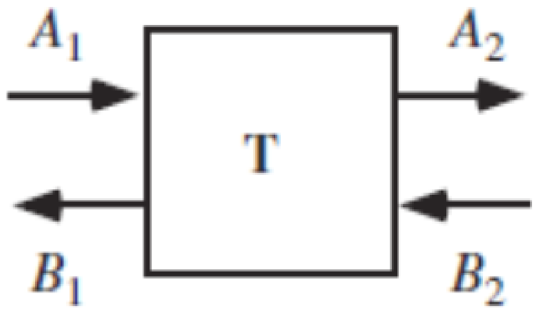


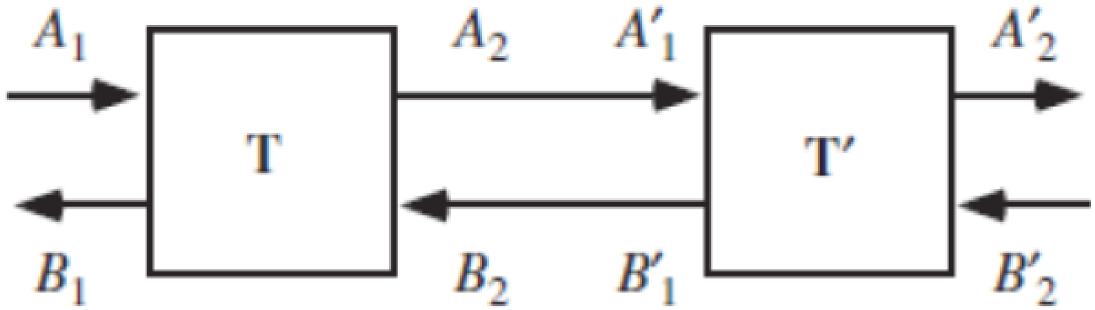
If network is reciprocal, $\mathbf{S}_t = \mathbf{S}$

If network is lossless, $\mathbf{S}$ is unitary: $\mathbf{S}_t^* \mathbf{S} = 1$

Definition of the Transmission Matrix

T allows you to cascade networks





Left side: $A_1 = a_1, B_1 = b_1$

Right side: $A_2 = b_2, B_2 = a_2$

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} T'_{11} & T'_{12} \\ T'_{21} & T'_{22} \end{bmatrix} \begin{bmatrix} A'_2 \\ B'_2 \end{bmatrix}$$

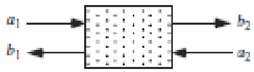
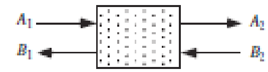
$$\begin{bmatrix} T_{11} = \dfrac{1}{S_{21}} & T_{12} = -\dfrac{S_{22}}{S_{12}} \\ T_{21} = \dfrac{S_{11}}{S_{21}} & T_{22} = -\dfrac{S_{11}S_{22} - S_{12}S_{21}}{S_{21}} \end{bmatrix}$$

If network is reciprocal, scattering matrix is symmetric and $\det \mathbf{T} = 1$

If network is lossless, $\mathbf{S}$ is unitary: $\mathbf{S}_t^* \mathbf{S} = 1$

Relations Between S and T

TABLE 3.1: Relations Between Scattering and Transmission Matrices

Scattering Matrix	Transmission Matrix
Definition	*Definition*
	
$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$	$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$
$b_1 = S_{11}a_1 + S_{12}a_2$	$A_1 = T_{11}A_2 + T_{12}B_2$
$b_2 = S_{21}a_1 + S_{22}a_2$	$B_1 = T_{21}A_2 + T_{22}B_2$
Relation to r and t	*Relation to r and t*
$r_{12} = \frac{b_1}{a_1}\Big\rvert_{a_2=0} = S_{11}$	$r_{12} = \frac{B_1}{A_1}\Big\rvert_{B_2=0} = \frac{T_{21}}{T_{11}}$
$t_{12} = \frac{b_2}{a_1}\Big\rvert_{a_2=0} = S_{21}$	$t_{12} = \frac{A_2}{A_1}\Big\rvert_{B_2=0} = \frac{1}{T_{11}}$
$r_{21} = \frac{b_2}{a_2}\Big\rvert_{a_1=0} = S_{22}$	$r_{21} = \frac{A_2}{B_2}\Big\rvert_{A_1=0} = -\frac{T_{12}}{T_{11}}$
$t_{21} = \frac{b_1}{a_2}\Big\rvert_{a_1=0} = S_{12}$	$t_{21} = \frac{B_1}{B_2}\Big\rvert_{A_1=0} = \frac{\det \mathbf{T}}{T_{11}}$
$\mathbf{S} = \begin{bmatrix} r_{12} & t_{21} \\ t_{12} & r_{21} \end{bmatrix}$	$\mathbf{T} = \frac{1}{t_{12}} \begin{bmatrix} 1 & -r_{21} \\ r_{12} & t_{12}t_{21} - r_{12}r_{21} \end{bmatrix}$
$\det \mathbf{S} = S_{11}S_{22} - S_{12}S_{21} = r_{12}r_{21} - t_{12}t_{21}$	$\det \mathbf{T} = T_{11}T_{22} - T_{12}T_{21} = t_{21}/t_{12}$
Relation to **T***-Matrix*	*Relation to* **S***-Matrix*
$\mathbf{S} = \frac{1}{T_{11}} \begin{bmatrix} T_{21} & \det \mathbf{T} \\ 1 & -T_{12} \end{bmatrix}$	$\mathrm{T} = \frac{1}{S_{21}} \begin{bmatrix} 1 & -S_{22} \\ S_{11} & -\det \mathbf{S} \end{bmatrix}$

Network Properties

TABLE 3.2: Network Properties and Their Consequences on the Matrix Coefficients

Reciprocal Network (valid for normalized fields with and without loss)

$$\mathbf{S}_t = \mathbf{S} \rightarrow \begin{matrix} S_{12} = S_{21} \\ \det \mathbf{T} = 1 \end{matrix}$$

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{21} \\ S_{21} & S_{22} \end{bmatrix} = \frac{1}{T_{11}} \begin{bmatrix} T_{21} & 1 \\ 1 & -T_{12} \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & (T_{12}T_{21} + 1)/T_{11} \end{bmatrix} = \frac{1}{S_{21}} \begin{bmatrix} 1 & -S_{22} \\ S_{11} & S_{21}^2 - S_{11}S_{22} \end{bmatrix}$$

Lossless Reciprocal Network

$$|S_{11}|^2 + |S_{21}|^2 = 1 \qquad |T_{21}|^2 + 1 = |T_{11}|^2$$

$$\mathbf{S}_t^*\mathbf{S} = 1 \rightarrow |S_{12}|^2 + |S_{22}|^2 = 1 \rightarrow 1 + |T_{12}|^2 = |T_{11}|^2$$

$$S_{11}^*S_{12} + S_{21}^*S_{22} = 0 \qquad T_{21}^* - T_{12} = 0$$

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{21} \\ S_{21} & -S_{11}^*(S_{21}/S_{21}^*) \end{bmatrix} = \frac{1}{T_{11}} \begin{bmatrix} T_{21} & 1 \\ 1 & -T_{21}^* \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{21}^* \\ T_{21} & T_{11}^* \end{bmatrix} = \begin{bmatrix} 1/S_{21} & S_{11}^*/S_{21}^* \\ S_{11}/S_{21} & 1/S_{21}^* \end{bmatrix}$$

Lossless Reciprocal Network with r and t Phase Shifts of 0 or π

$$S_{22} = -S_{11}$$

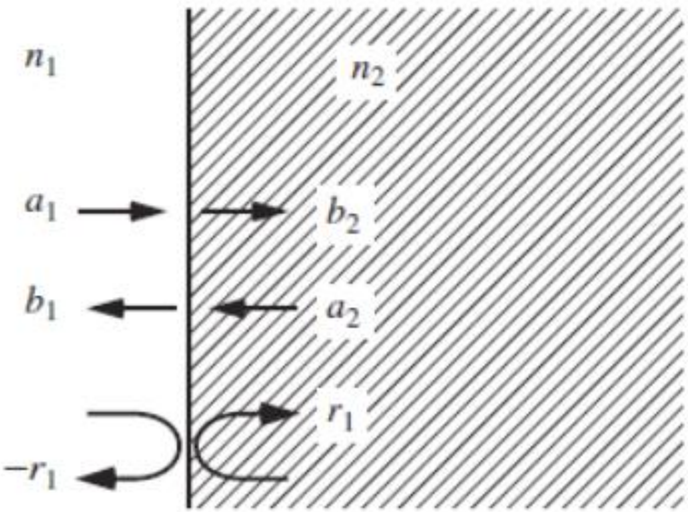
$$\begin{matrix} S_{11} = S_{11}^* \\ S_{21} = S_{21}^* \end{matrix} \rightarrow \det \mathbf{S} = -1$$

$$T_{22} = T_{11}, \quad T_{12} = T_{21}$$

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{21} \\ S_{21} & -S_{11} \end{bmatrix} = \frac{1}{T_{11}} \begin{bmatrix} T_{21} & 1 \\ 1 & -T_{21} \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{21} \\ T_{21} & T_{11} \end{bmatrix} = \frac{1}{S_{21}} \begin{bmatrix} 1 & S_{11} \\ S_{11} & 1 \end{bmatrix}$$

Dielectric Interface



$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} = -r_1 = \frac{n_1 - n_2}{n_1 + n_2}$$

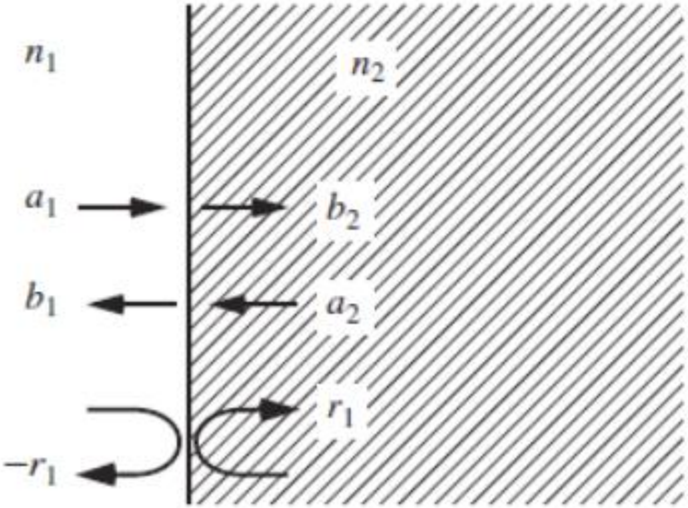
Note r_1 is a positive real number if $n_2 > n_1$.

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} = r_2 = -(-r_1) = r_1$$

using EM wave impedances $\Rightarrow$ $$\frac{\eta_L - \eta_0}{\eta_L + \eta_0} = \frac{\sqrt{\frac{\mu_0}{\varepsilon_2}} - \sqrt{\frac{\mu_0}{\varepsilon_1}}}{\sqrt{\frac{\mu_0}{\varepsilon_1}} + \sqrt{\frac{\mu_0}{\varepsilon_2}}} = \frac{\frac{1}{\sqrt{\varepsilon_2}} - \frac{1}{\sqrt{\varepsilon_1}}}{\frac{1}{\sqrt{\varepsilon_2}} + \frac{1}{\sqrt{\varepsilon_1}}}$$

$$= \frac{\frac{1}{n_2} - \frac{1}{n_1}}{\frac{1}{n_2} + \frac{1}{n_1}} = \frac{\frac{n_1 - n_2}{n_1 n_2}}{\frac{n_1 + n_2}{n_1 n_2}} = \frac{n_1 - n_2}{n_1 + n_2}$$

Dielectric Interface



$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} = -r_1 = \frac{n_1 - n_2}{n_1 + n_2}$$

Note r_1 is a positive real number if $n_2 > n_1$.

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} = r_2 = -(-r_1) = r_1$$

Note r_2 is a negative real number if $n_2 > n_1$ (π-phase shift).

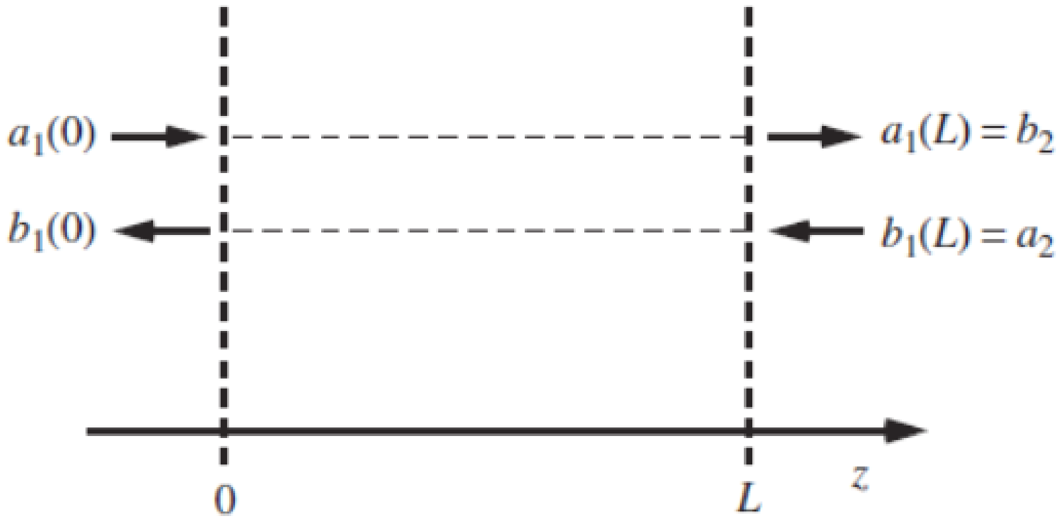
$$S_{12} = S_{21} = t = \sqrt{1 - r_1^2}$$

$$\mathbf{S} = \begin{bmatrix} -r_1 & t \\ t & r_1 \end{bmatrix} \Rightarrow$$

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{21} \\ T_{21} & T_{11} \end{bmatrix} = \frac{1}{S_{21}} \begin{bmatrix} 1 & S_{11} \\ S_{11} & 1 \end{bmatrix}$$

$$\mathbf{T} = \frac{1}{t} \begin{bmatrix} 1 & -r_1 \\ -r_1 & 1 \end{bmatrix}$$

Transmission Line



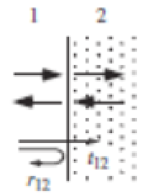
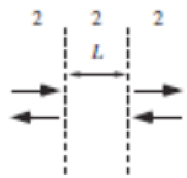
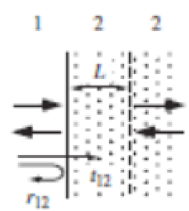
$$b_2 = a_1 e^{-j\tilde{\beta}L}; \quad a_2 = b_1 e^{+j\tilde{\beta}L}$$

$$\tilde{\beta} = \beta + j\beta_i$$

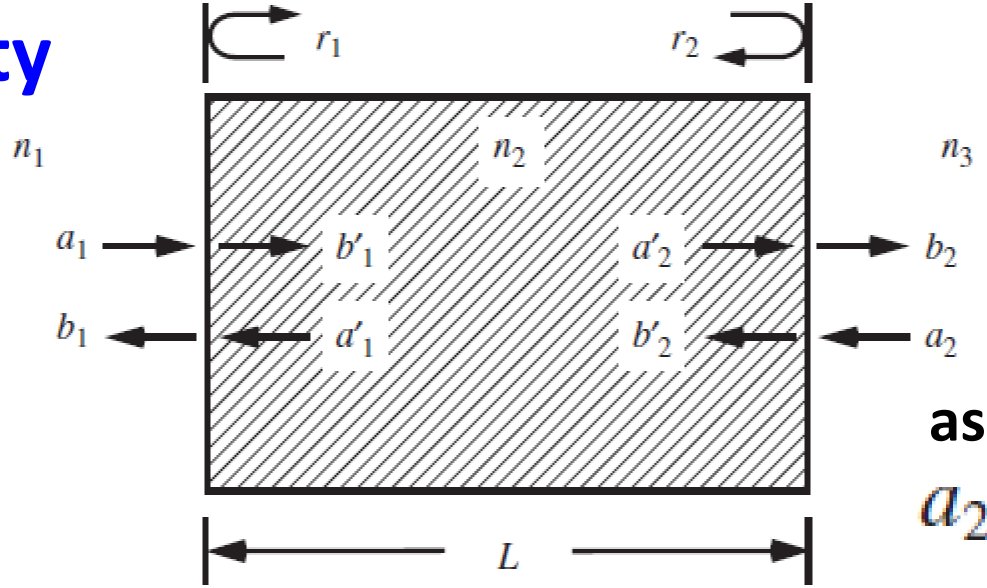
$$\mathbf{S} = \begin{bmatrix} 0 & e^{-j\tilde{\beta}L} \\ e^{-j\tilde{\beta}L} & 0 \end{bmatrix} \qquad \mathbf{T} = \begin{bmatrix} e^{j\tilde{\beta}L} & 0 \\ 0 & e^{-j\tilde{\beta}L} \end{bmatrix}$$

Summary of Building Blocks for S and T

TABLE 3.3: Summary of S- and T-matrices for Simple "Building-Block" Components

Scattering Matrix	Structure	Transmission Matrix
$\begin{bmatrix} r_{12} & t_{12} \\ t_{12} & -r_{12} \end{bmatrix}$		$\frac{1}{t_{12}}\begin{bmatrix} 1 & r_{12} \\ r_{12} & 1 \end{bmatrix}$
	r_{21}=-r_{12} *and* t_{21}=t_{12}	$r_{12}^2 + t_{12}^2 = 1$
$\begin{bmatrix} 0 & e^{-j\phi} \\ e^{-j\phi} & 0 \end{bmatrix}$		$\begin{bmatrix} e^{j\phi} & 0 \\ 0 & e^{-j\phi} \end{bmatrix}$
	$\phi = \tilde{\beta}_2 L$	
$\begin{bmatrix} r_{12} & t_{12}e^{-j\phi} \\ t_{12}e^{-j\phi} & -r_{12}e^{-j2\phi} \end{bmatrix}$		$\frac{1}{t_{12}}\begin{bmatrix} e^{j\phi} & r_{12}e^{-j\phi} \\ r_{12}e^{j\phi} & e^{-j\phi} \end{bmatrix}$
		$r_{12}^2 + t_{12}^2 = 1$

Fabry-Perot Cavity



assume

$$a_2 = 0$$

We can write these relations

$$b_1 = -a_1 r_1 + a_1' t_1,$$

$$b_1' = a_1 t_1 + a_1' r_1,$$

$$b_2 = a_2' t_2,$$

$$b_2' = a_2' r_2.$$

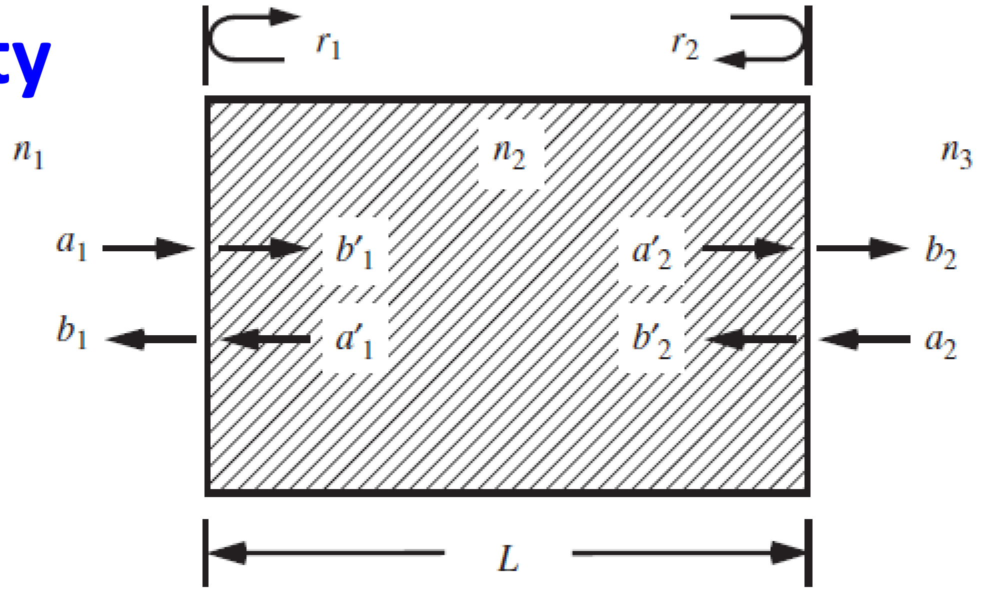
$$a_1' = b_2' e^{-j\tilde{\beta} L},$$

$$a_2' = b_1' e^{-j\tilde{\beta} L}.$$

Solve for

$$S_{11} = b_1/a_1 \qquad S_{21} = b_2/a_1$$

Fabry-Perot Cavity



After a few manipulations:

$$a_2 = 0$$

$$S_{11} = -r_1 + \frac{t_1^2 r_2 e^{-2j\tilde{\beta}L}}{1 - r_1 r_2 e^{-2j\tilde{\beta}L}},$$

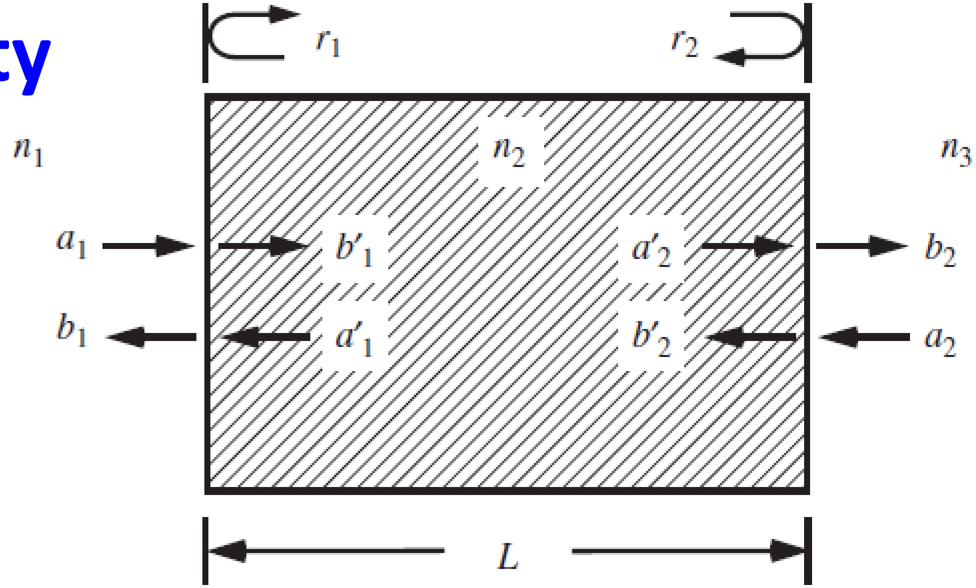
$$S_{21} = \frac{t_1 t_2 e^{-j\tilde{\beta}L}}{1 - r_1 r_2 e^{-2j\tilde{\beta}L}}.$$

$$a_1 = 0$$

$$S_{22} = -r_2 + \frac{t_2^2 r_1 e^{-2j\tilde{\beta}L}}{1 - r_1 r_2 e^{-2j\tilde{\beta}L}},$$

$$S_{12} = S_{21}.$$

Fabry-Perot Cavity



The Transmission Matrix for the cavity can be obtained from the Scattering Matrix or by multiplication of elementary Transmission Matrices for the interfaces and the transmission line

$$a_2 = 0$$

$$a_1 = 0$$

$$T_{11} = \frac{1}{t_1 t_2}[e^{j\tilde{\beta}L} - r_1 r_2 e^{-j\tilde{\beta}L}],$$

$$T_{21} = -\frac{1}{t_1 t_2}[r_1 e^{j\tilde{\beta}L} - r_2 e^{-j\tilde{\beta}L}].$$

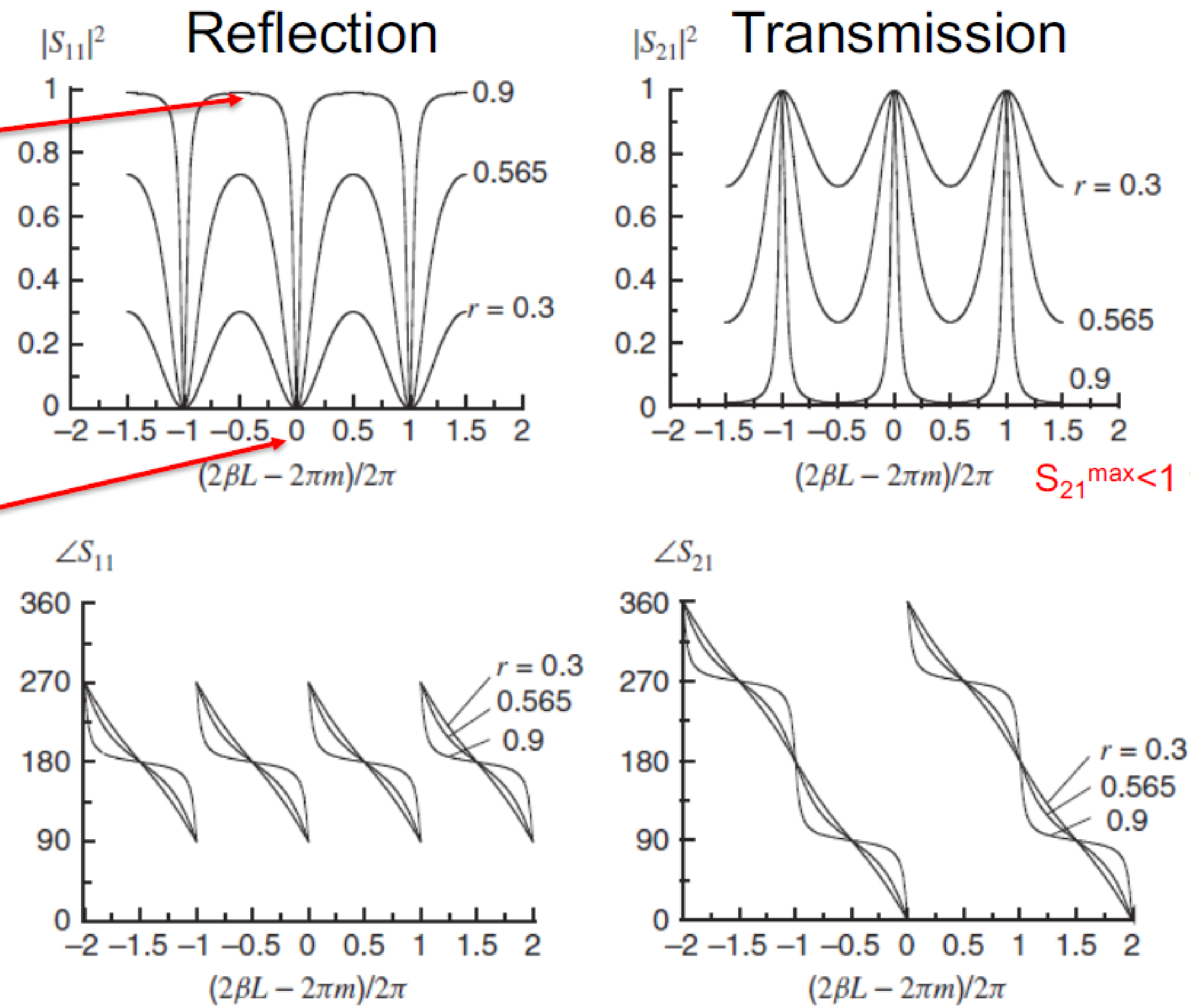
$$T_{12} = -\frac{1}{t_1 t_2}[r_1 e^{-j\tilde{\beta}L} - r_2 e^{j\tilde{\beta}L}].$$

$$T_{22} = \frac{1}{t_1 t_2}[e^{-j\tilde{\beta}L} - r_1 r_2 e^{j\tilde{\beta}L}],$$

Lossless Fabry-Perot Cavity Spectra for S

Etalon is highly reflective if L=odd quarter-integer multiple of λ/n_2

Etalon is 100% transmissive If L = half-integer multiple of λ/n_2

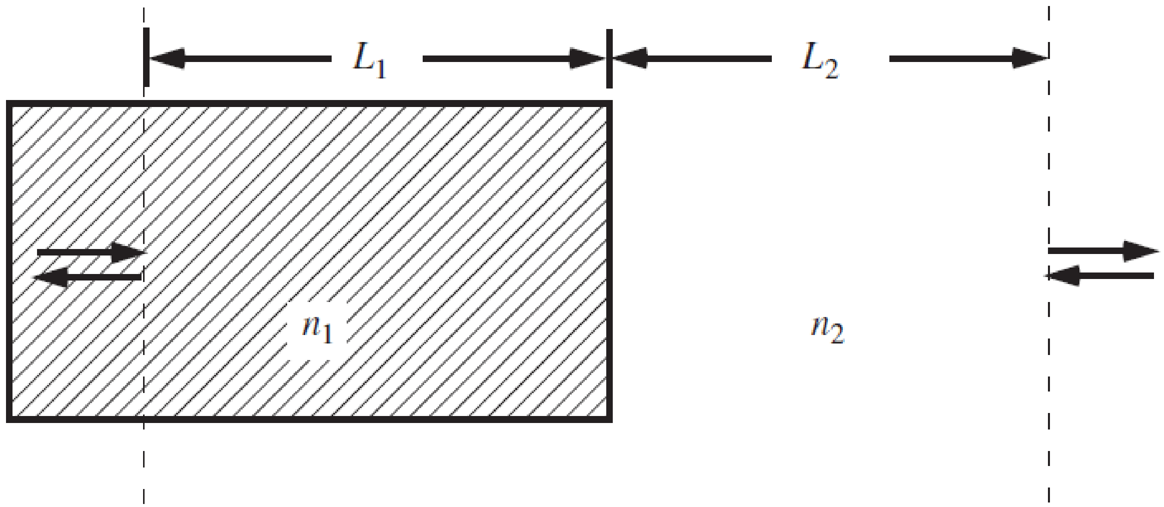


S_{21}^{max}<1 with loss

Cleaved cavity r=0.565

Transmission Line – Interface – Transmission Line

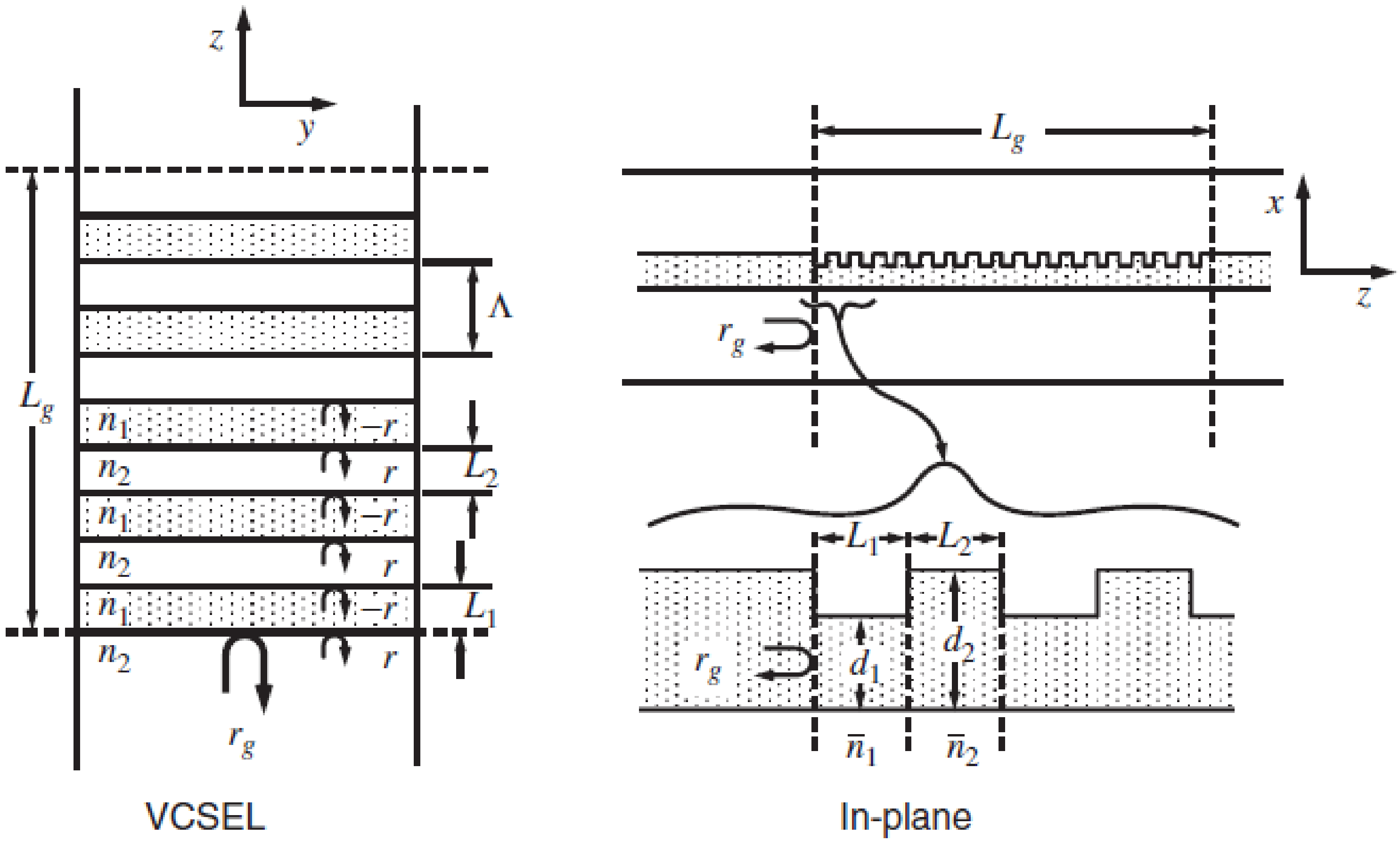
(Example 3.1 in Coldren, Corzine and Mašanović)



$$\mathbf{T} = \mathbf{T_1} \cdot \mathbf{T_2} \cdot \mathbf{T_3} = \begin{bmatrix} e^{j\phi_1} & 0 \\ 0 & e^{-j\phi_1} \end{bmatrix} \cdot \frac{1}{t_{12}} \begin{bmatrix} 1 & -r_{12} \\ -r_{12} & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{j\phi_2} & 0 \\ 0 & e^{-j\phi_2} \end{bmatrix}$$

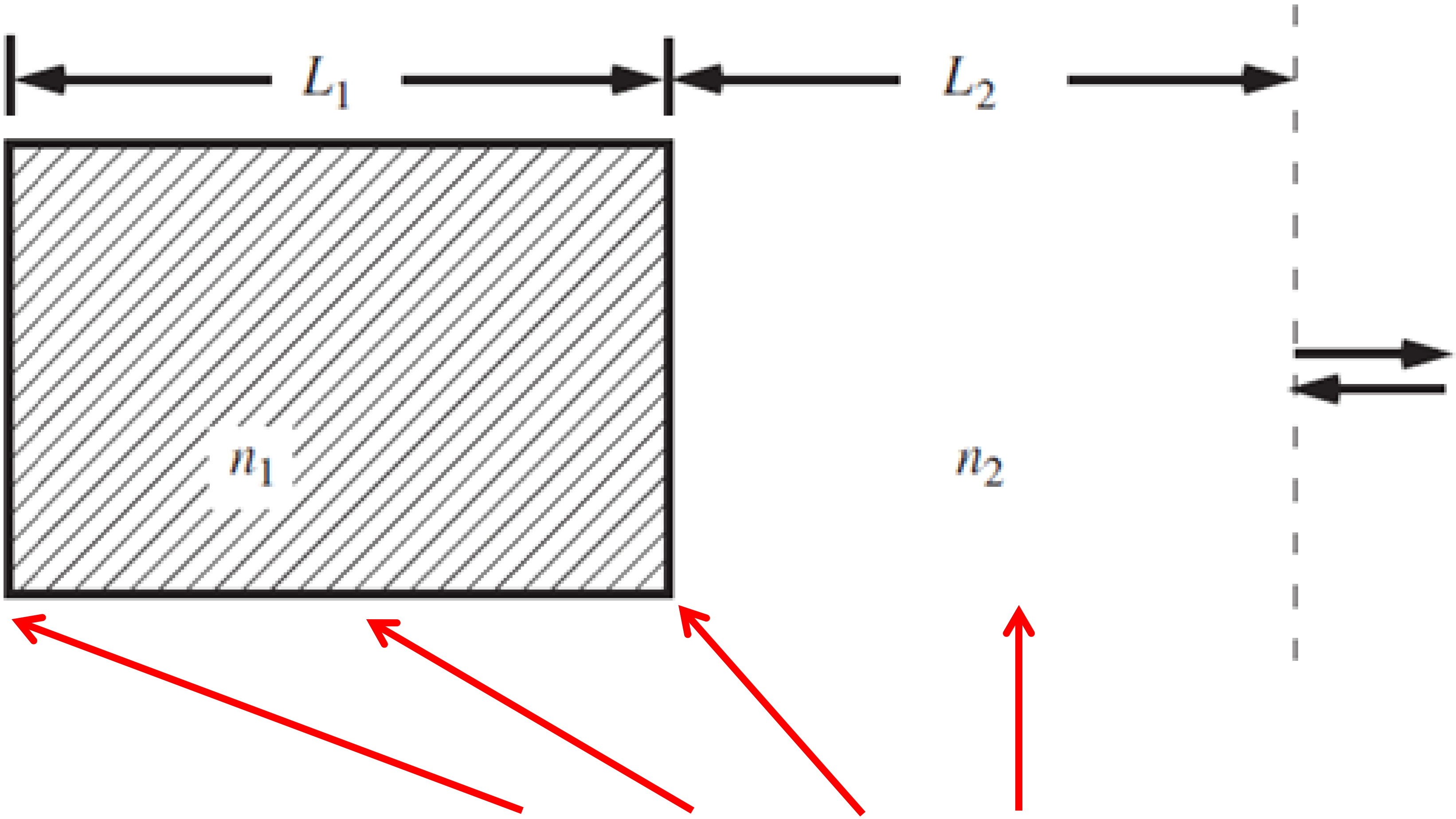
$$= \frac{1}{t_{12}} \begin{bmatrix} e^{j(\phi_1+\phi_2)} & -r_{12}e^{j(\phi_1-\phi_2)} \\ -r_{12}e^{j(\phi_2-\phi_1)} & e^{-j(\phi_1+\phi_2)} \end{bmatrix}.$$

Application to Gratings (Distributed Bragg Reflectors)



- **At the Bragg frequency, the period of the grating is half of the average optical wavelength in the medium.**
- **For each period, multiply 4 simple T-Matrices together**

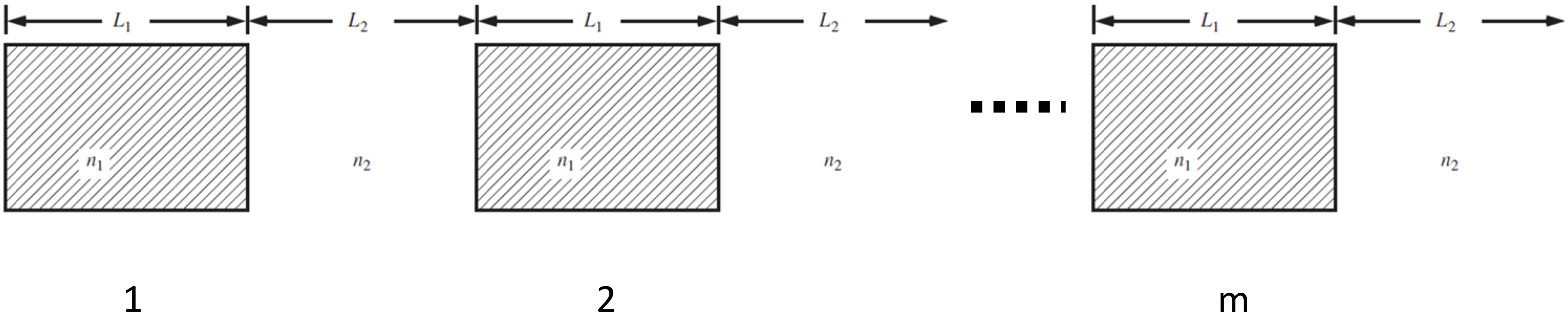
Period of a uniform grating structure



$$T_i = T_1 \cdot T_2 \cdot T_3 \cdot T_4$$

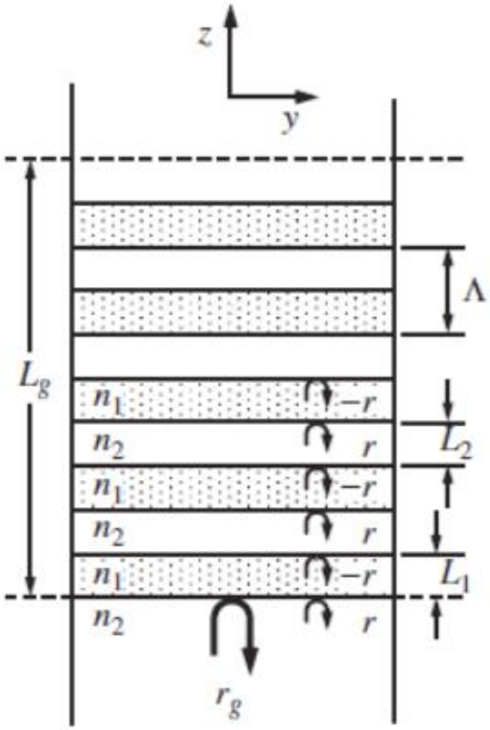
Add one matrix to the previous case to account for the additional interface

T-matrix for the complete grating structure

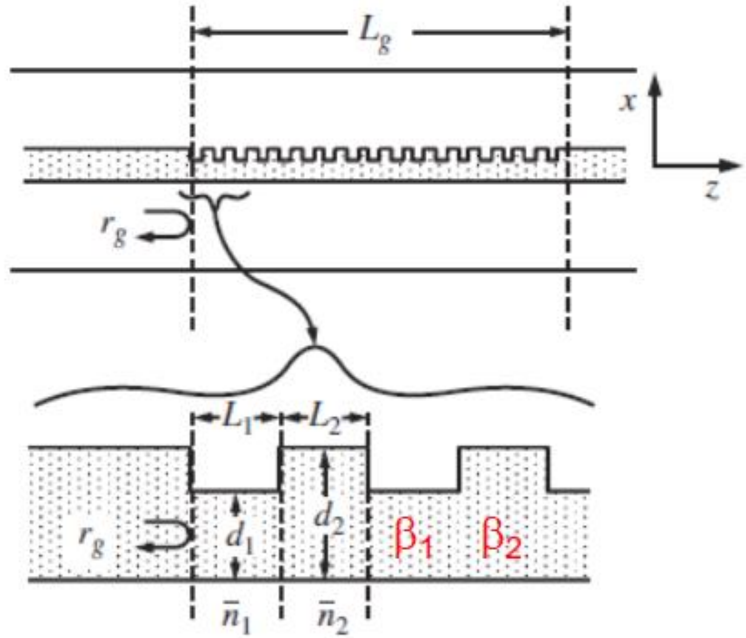


$$T_g = [T_i]^m = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}^m$$

Application to Gratings (Distributed Bragg Reflectors)



VCSEL



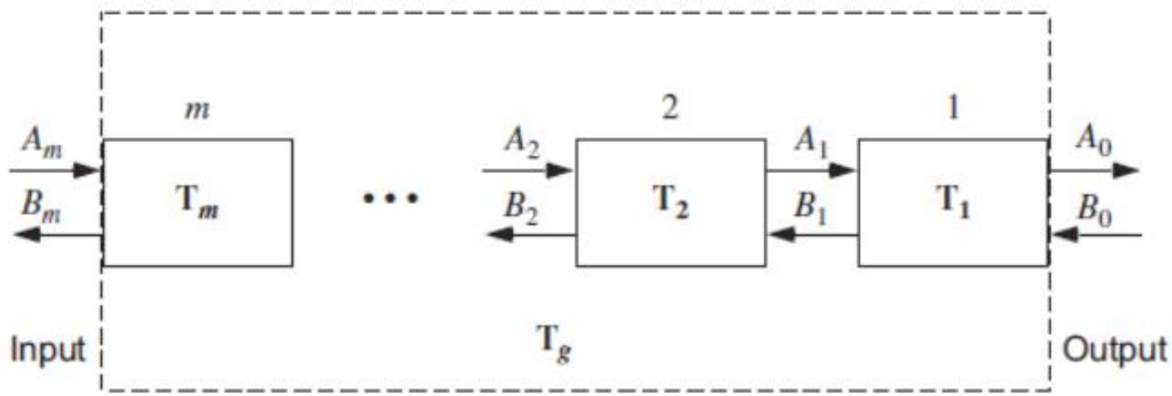
In-plane

one period
$L_1 + L_2$

$$L_1 = \frac{\lambda_{design}}{4n_1}$$

$$L_2 = \frac{\lambda_{design}}{4n_2}$$

$$T_g = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}^m$$



Application to Gratings (Distributed Bragg Reflectors)

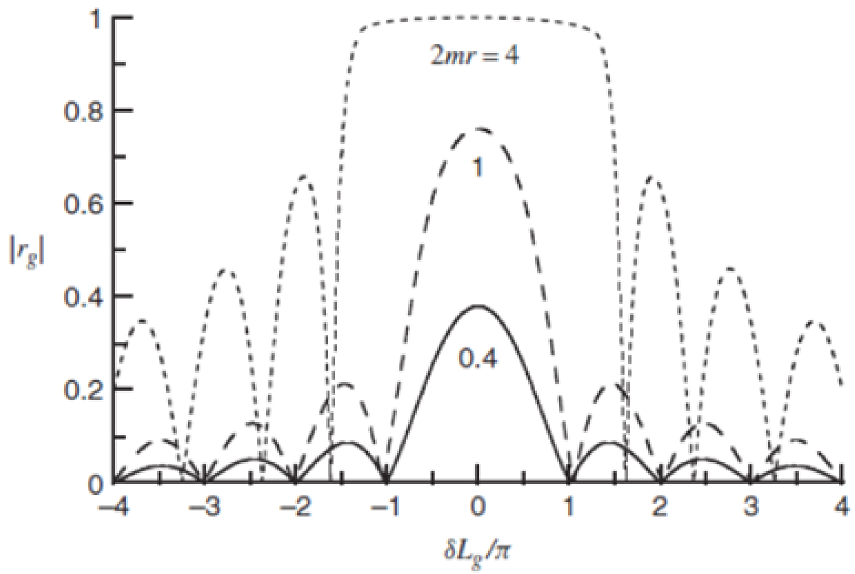
At the Bragg condition, elements for the T-matrix of one period are

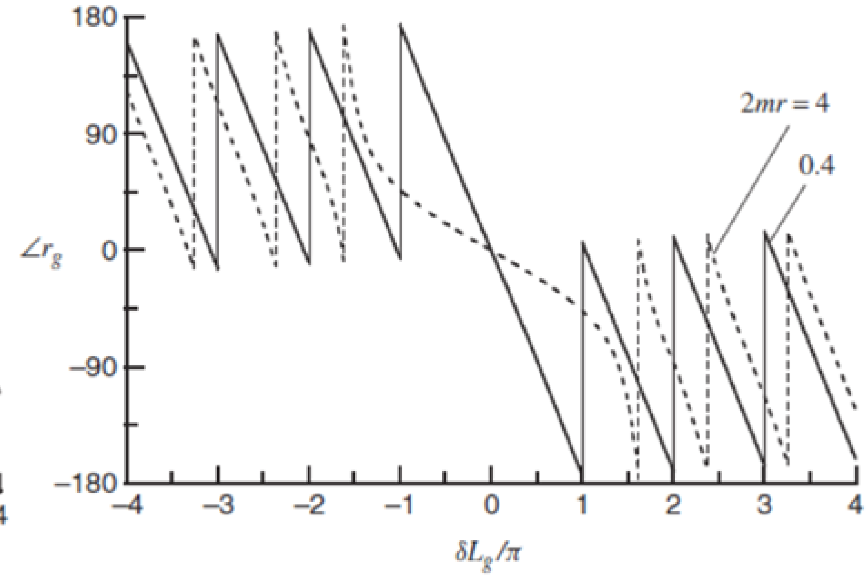
$$
\begin{aligned}
T_{11} &= \frac{1}{t^2}[e^{j\phi+} - r^2 e^{-j\phi-}] \rightarrow -\frac{1+r^2}{t^2}, \\
T_{21} &= \frac{r}{t^2}[e^{j\phi+} - e^{-j\phi-}] \rightarrow -\frac{2r}{t^2}, \\
T_{12} &= \frac{r}{t^2}[e^{-j\phi+} - e^{j\phi-}] \rightarrow -\frac{2r}{t^2}, \\
T_{22} &= \frac{1}{t^2}[e^{-j\phi+} - r^2 e^{j\phi-}] \rightarrow -\frac{1+r^2}{t^2},
\end{aligned}
$$

$\phi_\pm \equiv \tilde{\beta}_1 L_1 \pm \tilde{\beta}_2 L_2$ becomes either π or 0 at the Bragg condition.

Reflected Amplitude/Phase for Gratings

Example with $m = 20$ and $r = 0.1, 0.025, 0.01$





$\delta \equiv \beta - \beta_0$ = detuning parameter

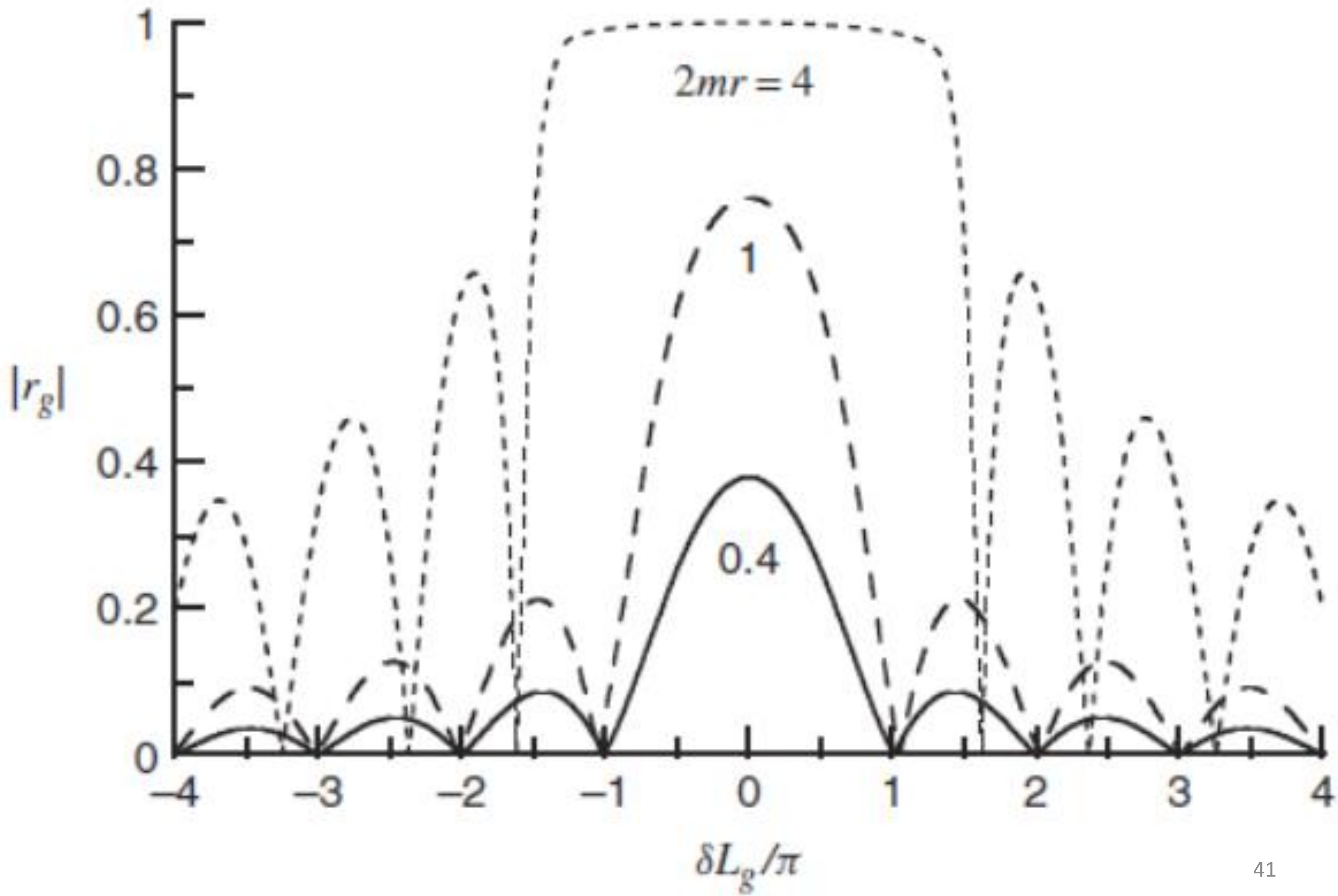
β is the average propagation constant of the grating

L_g is the grating length

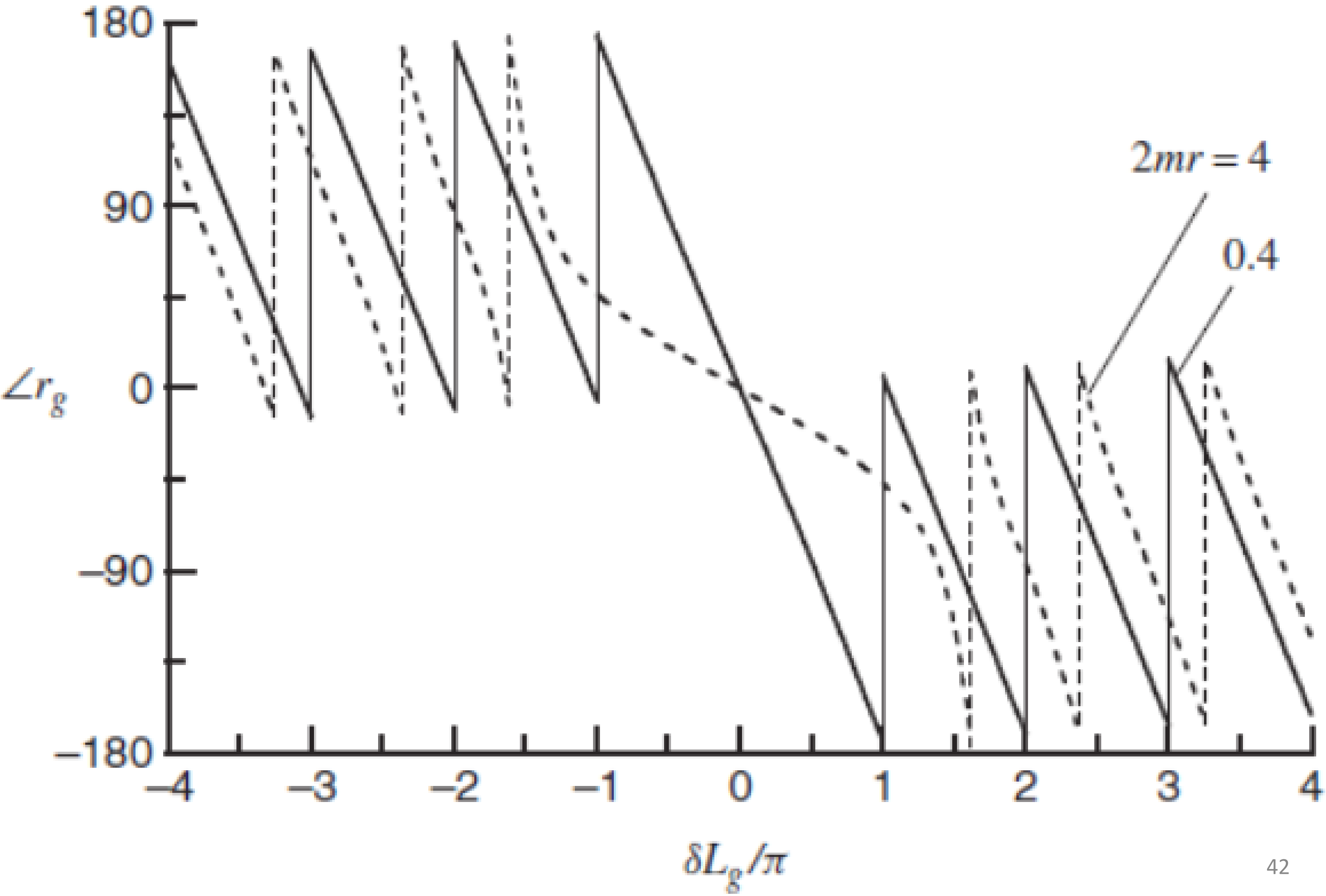
$$\beta = \frac{\beta_1/n_1 + \beta_2/n_2}{1/n_1 + 1/n_2}$$

the phase delay of each layer is $\beta_1 L_1 = \beta_2 L_2 = \pi/2$.

Reflected Amplitude for Gratings



Reflected Phase for Gratings

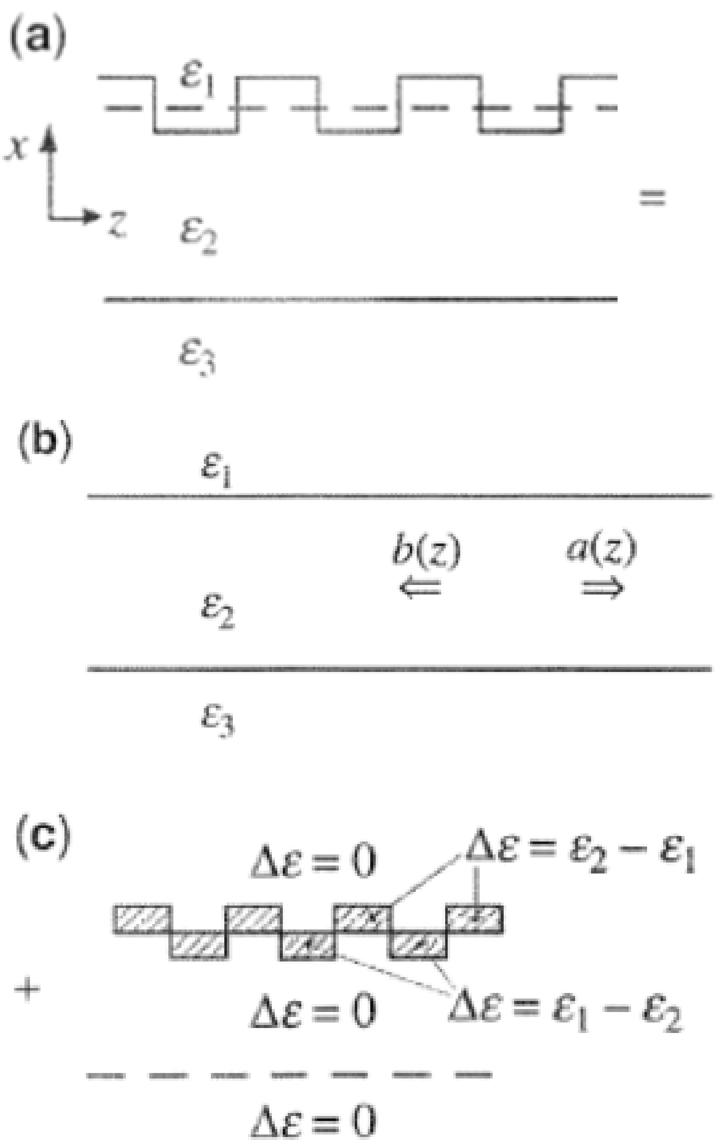


Reading Assignments:

- Section 10.5 of Chuang's book

- Sections 3.1-3.5 in Coldren, Corzine and Mašanović
- Section 8.2.5 in Coldren, Corzine and Mašanović

Next time: Gratings Structure with Coupled Modes Theory



A grating structure can be considered as a perturbation on a uniform slab waveguide:

$$\varepsilon(x,z) = \varepsilon^{(0)}(x) + \Delta\varepsilon(x,z)$$

For the unperturbed slab waveguide

TE_0 mode

Forward: $A_0 e^{i\beta_0 z}$ Backward: $B_0 e^{-i\beta_0 z}$

$\varepsilon(x,z)$ is periodic in z

Fourier transform:

$$\Delta\varepsilon(x,z) = \varepsilon_0 \sum_{p=-\infty}^{\infty} \Delta\varepsilon_p(x)\, e^{ip\frac{2\pi}{\Lambda}z}$$

For a lossless structure:

$$\Delta\varepsilon_p^*(x) = \Delta\varepsilon_{-p}(x)$$