ECE 536 – Integrated Optics and Optoelectronics
Lecture 22 – April 7, 2022

Spring 2022

Tu-Th 11:00am-12:20pm

Prof. Umberto Ravaioli

ECE Department, University of Illinois

Lecture 22 Outline

- Finish Coupled Mode Theory
- Distributed Feedback (DFB) Laser
- Distributed Bragg Reflector Lasers
  – In plane edge emitting DBR Laser
  – Vertical Cavity Surface Emitting (VCSEL) Laser

Distributed Feedback Structures

Coupled-mode theory

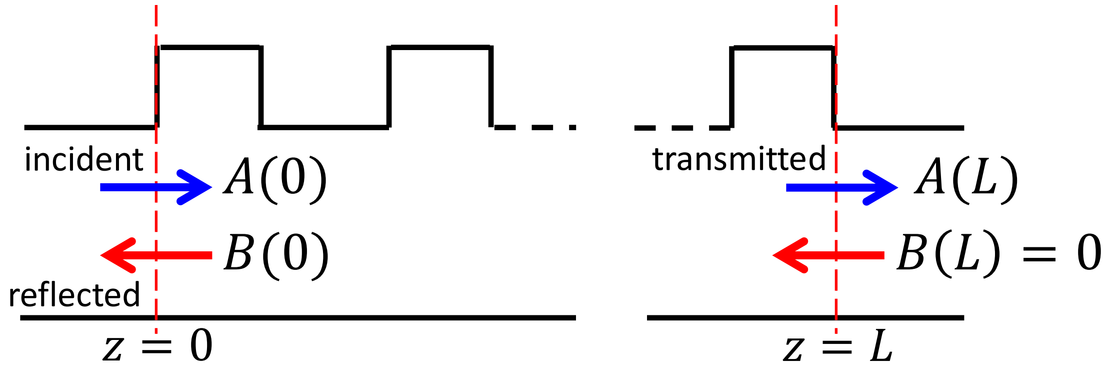
Solution for index grating DFB structure

$$\frac{dA(z)}{dz} = i\Delta\beta A(z) + iK_{ab}B(z)$$
$$\frac{dB(z)}{dz} = iK_{ba}A(z) - i\Delta\beta B(z)$$

or equivalently in matrix form:

Coupled Mode Equations:

$$\frac{d}{dz}\begin{bmatrix} A(z) \\ B(z) \end{bmatrix} = i\begin{bmatrix} \Delta\beta & K_{ab} \\ K_{ba} & -\Delta\beta \end{bmatrix}\begin{bmatrix} A(z) \\ B(z) \end{bmatrix}$$



Solution steps

Coupled Mode Equations:

$$\frac{d}{dz}\begin{bmatrix} A(z) \\ B(z) \end{bmatrix} = i\begin{bmatrix} \Delta\beta & K_{ab} \\ K_{ba} & -\Delta\beta \end{bmatrix}\begin{bmatrix} A(z) \\ B(z) \end{bmatrix}$$

Assume lossless gratings ($K \equiv K_{ab}$ and $K_{ba} = -K^*$) and eigensolution with form

$$\begin{bmatrix} A(z) \\ B(z) \end{bmatrix} = \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} e^{iqz}$$

The corresponding eigenequation is

$$\begin{bmatrix} \Delta\beta - q & K \\ -K^* & -\Delta\beta - q \end{bmatrix}\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = 0$$

with eigenvalues

$$q_{\pm} = \pm\sqrt{(\Delta\beta)^2 - |K|^2} = \pm iS$$

$$S = \sqrt{|K|^2 - (\Delta\beta)^2}$$

Eigenvalues of the matrix

$$\begin{bmatrix} \Delta\beta - q & K \\ -K^* & -\Delta\beta - q \end{bmatrix}$$

Characteristic equation

$$(\Delta\beta - q)(-\Delta\beta - q) - (-KK^*) = 0$$

$$(\Delta\beta)^2 + q^2 - \Delta\beta q + \Delta\beta q + |K|^2 = 0$$

$$q^2 = (\Delta\beta)^2 - |K|^2 = 0$$

with eigenvalues $q_\pm = \pm\sqrt{(\Delta\beta)^2 - |K|^2} = \pm iS$

$$S = \sqrt{|K|^2 - (\Delta\beta)^2}$$

Solution steps

We keep only two counterpropagating modes for forward and backward TE_0 propagation with wave vectors

$$\beta_{\pm} = \frac{\pi}{\Lambda} \pm \sqrt{(\Delta\beta)^2 - |K|^2}$$

For $$|\Delta\beta| = \left|\beta(\omega) - \frac{\pi}{\Lambda}\right| < K$$

eigenvalues $$\beta_{\pm} = \frac{\pi}{\Lambda} \pm i\sqrt{|K|^2 - (\Delta\beta)^2} = \frac{\pi}{\Lambda} \pm iS$$

where $S = \sqrt{|K|^2 - (\Delta\beta)^2}$ **defines a circle of radius** $|K|$

In momentum space this circle defines a "stopband" for solutions. This is analogous to the forbidden energy gap in the band structure of semiconductors (the grating behaves like a 1D crystal).

Remember the mode dispersion curves for symmetric slab WG

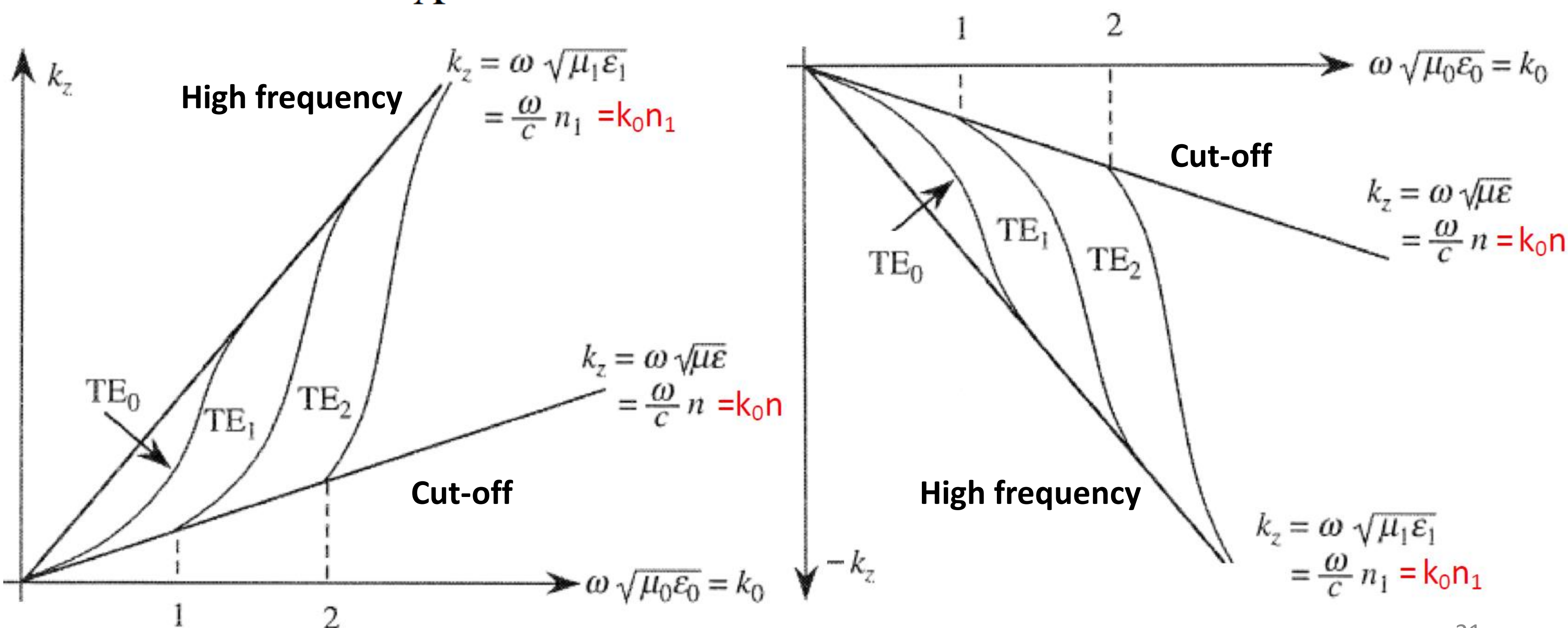
Notation switch in Chuang's book $e^{ik_z z} \longrightarrow e^{i\beta z}$

effective index $n_{eff} = \dfrac{k_z}{k_o}$

$$\beta_a = \beta_0(\omega) = \frac{\omega n_{eff}}{c}$$ forward TE_0 mode

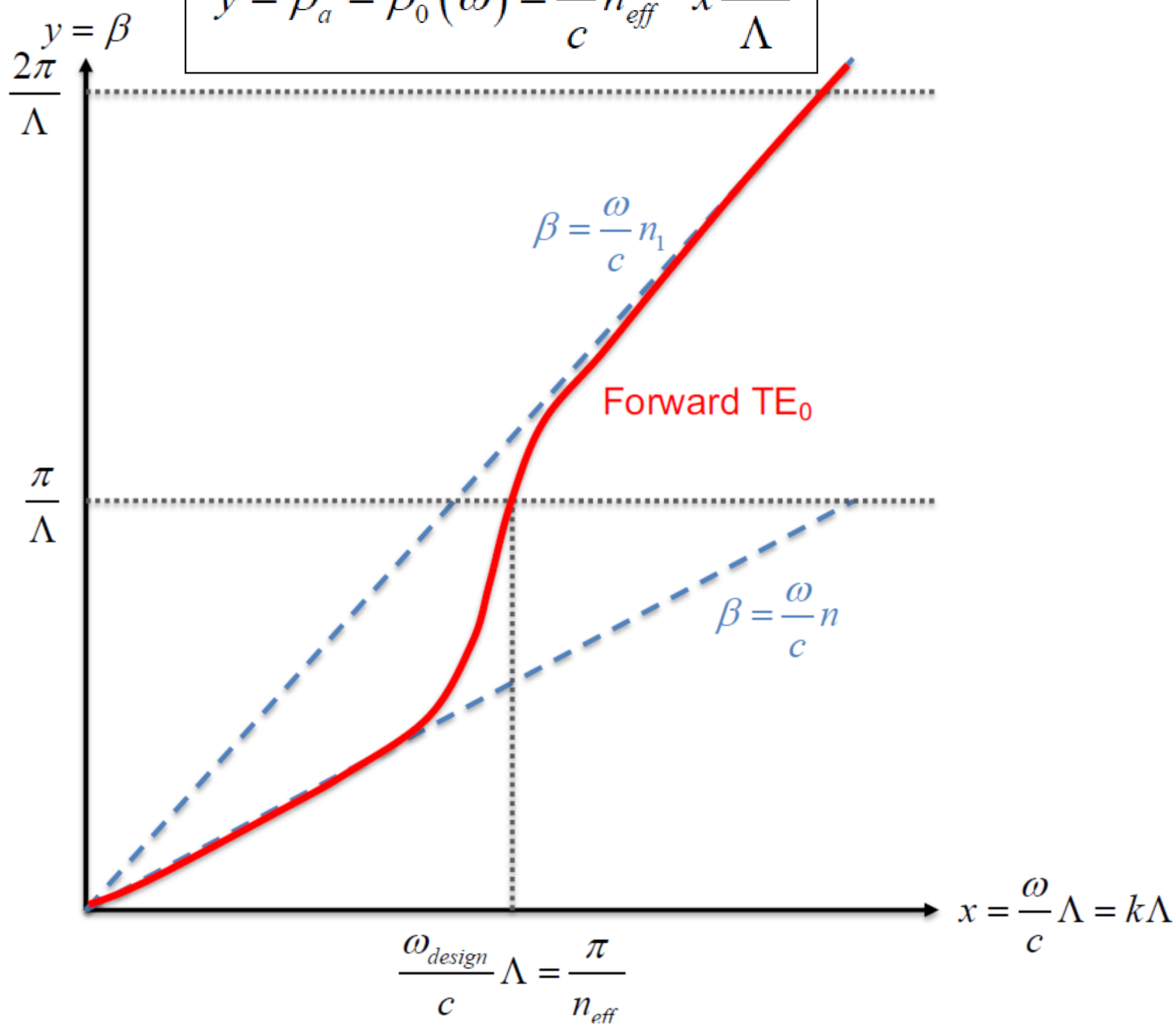
$$\beta_b = -\beta_0(\omega) + \frac{2\pi}{\Lambda}$$ backward TE_0 mode

No Coupling (unperturbed)



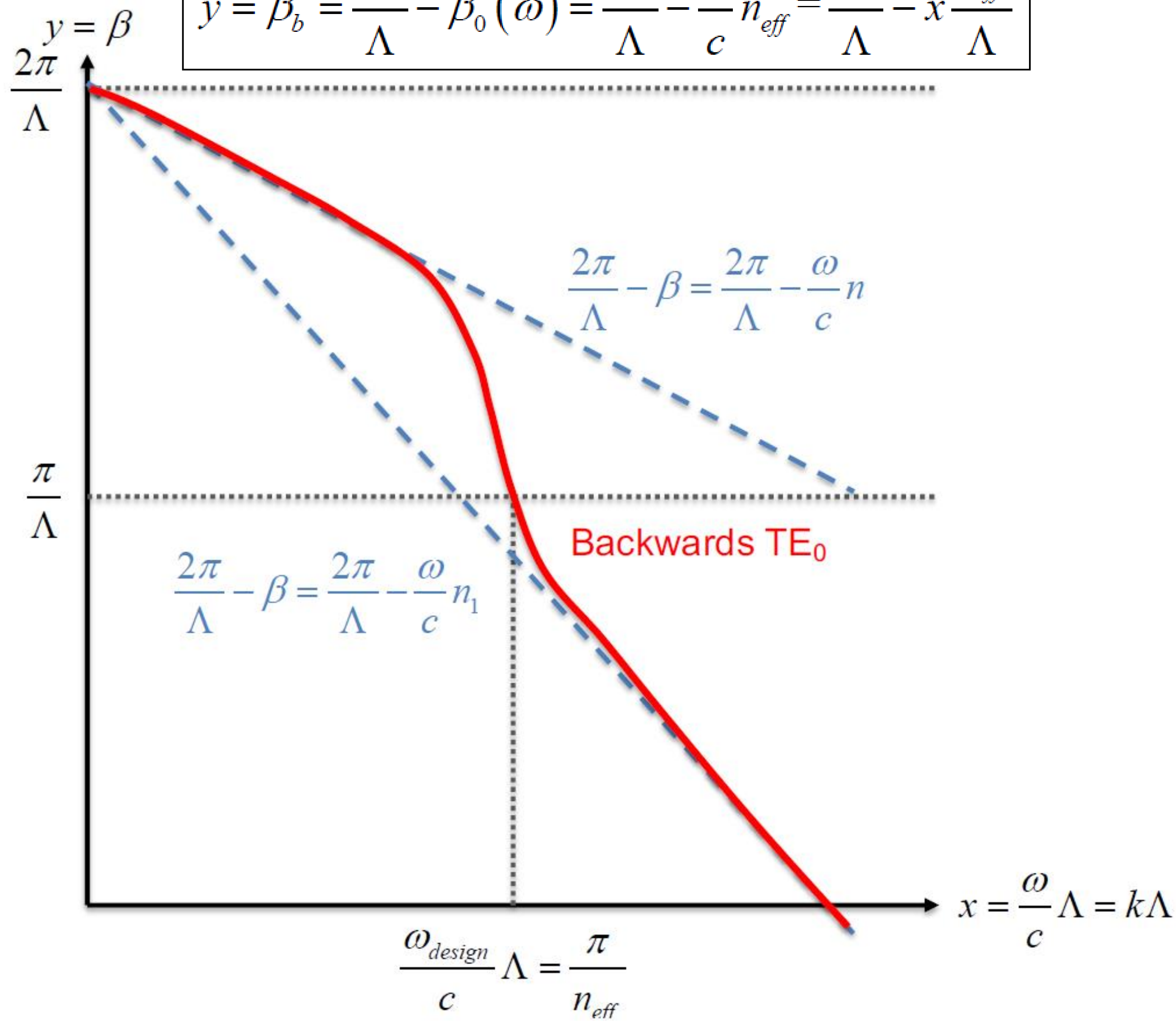
In the reference system for the grating

$$y = \beta_a = \beta_0(\omega) = \frac{\omega}{c} n_{eff} = x \frac{n_{eff}}{\Lambda}$$

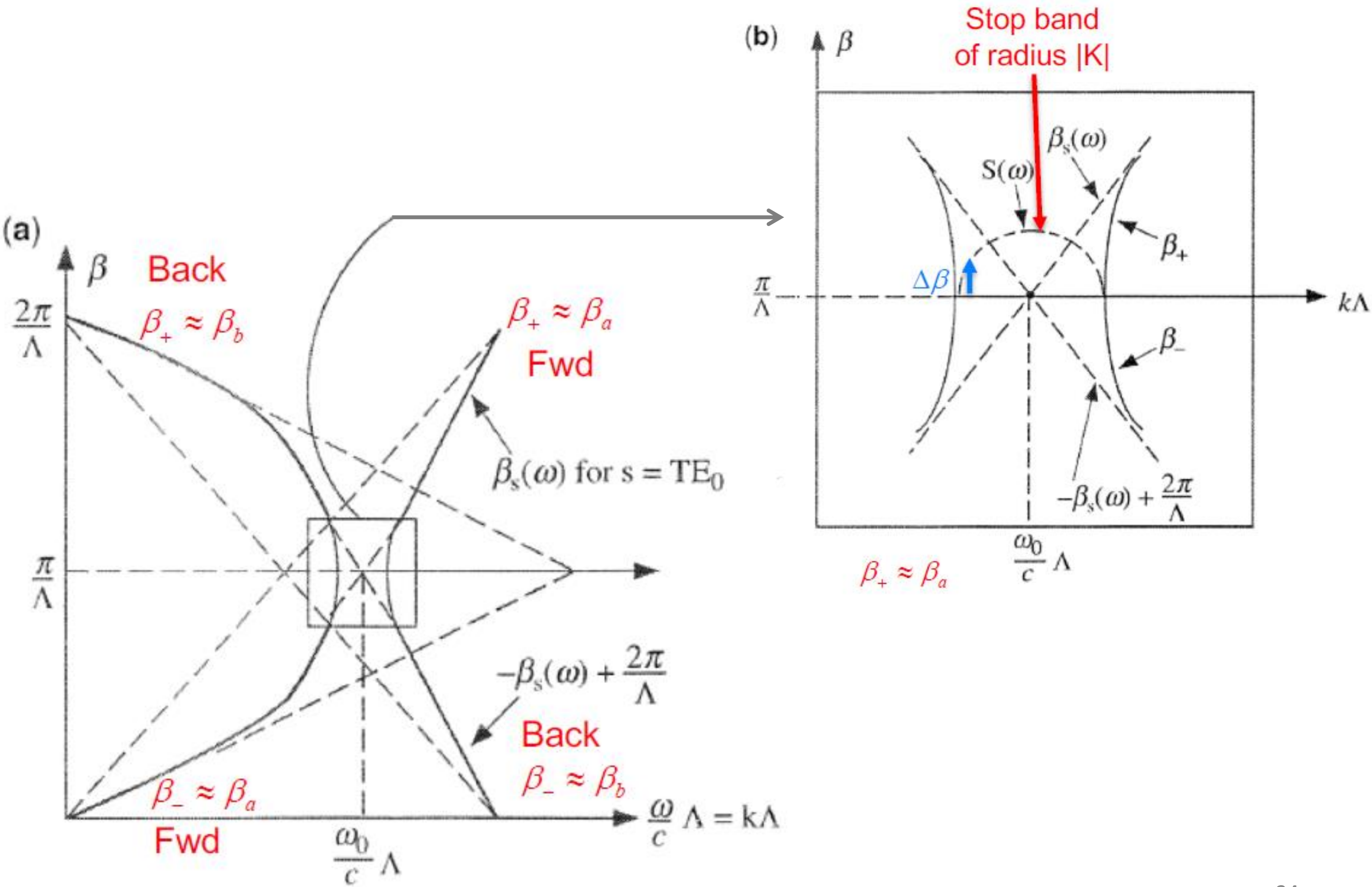


In the reference system for the grating

$$y = \beta_b = \frac{2\pi}{\Lambda} - \beta_0(\omega) = \frac{2\pi}{\Lambda} - \frac{\omega}{c} n_{eff} = \frac{2\pi}{\Lambda} - x\frac{n_{eff}}{\Lambda}$$



With coupling there is splitting where the unperturbed curves cross



$$\left.\begin{array}{l}\delta \to \Delta\beta \\ \beta_0 \to \beta_B\end{array}\right\}$$ **in Chuang**

from Chapter 6 in Coldren, Corzine and Mašanović

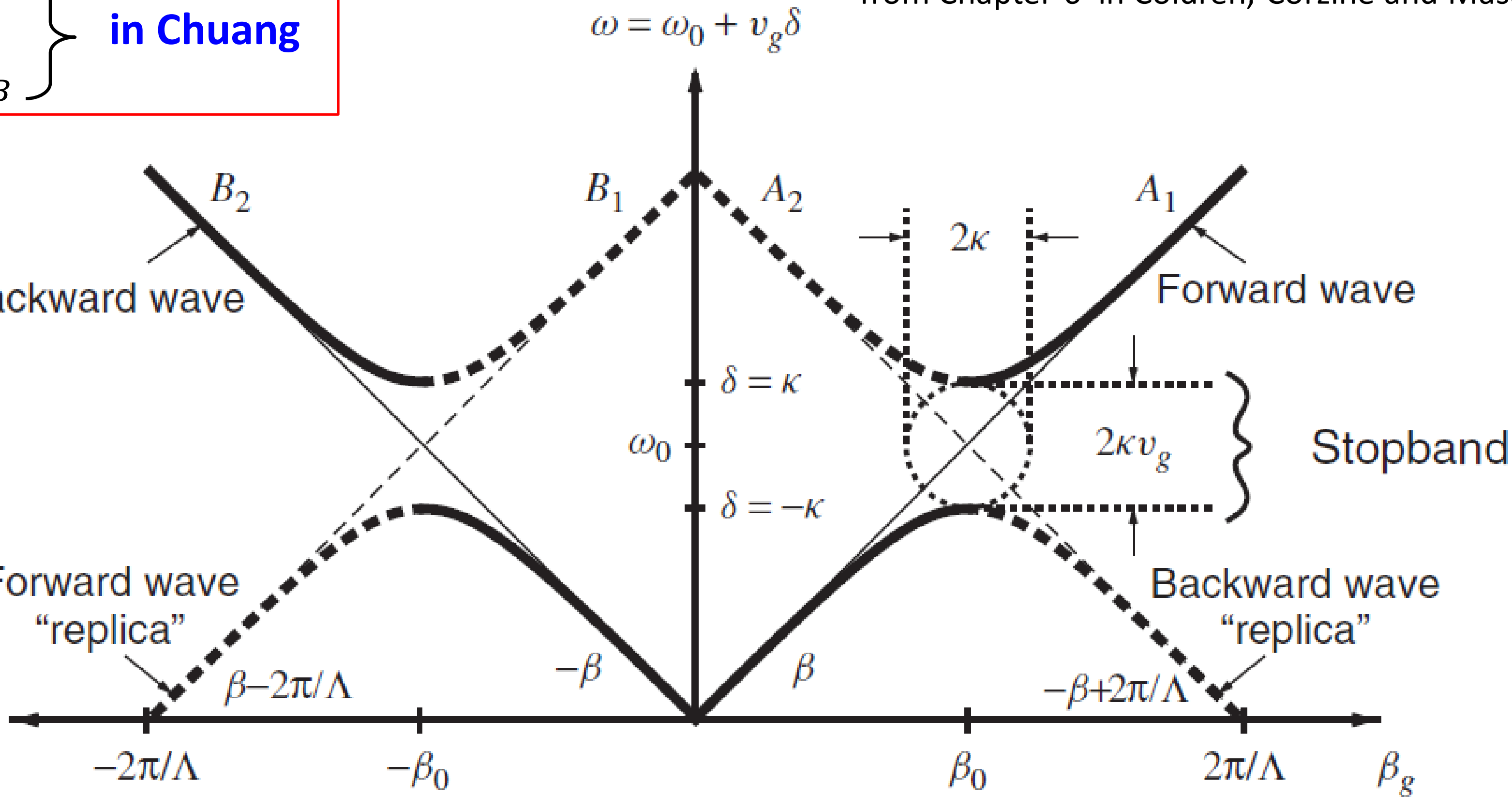
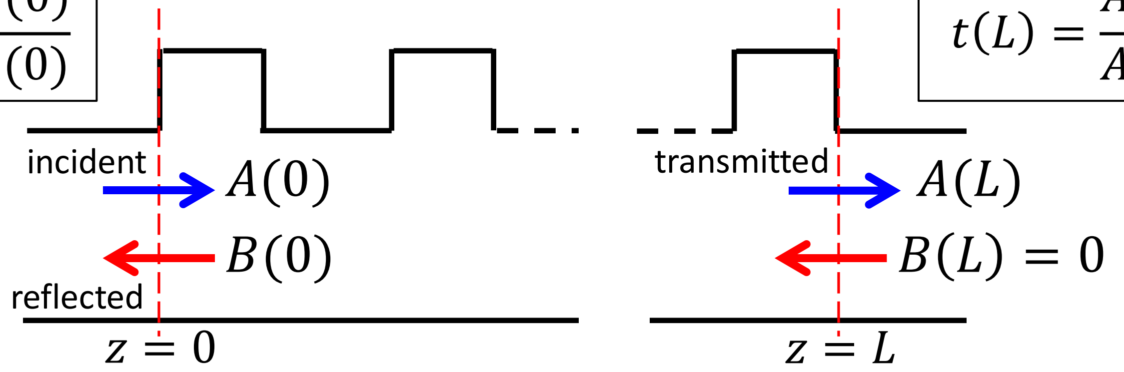


FIGURE 6.4: An $\omega - \beta$ diagram for the coupled β_g-solutions to contradirectional coupling in a grating, with the uncoupled solutions denoted by thinner straight lines. Each of the grating-generated replica solutions (dashed) and ordinary forward and backward wave solutions (solid) correspond to one of the A or B coefficients indicated. The extent of the stopband in both ω and β directions is also shown (where v_g is the group velocity of the unperturbed mode). However, the complex $\tilde{\beta}_g$-solutions which exist throughout the stopband are not shown. Finally, the scale of the stopband has been exaggerated somewhat, considering that $\kappa \ll \pi/\Lambda$ to satisfy the weak coupling criterion.

Analytical solution for reflection and transmission coefficients

$$\Gamma(0) = \frac{B(0)}{A(0)}$$



$$t(L) = \frac{A(L)}{A(0)}$$

$$\Gamma(0) = \frac{-iK_{ba}\sin qL}{q\cos qL - i\Delta\beta\sin qL}$$

Reflection Coefficient

$$t(L) = \frac{q}{a\cos qL - i\Delta\beta\sin qL}$$

Transmission Coefficient

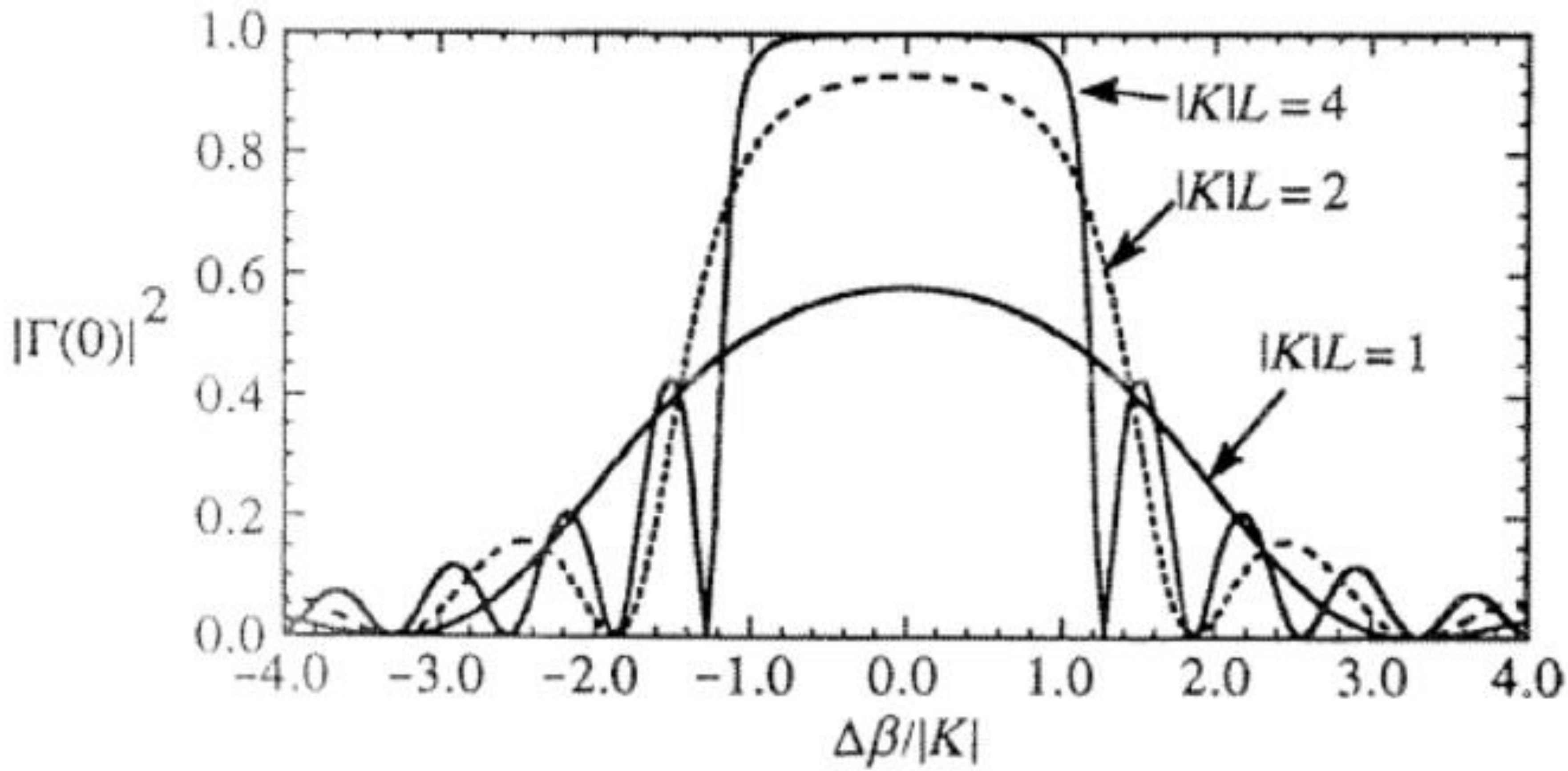
within the stop band:

$$\Gamma(0) = \frac{-iK^{*}\sinh SL}{S\cosh SL - i\Delta\beta\sinh SL}$$

$$t(L) = \frac{iS}{\Delta\beta\sinh SL + iS\cosh SL}$$

at $\Delta\beta = 0,\ S = |K|$

$$|\Gamma(0)| = \tanh|KL|$$



1.0
0.8
0.6
0.4
0.2
0.0
$|\Gamma(0)|^2$
$|K|L = 4$
$|K|L = 2$
$|K|L = 1$
−4.0
−3.0
−2.0
−1.0
0.0
1.0
2.0
3.0
4.0
$\Delta\beta/|K|$

Example of Coupling Coefficients: Simple Index Grating

Consider a waveguide with a periodic index:

$$n = n_0 + \Delta n(z) \quad \text{for} \quad |x| \le d/2$$

For a small $\Delta n(z)$:

$$n^2 \simeq n_0^2 + 2n_0 \Delta n(z) = \varepsilon^{(0)}(x) + \Delta\varepsilon(x,z)$$

n_0^2 is the unperturbed portion associated with $\varepsilon^{(0)}(x)$

$$\Delta\varepsilon(x,z) = 2n_0 \Delta n(z) \quad \text{for} \quad |x| \le d/2$$

Since $\Delta n(z)$ is periodic, we can express it as a Fourier series:

$$\Delta n(z) = \sum_{p=-\infty}^{\infty} \Delta n_p e^{ip\frac{2\pi}{\Lambda}z}$$

$$\Delta\varepsilon_p = \begin{cases} 2n_0 \Delta n_p & |x| \le d/2 \\ 0 & |x| > d/2 \end{cases}$$

$$\Delta\varepsilon_p = \begin{cases} 2n_0 \Delta n_p & |x| \le d/2 \\ 0 & |x| > d/2 \end{cases}$$

wave number of TE_0 mode in material

wave number in vacuum

$$k_0 = \omega\sqrt{\mu_0 \varepsilon_0} \quad \text{and} \quad \beta_0 = k_0 n_0$$

$$K_{ab} = \frac{\omega\varepsilon_0}{4} \int_{|x|\le d/2} \Delta\varepsilon_p(x) \left|E_y^{(0)}(x)\right|^2 dx = \frac{\omega\varepsilon_0}{4}(2n_0\Delta n_{+1}) \frac{2\omega\mu}{\beta_0}\Gamma$$

$$K_{ba} = -\frac{\omega}{4}\varepsilon_0 \int_{-\infty}^{\infty} \Delta\varepsilon_{-\ell}(x) \left|E_y^{(0)}(x)\right|^2 dx$$

$$K_{ab} \simeq \Gamma k_0 \Delta n_{+1}$$

$$K_{ba} \simeq -\Gamma k_0 \Delta n_{-1}$$

Example of Coupling Coefficients: Index and Phase Grating

magnitude phase

$$\Delta n(z) = \underset{\text{magnitude}}{\Delta n} \cos\left(\frac{2\pi}{\Lambda} z + \underset{\text{phase}}{\varphi}\right)$$

$$= \frac{\Delta n}{2} e^{i\varphi} e^{i\frac{2\pi}{\Lambda}z} + \frac{\Delta n}{2} e^{-i\varphi} e^{-i\frac{2\pi}{\Lambda}z}$$

$$= \Delta n_{+1} e^{i\frac{2\pi}{\Lambda}z} + \Delta n_{-1} e^{-i\frac{2\pi}{\Lambda}z}$$

Coupling Coefficients

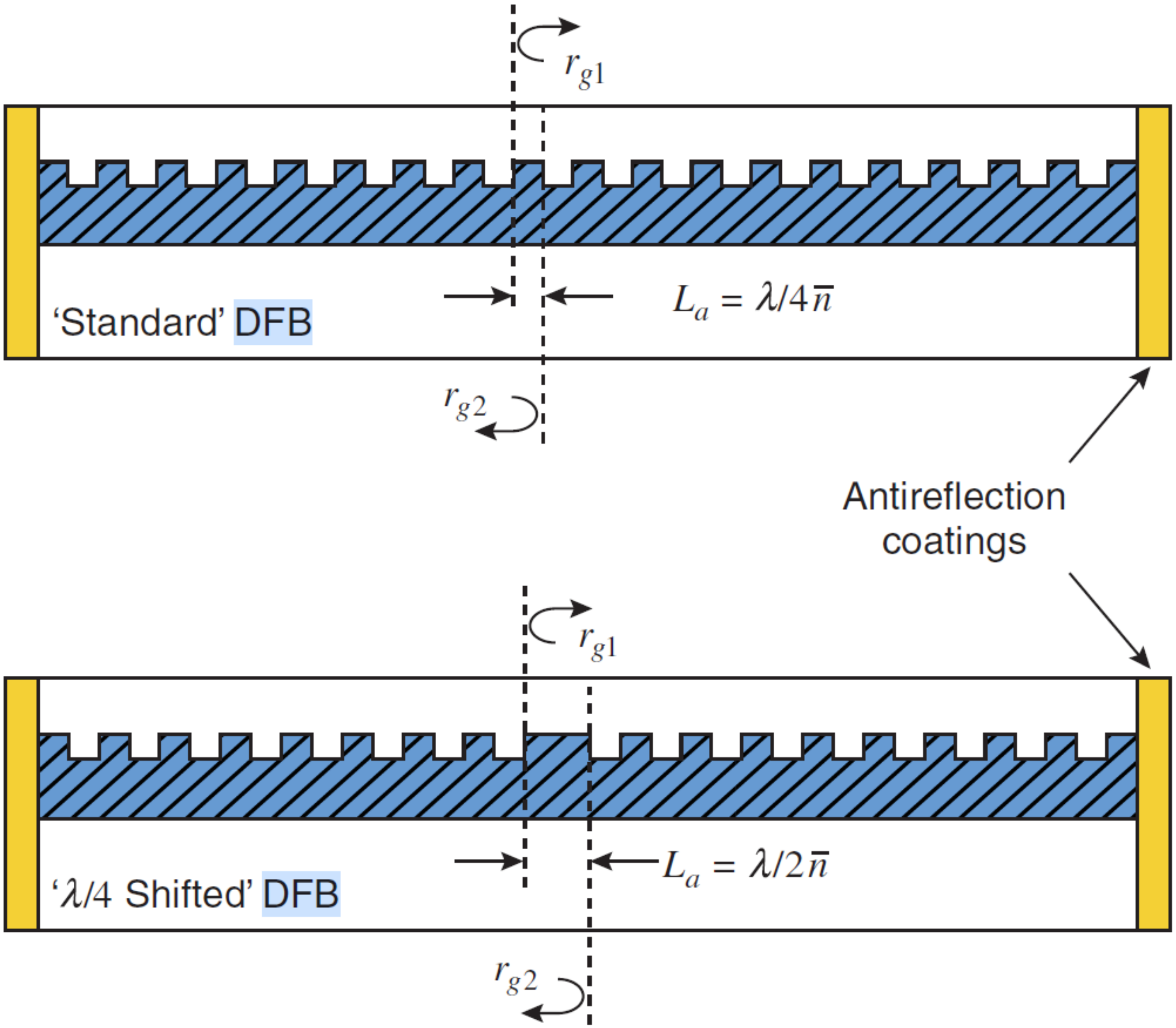
$$\boxed{K_{ab} = \Gamma k_0 \frac{\Delta n}{2} e^{i\varphi}}$$

$$\boxed{K_{ba} = -\Gamma k_0 \frac{\Delta n}{2} e^{-i\varphi}}$$

$$= -K_{ab}^{*}$$

still valid because lossless medium

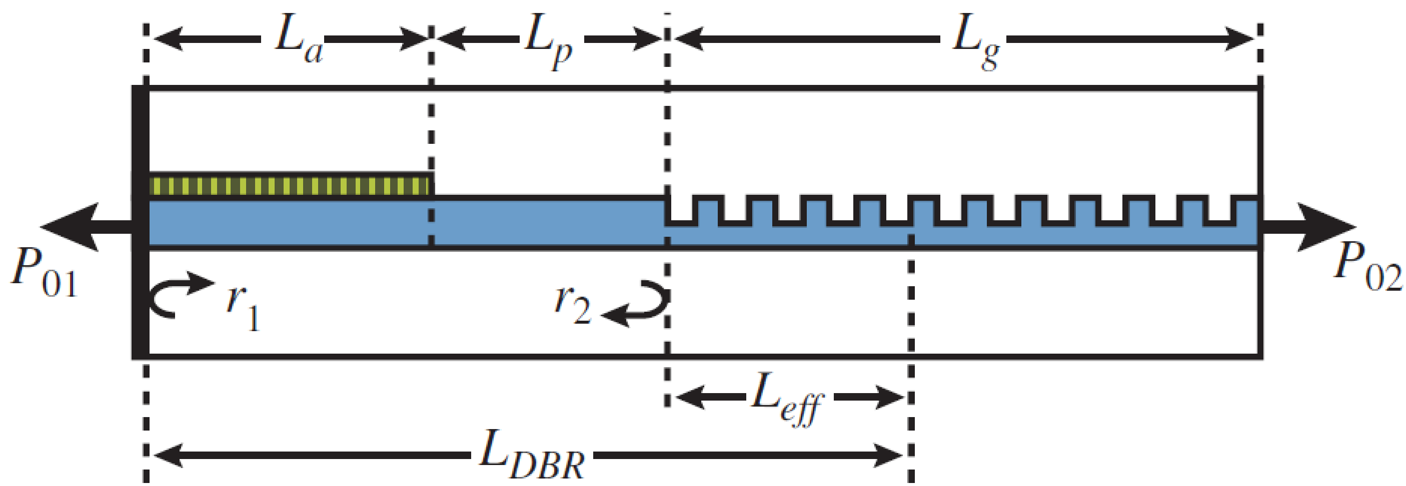
Applications of gratings in lasers – Distributed Feedback (DFB) Laser

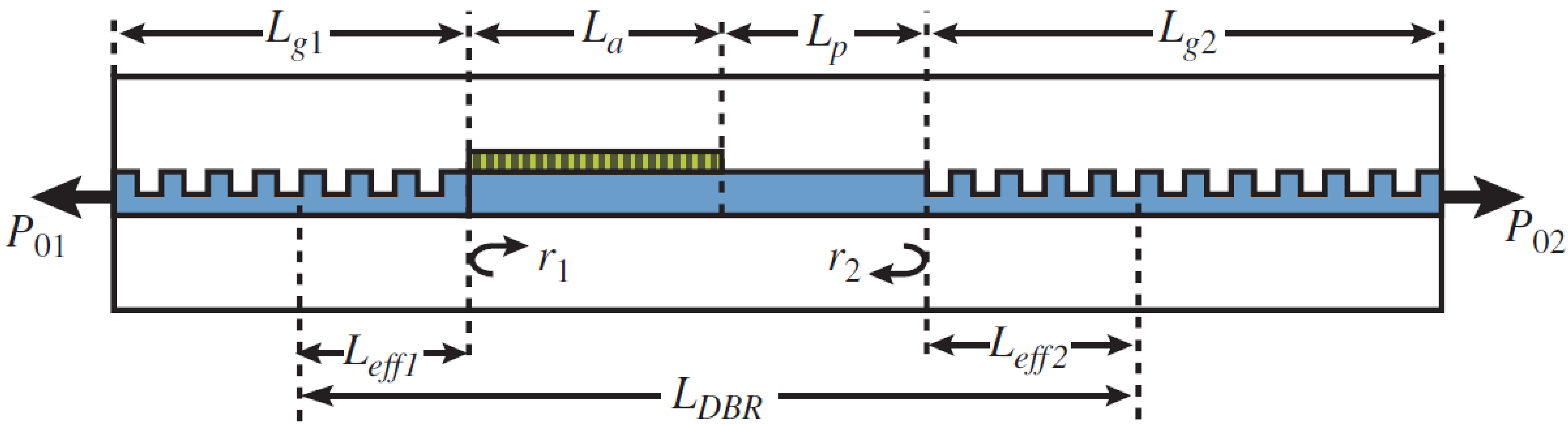


from Coldren, Corzine and Mašanović

Applications of gratings in lasers – Distributed Bragg Reflector (DBR) laser

Edge emitting lasers

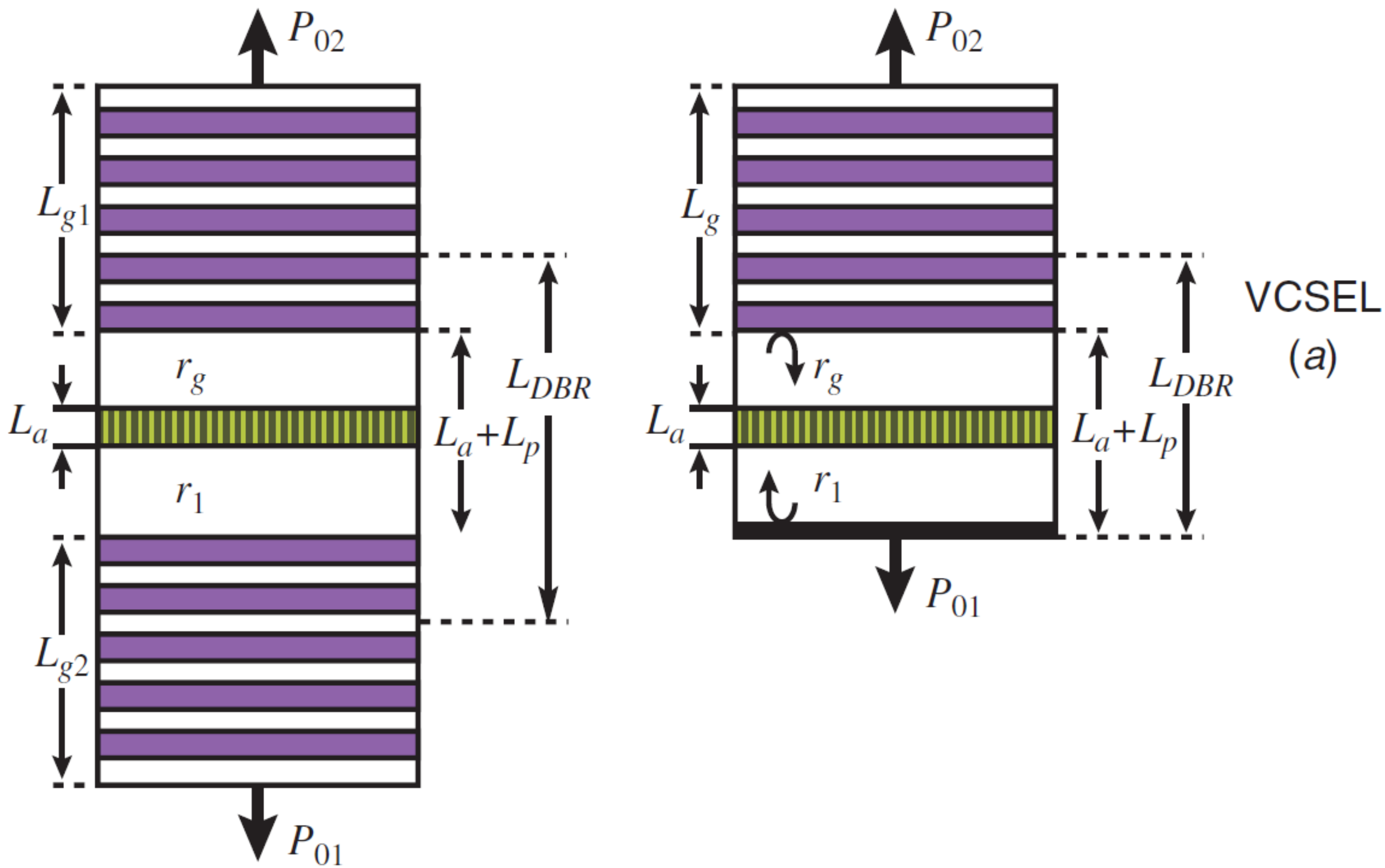




from Coldren, Corzine and Mašanović

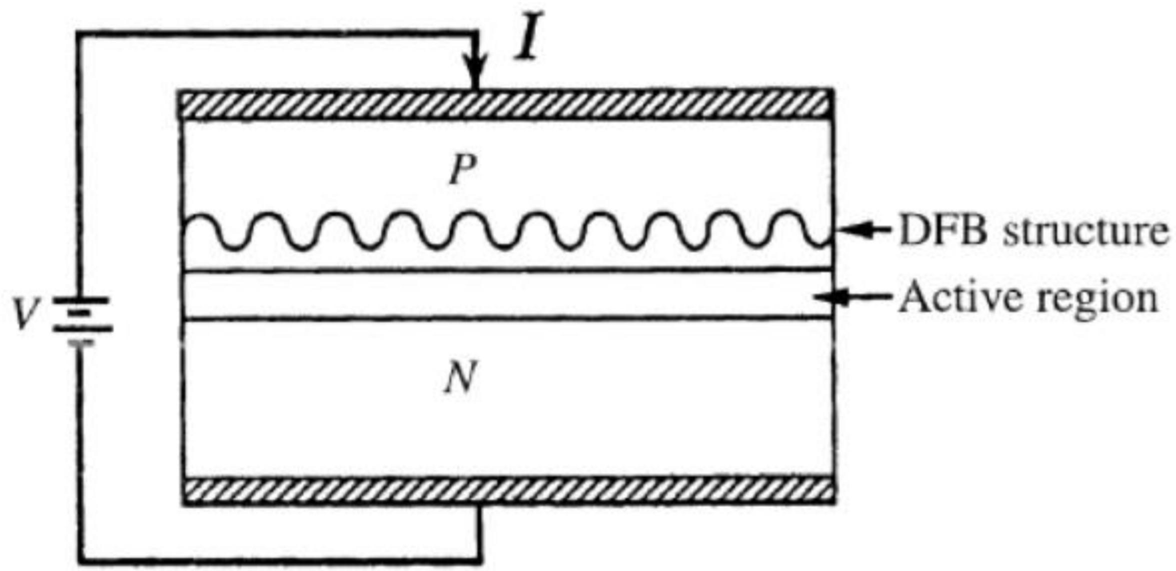
Applications of gratings in lasers – Distributed Bragg Reflector (DBR) laser

Vertical Cavity Surface Emitting (VCSEL) lasers



from Coldren, Corzine and Mašanović

Distributed Feedback (DFB) Laser



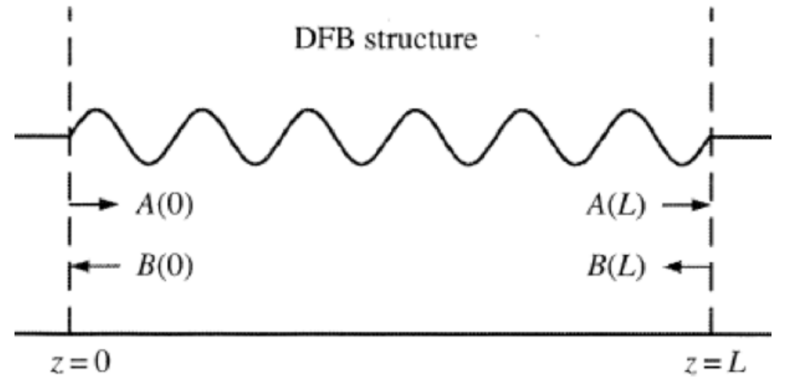
DFB Lasers provide tight control of channel wavelength and minimize chromatic dispersion (important in systems for long distance optical communications)

The reflection coefficient depends on

K Coupling coefficient
δ Detuning parameter
$\beta_B = \ell\,\pi/\Lambda$ Bragg wave number
ℓ Order of grating (integer)

Distributed Feedback (DFB) Laser

$$E_y(x) = \left(A(z)e^{i\beta_B z} + B(z)e^{-i\beta_B z} \right) E_y^{(TE_0)}(x)$$



$\beta_B = \ell\pi/\Lambda$ Bragg wave number

$A(z)$ and $B(z)$ satisfy the coupled-mode equation:

$$\frac{d}{dz}\begin{bmatrix} A(z) \\ B(z) \end{bmatrix} = i\begin{bmatrix} \Delta\beta & K_{ab} \\ K_{ba} & -\Delta\beta \end{bmatrix}\begin{bmatrix} A(z) \\ B(z) \end{bmatrix}$$

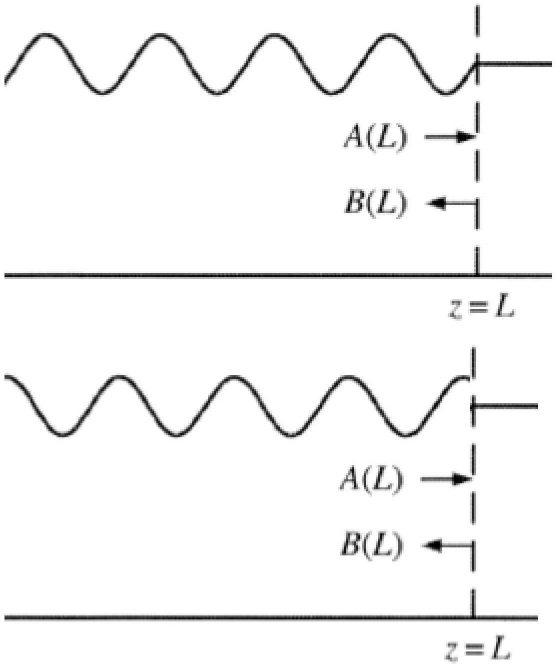
Consider this index profile for the grating

$$\boxed{\Delta n(z) = \Delta n \cos\left(\frac{2\pi}{\Lambda}z + \varphi\right)}$$

We found earlier:

$$\boxed{K_{ab} = \Gamma k_0 \frac{\Delta n}{2} e^{i\varphi}} \qquad \boxed{K_{ba} = -\Gamma k_0 \frac{\Delta n}{2} e^{-i\varphi}} = -K_{ab}^*$$

DFB Structure – Reflection and Transmission



The phase of the grating φ depends on the location of the grating relative to the facet location

Recall that:

$$A(z) = Ae^{iqz}$$

From the eigenequations of the coupled mode system

$$q = \sqrt{(\Delta\beta)^2 + K_{ab}K_{ba}} = \sqrt{(\Delta\beta)^2 - |K|^2}$$

When $|\Delta\beta| < |K|$ the eigenvalue q becomes imaginary and there is strong coupling between forward and backward waves.

DFB Structures with Gain

The detuning parameter is modified to include gain or loss contribution.

If we indicate with $\delta = \beta_0 - \beta_B$ **the value of** $\Delta\beta$ **in the case of no gain or loss,**

$$\Delta\beta = \left(\beta_0 - i\frac{g_n}{2}\right) - \beta_B = \beta_0 - \beta_B - i\frac{g_n}{2} = \delta - i\frac{g_n}{2}$$

where as usual

$$g_n = \Gamma g - \alpha \quad \Leftarrow \ \text{Net Modal Gain}$$

$$\alpha = \Gamma\alpha_i + (1 - \Gamma)\alpha_s$$

General case solution of the Coupled Mode equations

Consider again forward and backward TE modes

$$E_y(x,z) = \left[A_0(z)e^{i\beta_0 z} + B_0(z)e^{-i\beta_0 z} \right] E_y^{(0)}(x)$$

$$= \left[A(z)e^{i\beta_B z} + B(z)e^{-i\beta_B z} \right] E_y^{(0)}(x)$$

$$\boxed{\beta_B = \frac{\ell\pi}{\Lambda}}$$

Expand with eigenvectors to express general solutions

$$A(z) = c_1\mathbf{v}_1 e^{iq_+ z} + c_2\mathbf{v}_2 e^{iq_- z}$$
$$B(z) = c_1\mathbf{v}_1 e^{iq_+ z} + c_2\mathbf{v}_2 e^{iq_- z}$$

We can have initial condition $c_1 = 1,\ c_2 = 0.$

$$\Rightarrow \quad E_y(x,z) = \left[\mathbf{v}_1 e^{iq_+ z} e^{i\beta_B z} + \mathbf{v}_1 e^{iq_+ z} e^{-i\beta_B z} \right] E_y^{(0)}(x).$$

The general solution is the sum of the eigenmodes

(Sections 8.22 and 8.52 provide the mathematical details of eigenstate solution method)

$$\begin{bmatrix} A(z) \\ B(z) \end{bmatrix} = c_1 \begin{bmatrix} K_{ab} \\ q - \Delta\beta \end{bmatrix} e^{iqz} + c_2 \begin{bmatrix} K_{ab} \\ -q - \Delta\beta \end{bmatrix} e^{-iqz}$$

$$= \begin{bmatrix} A^{+} \\ B^{+} \end{bmatrix} e^{iqz} + \begin{bmatrix} A^{-} \\ B^{-} \end{bmatrix} e^{-iqz}$$

from which

$$\frac{B^{+}}{A^{+}} = \frac{c_1 \left(q - \Delta\beta\right)}{c_1 K_{ab}} = \frac{\Delta\beta - q}{-K_{ab}} = \frac{K_{ba}}{\Delta\beta + q} \quad (e^{+iqz}\ \text{eigenmode})$$

$$\frac{B^{-}}{A^{-}} = \frac{\Delta\beta + q}{-K_{ab}} = \frac{K_{ba}}{\Delta\beta - q} \quad (e^{-iqz}\ \text{eigenmode})$$

Eigenequation $\left(\Delta\beta - q\right)\left(-\Delta\beta - q\right) - K_{ba}K_{ab} = 0$

Let's define $$r_p(q) = \frac{B^+}{A^+} \qquad r_m(q) = \frac{A^-}{B^-}$$

With these rewrite

$$\begin{bmatrix} A(z) \\ B(z) \end{bmatrix} = A^+ \begin{bmatrix} 1 \\ r_p \end{bmatrix} e^{iqz} + B^- \begin{bmatrix} r_m \\ 1 \end{bmatrix} e^{-iqz}$$

$\boxed{z = 0}$

$$\begin{bmatrix} A(0) \\ B(0) \end{bmatrix} = A^+ \begin{bmatrix} 1 \\ r_p \end{bmatrix} + B^- \begin{bmatrix} r_m \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & r_m \\ r_p & 1 \end{bmatrix} \begin{bmatrix} A^+ \\ B^- \end{bmatrix}$$

$\boxed{z = L}$

$$\begin{bmatrix} A(L) \\ B(L) \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} A(0) \\ B(0) \end{bmatrix}$$

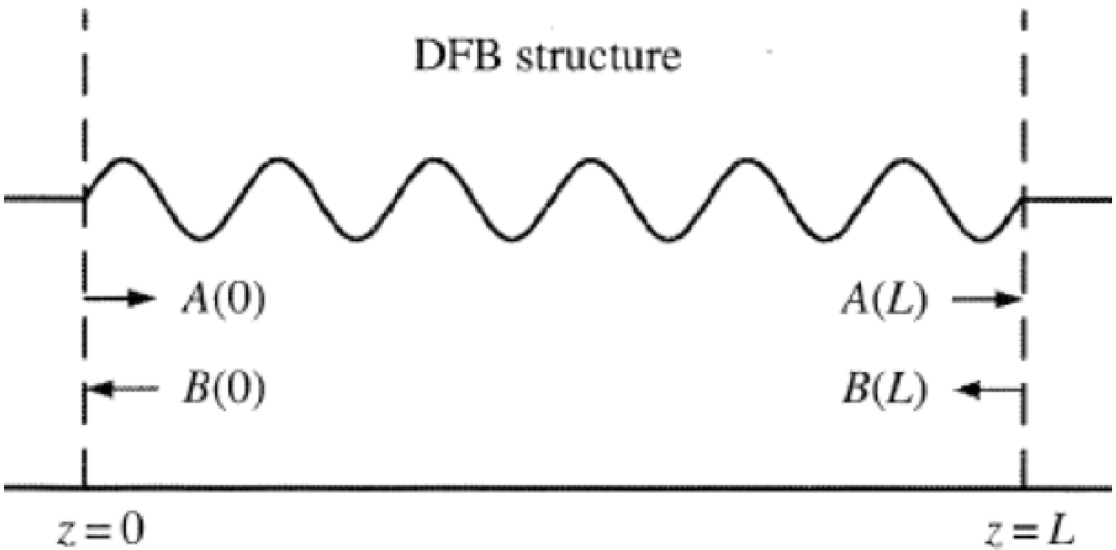
$$F_{11} = \frac{1}{1 - r_p r_m}\left(e^{iqL} - r_p r_m e^{-iqL}\right) \qquad F_{12} = \frac{-r_m}{1 - r_p r_m}\left(e^{iqL} - e^{-iqL}\right)$$

$$F_{12} = \frac{r_p}{1 - r_p r_m}\left(e^{iqL} - e^{-iqL}\right) \qquad F_{22} = \frac{1}{1 - r_p r_m}\left(-r_p r_m e^{iqL} + e^{-iqL}\right)$$

determinant $\boxed{F_{11}F_{22} - F_{12}F_{21} = 1}$

Reflection and Transmission in DFB structure

With no incident wave at $z = L$ and index matched material at the facet, there is no end reflection with $B(L) = 0$.



From

$$\begin{bmatrix} A(L) \\ B(L) \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} A(0) \\ B(0) \end{bmatrix}$$

$$0 = B(L) = F_{21} A\left(0\right) + F_{22} B\left(0\right)$$

Reflection and Transmission in DFB structure

Reflection coefficient at $z = 0$

$$\Gamma(0) = \frac{B(0)}{A(0)} = -\frac{F_{21}}{F_{22}} = \frac{-r_p\left(e^{iqL} - e^{-iqL}\right)}{-r_p r_m e^{iqL} + e^{-iqL}} = \frac{r_p\left(1 - e^{2iqL}\right)}{1 - r_p r_m e^{2iqL}}$$

Transmission coefficient at $z = L$

$$T = \left|\frac{A(L)}{A(0)}\right|^2 = \left|\frac{1}{F_{22}}\right|^2 = \left|\frac{1 - r_p r_m}{e^{-iqL} - r_p r_m e^{iqL}}\right|^2 = \left|\frac{1 - r_p r_m}{1 - r_p r_m e^{2iqL}}\right|^2$$

For the cased of index grating

$$qL = \sqrt{\left(\delta L - i\frac{g_n L}{2}\right)^2 - (KL)^2}$$

Transmission Characteristic for DFB

The lasing condition occurs when the denominator of the transmission coefficient approaches zero:

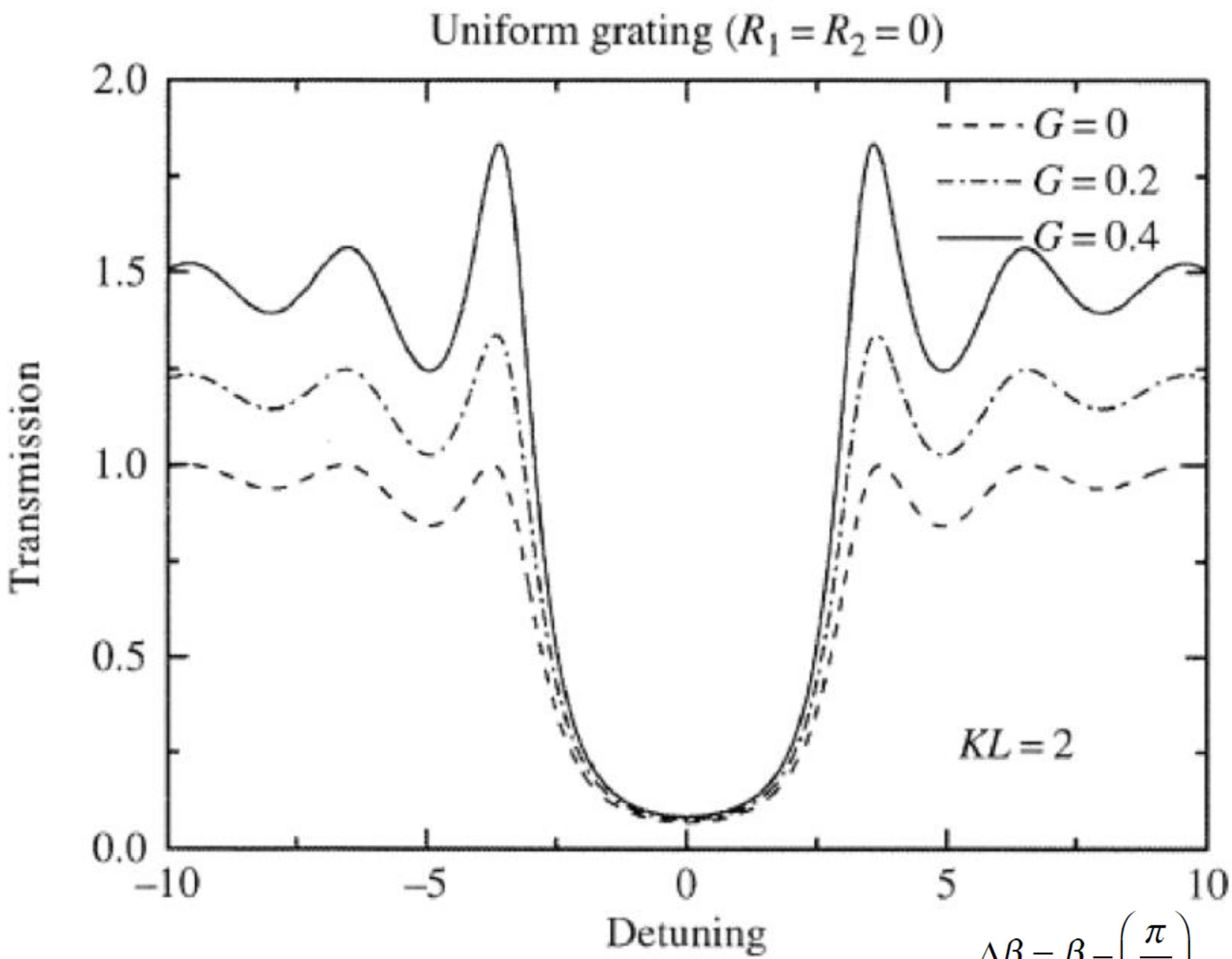
$$1 - r_p r_m e^{i2qL} = 0$$

Transmission spectra for the simple grating DFBs are symmetric with peaks on both sides of the detuning point.

These peaks correspond to two degenerate longitudinal modes that lase in the DFB structure.

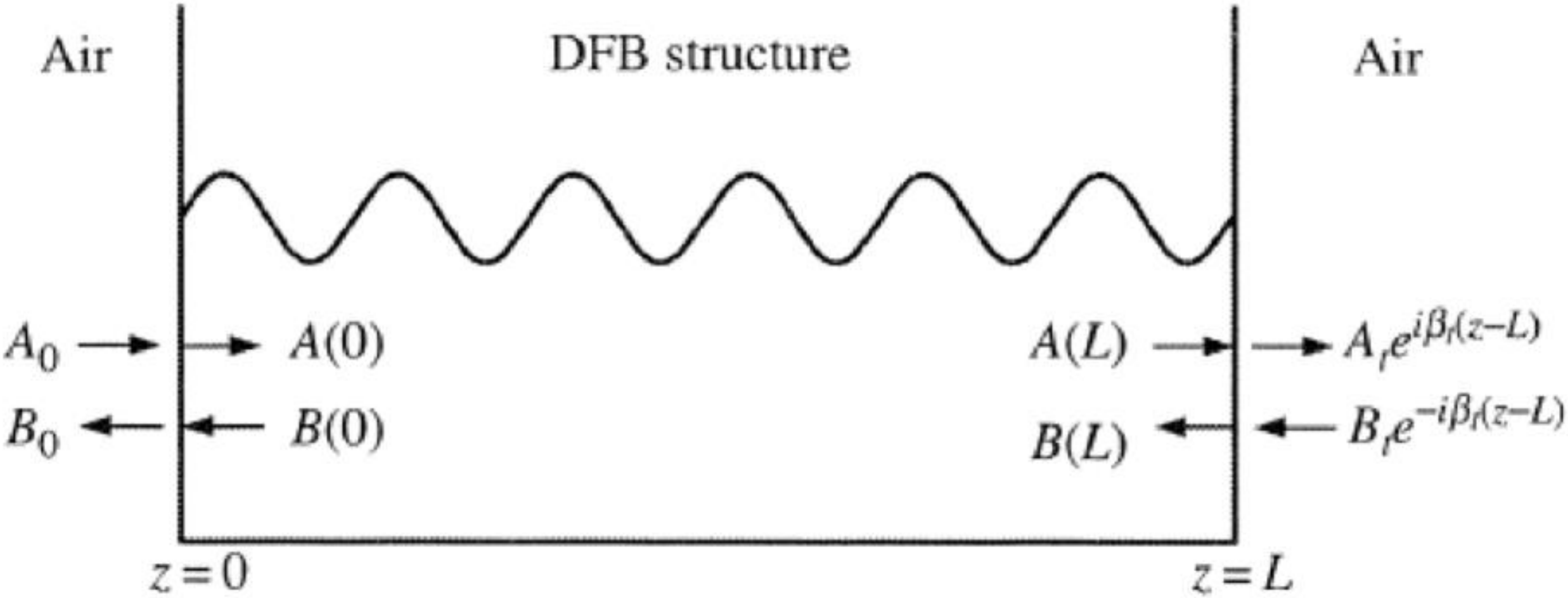
This explains the bimodal failure mechanism in simple grating DFB lasers.

Transmission Characteristic for DFB



$$\Delta\beta = \beta - \left(\frac{\pi}{\Lambda}\right)$$

DBF Structure with End Facets



The transmission matrix method can be applied to relate A_t with $A(O)$

DBF Structure with End Facets

$$\begin{bmatrix} A_t \\ B_t \end{bmatrix} = \frac{1}{t_{21}} \begin{bmatrix} 1 & r_{21} \\ r_{21} & 1 \end{bmatrix} \begin{bmatrix} A(L)e^{i\beta_B L} \\ B(L)e^{-i\beta_B L} \end{bmatrix}$$

$$= \frac{1}{t_{21}} \begin{bmatrix} 1 & r_{21} \\ r_{21} & 1 \end{bmatrix} \begin{bmatrix} e^{i\beta_B L} & 0 \\ 0 & e^{-i\beta_B L} \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} A(0) \\ B(0) \end{bmatrix}$$

$$= \frac{1}{\sqrt{1-|r_1|^2}\sqrt{1-|r_2|^2}} \begin{bmatrix} 1 & -r_2 \\ -r_2 & 1 \end{bmatrix} \begin{bmatrix} e^{i\beta_B L} & 0 \\ 0 & e^{-i\beta_B L} \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} A(0) \\ B(0) \end{bmatrix}$$

$$= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} A(0) \\ B(0) \end{bmatrix}$$

Lasing Condition with End Facets

$$\begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} A(0) \\ B(0) \end{bmatrix}$$

Transmission coefficient

$$T = \left| \frac{A_t}{A_0} \right|^2 = \left| \frac{m_{11}m_{22} - m_{12}m_{21}}{m_{22}} \right|^2$$

Lasing condition

$$m_{22} = \frac{1}{t_1 t_2} \left[-r_2 e^{i\beta_B L} \left(r_1 F_{11} + F_{12} \right) + e^{-i\beta_B L} \left(r_1 F_{21} + F_{22} \right) \right] \rightarrow 0$$

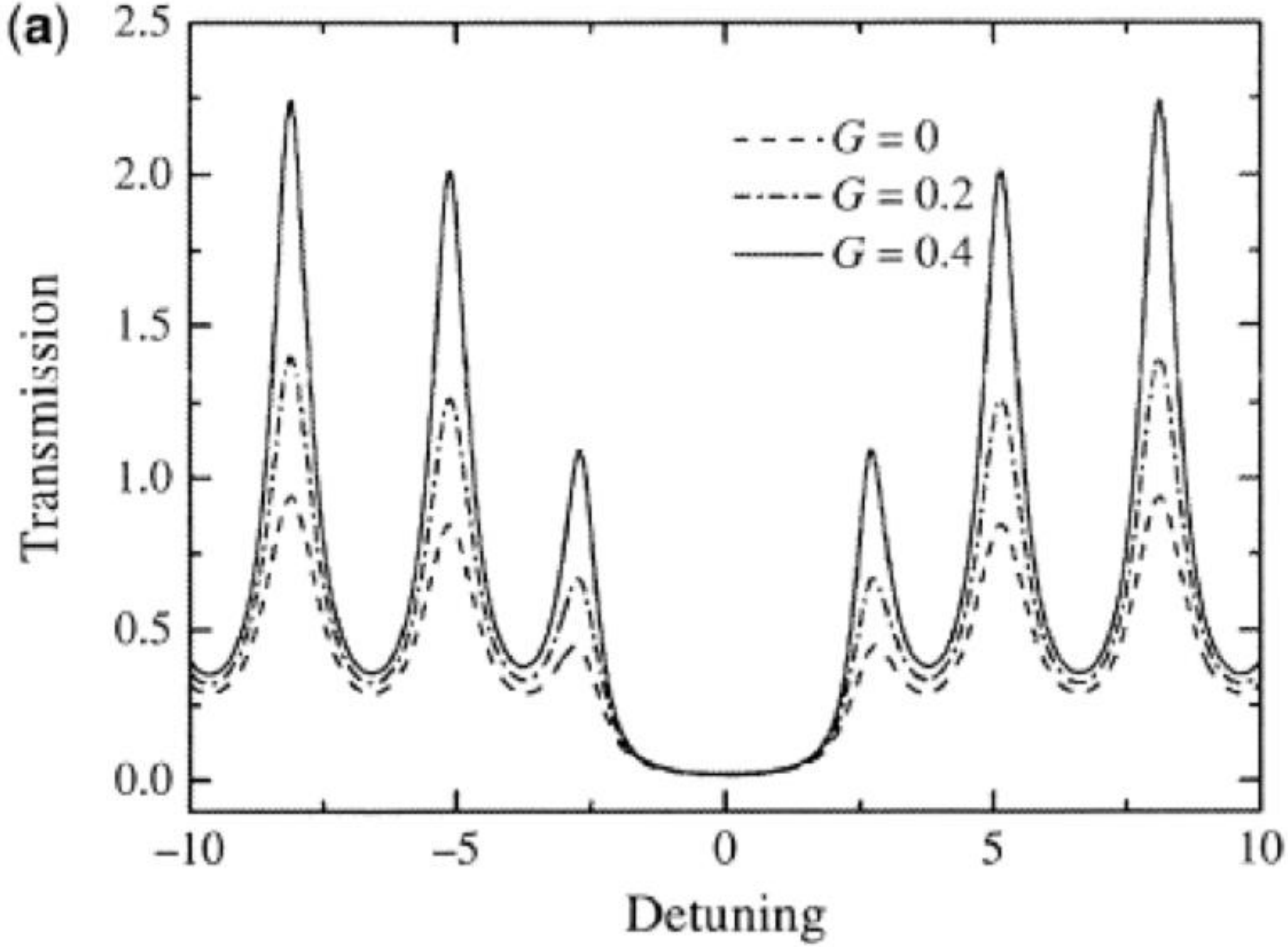
For the case of index grating, the oscillation condition can be shown to be

$$\left(\delta L - i\frac{g_n L}{2} \right) = -i\sqrt{\left(\delta L - i\frac{g_n L}{2} \right)^2 - (KL)^2} \left[\cot\sqrt{\left(\delta L - i\frac{g_n L}{2} \right)^2 - (KL)^2} \right]$$

Real and imaginary parts of the equation needs to be solved for δL and $g_n L$ for each value of KL

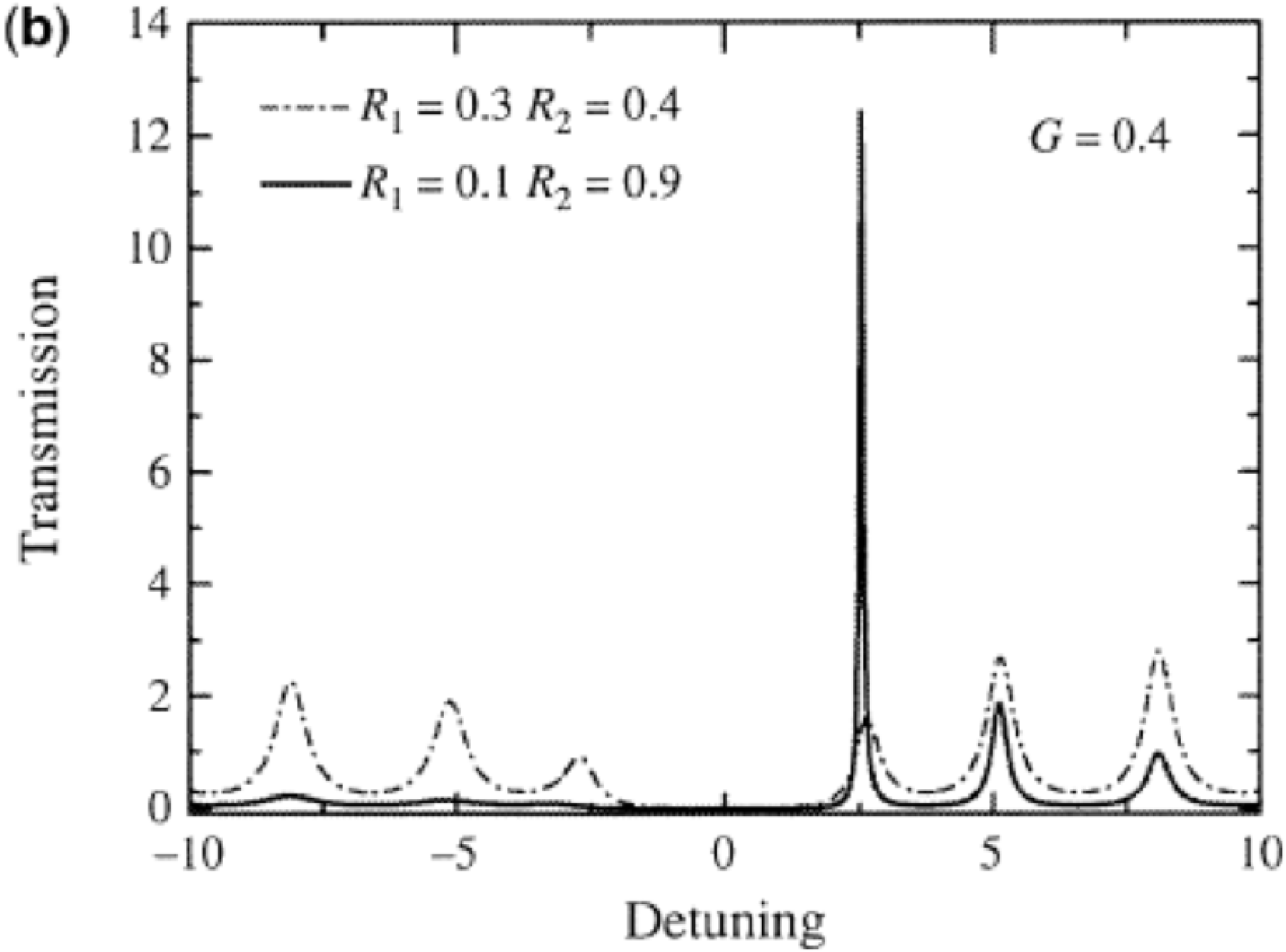
Effect of Cavity End Conditions: Facet Coating

Symmetrical Cavity

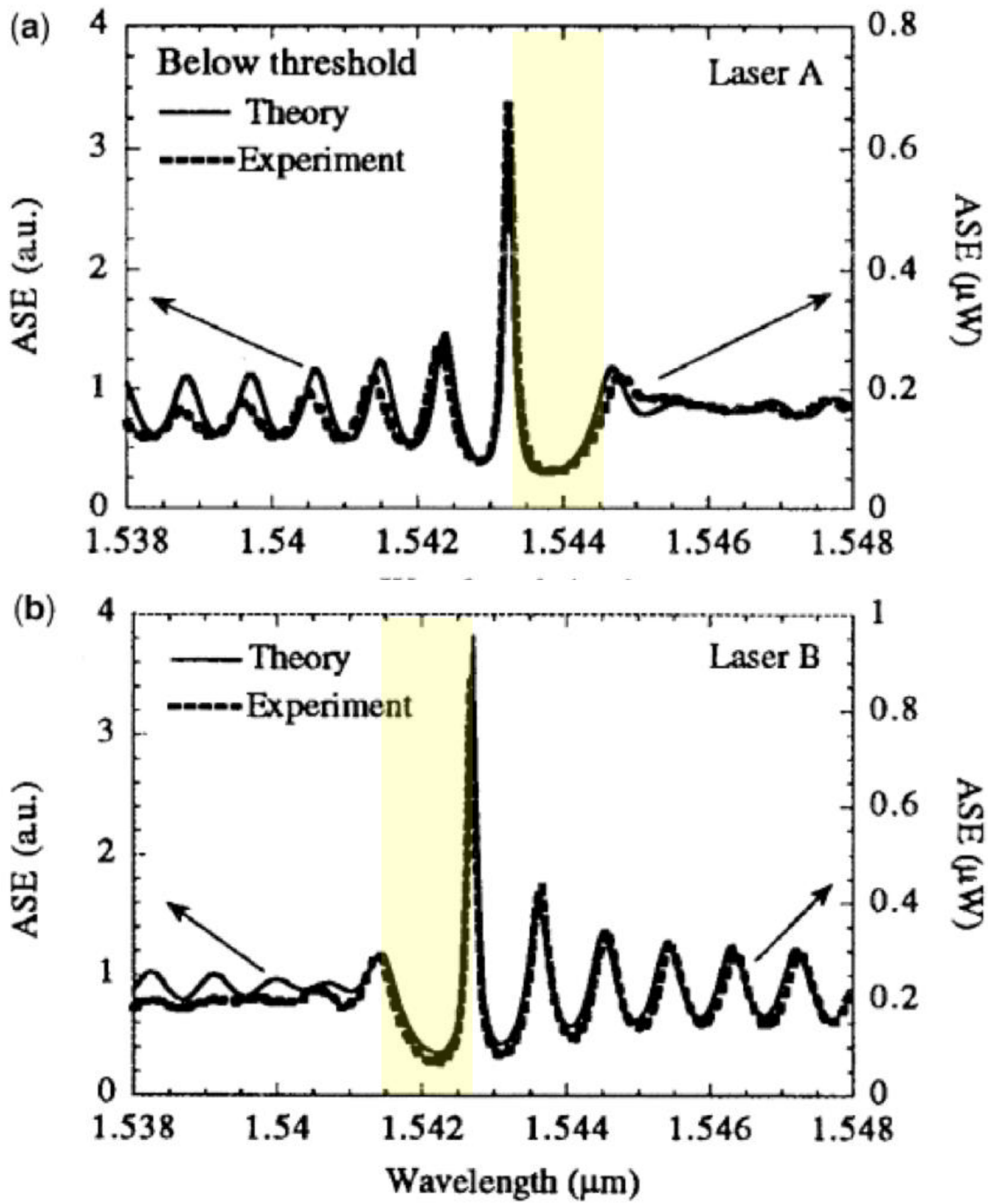


Effect of Cavity End Conditions: Facet Coating

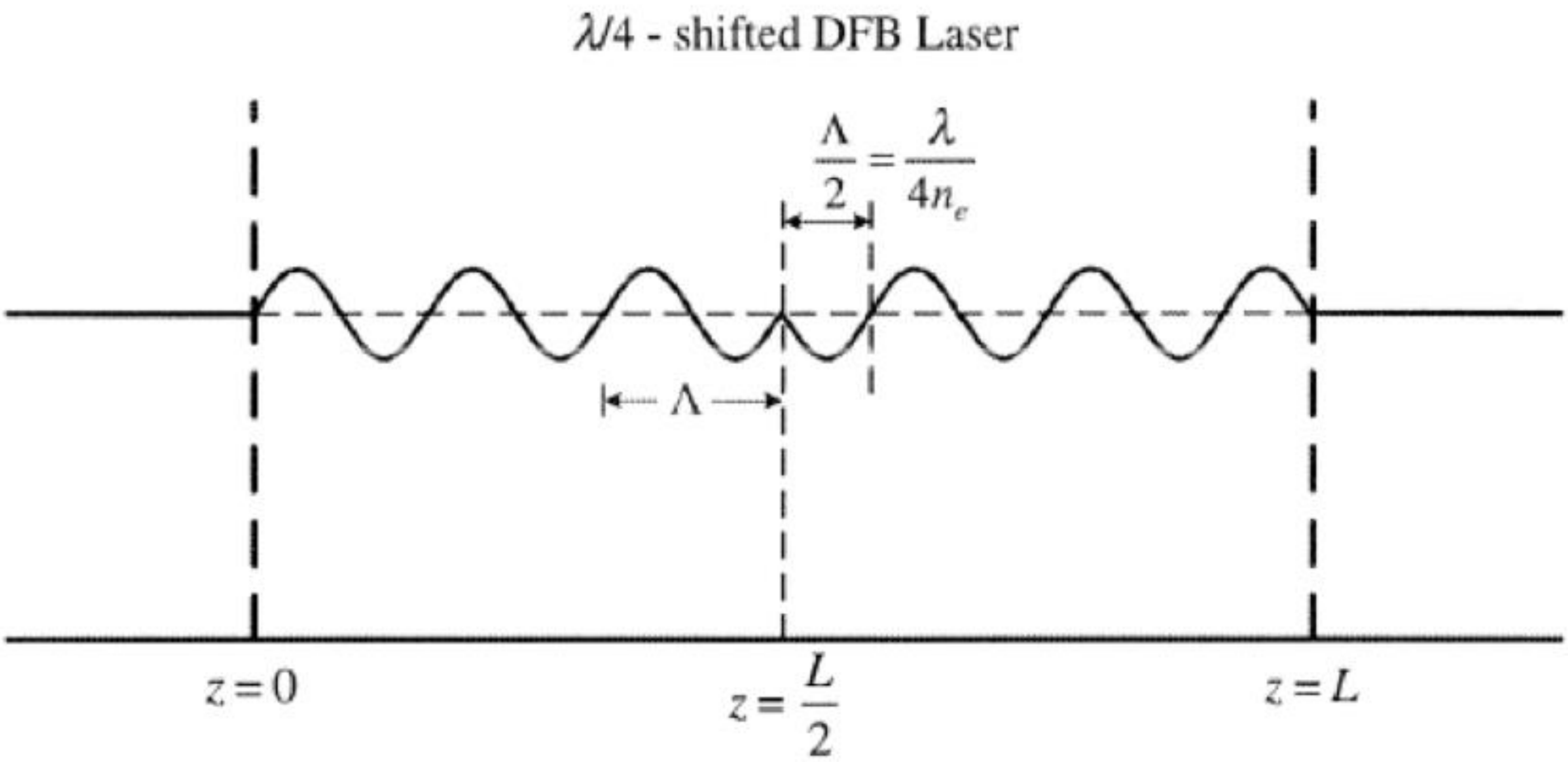
Asymmetrical Cavity



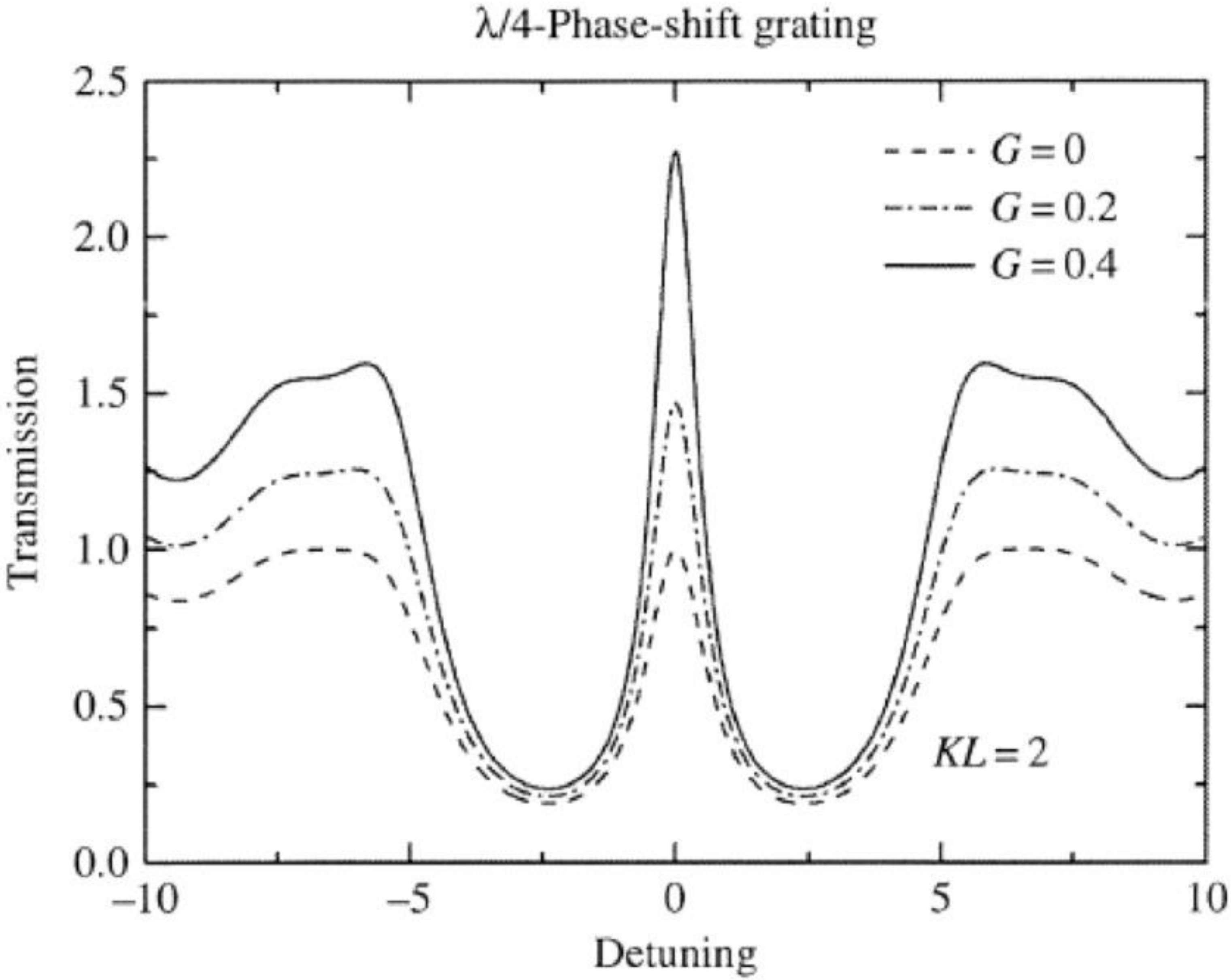
Lasing Peak can be obtained to operate on either side of stopband



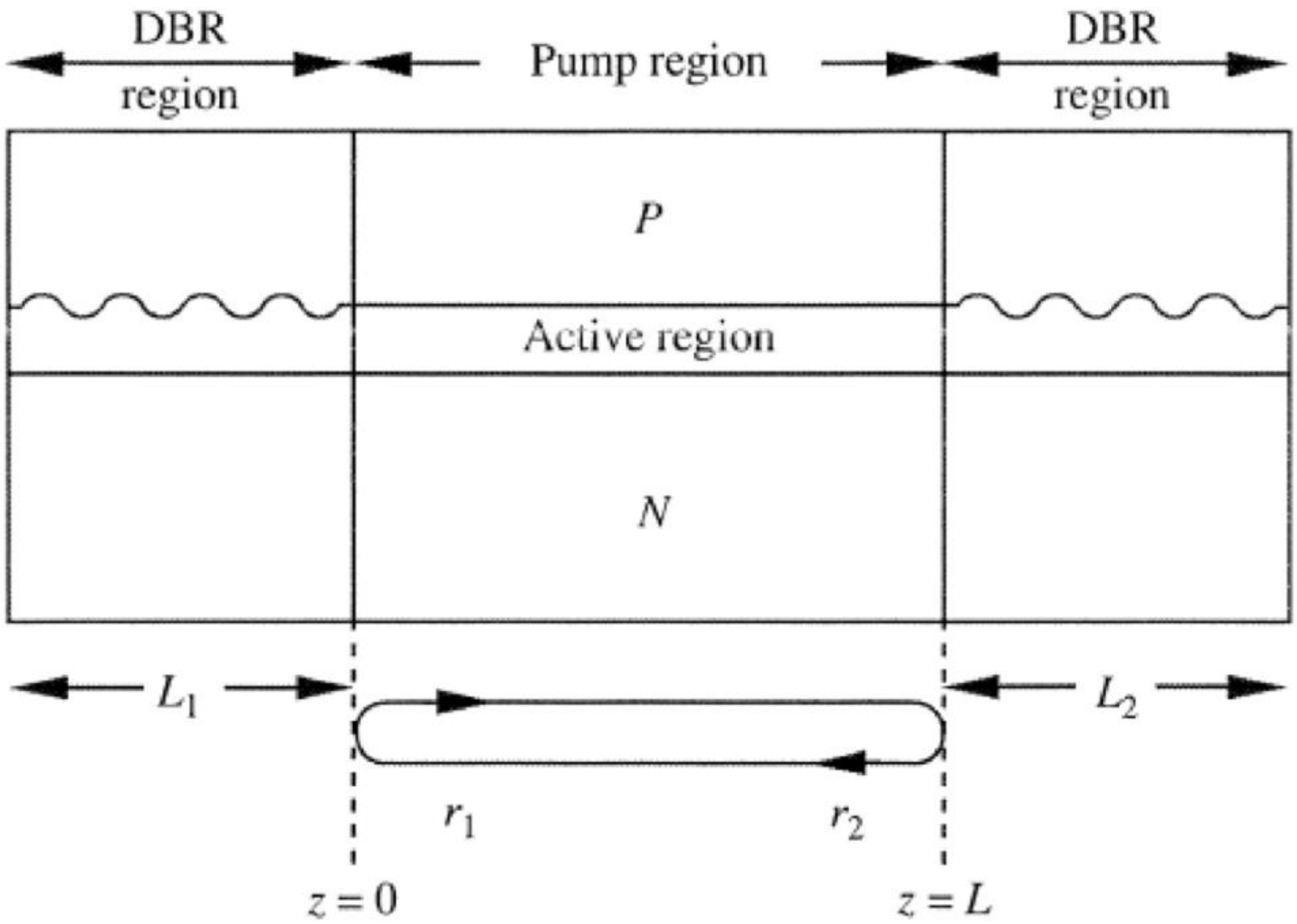
Phase Shifted DFB breaks the degeneracy



Phase Shifted DFB breaks the degeneracy



Distributed Bragg Reflector (DBR) Laser



Intuitive Picture

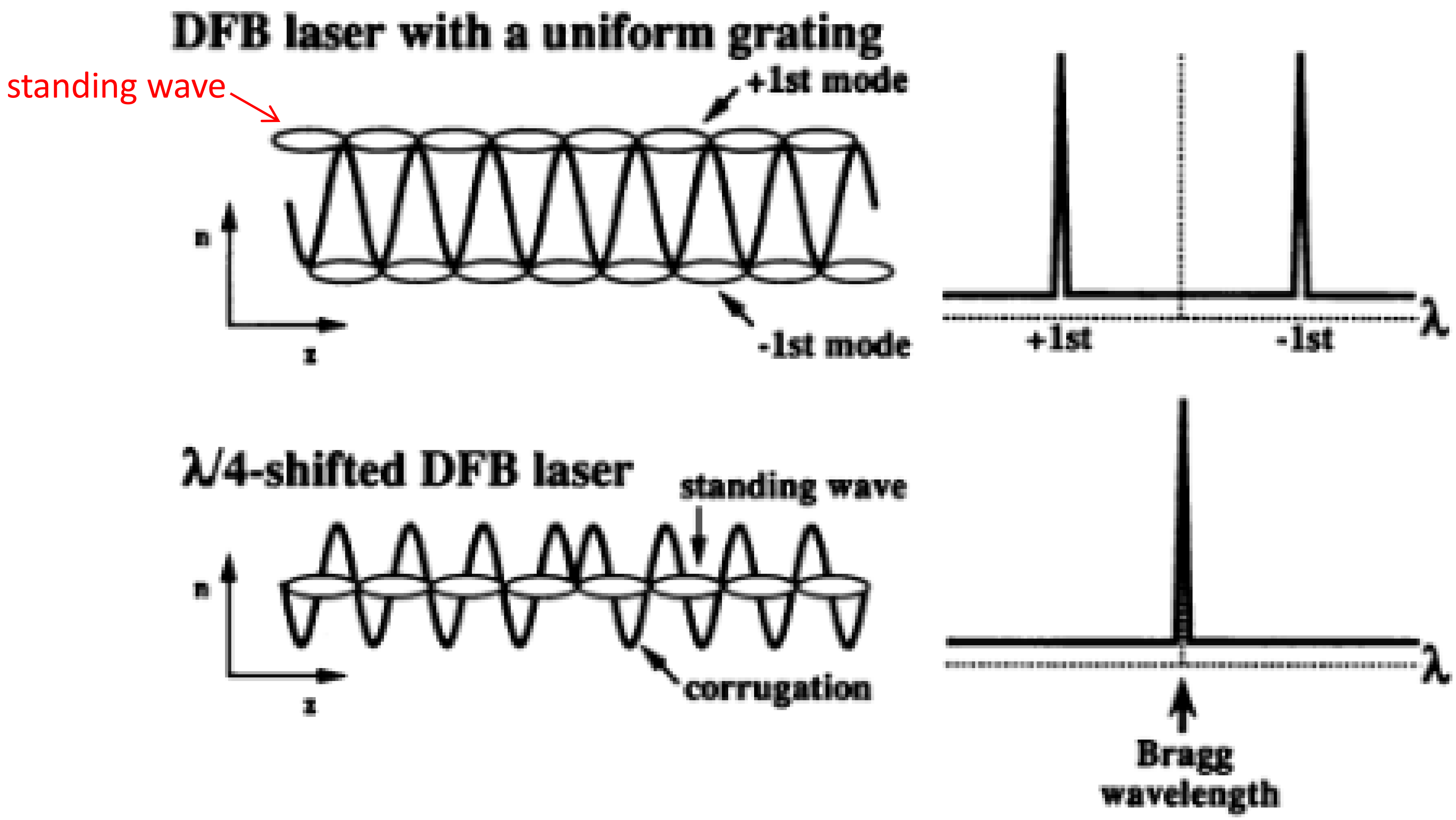
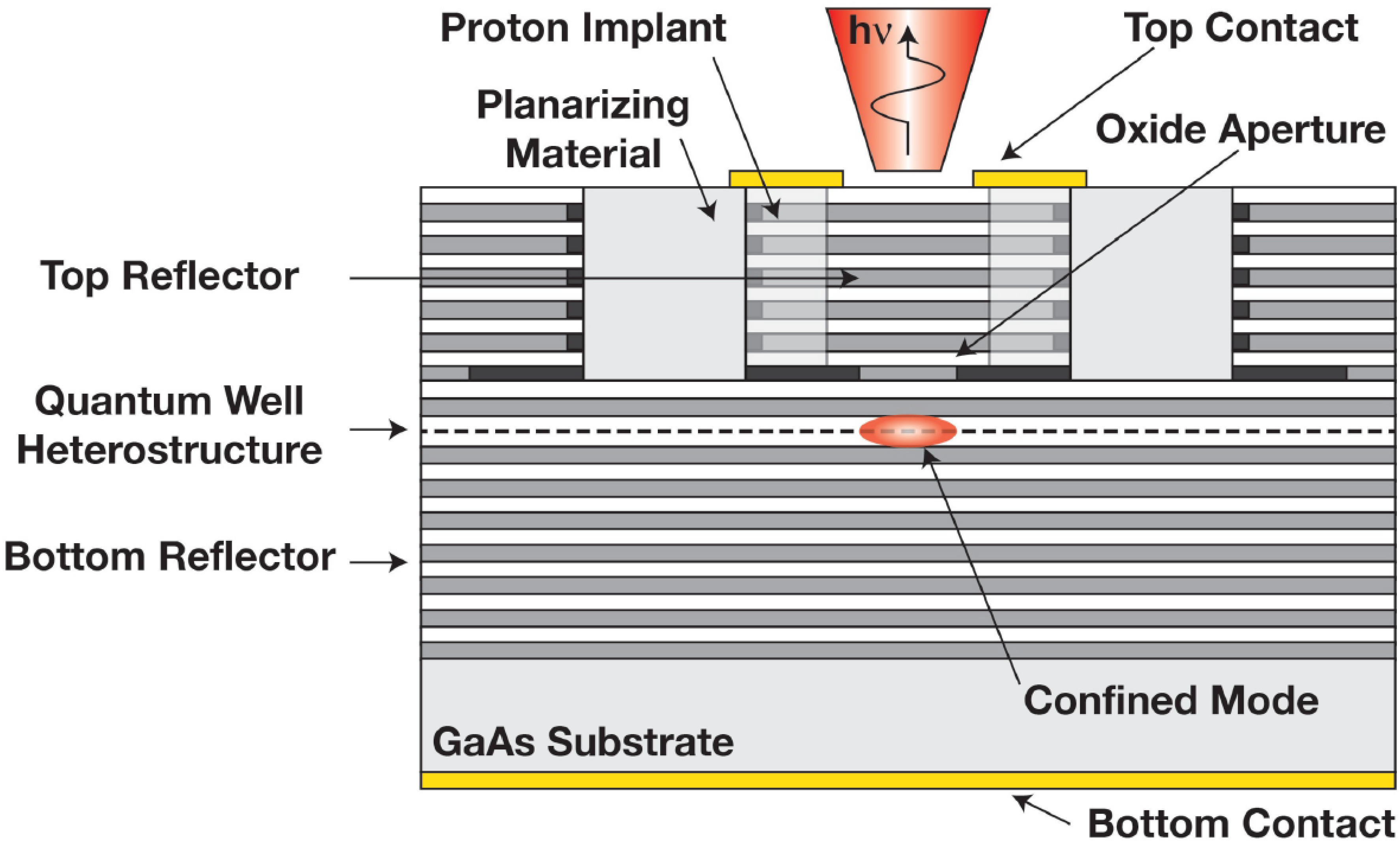


FIGURE 2.6. Schematic representation of longitudinal modes in DFB lasers with a uniform grating and λ/4-shifted DFB lasers. Corrugations and standing waves around the center of the laser axis are shown at the left-hand side of the figure.

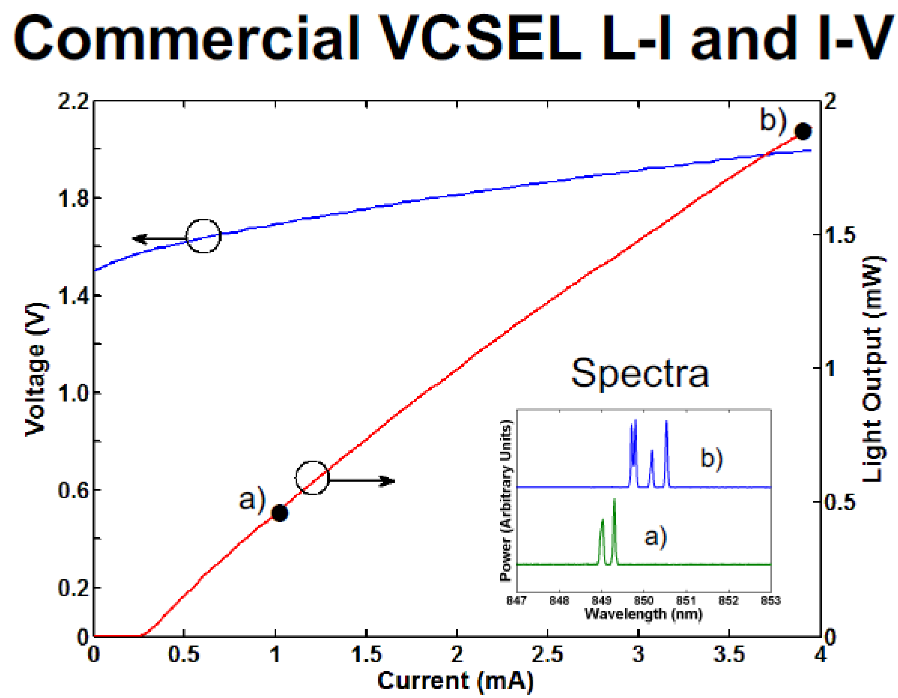
[N. Chinone and M. Okai, "Distributed Feedback Semiconductor Lasers" Chapter 2 in *Semiconductor Lasers: Past, Present, and Future*, G.P. Agrawal, editor (1995)]

The Modern VCSEL



VCSEL Properties

- Lower threshold current and power dissipation than edge-emitting lasers
  - Sub-mA versus several mA laser threshold current
  - mWs versus 10's of mWs power dissipation
- Good optical beam quality
  - Circular as opposed to elliptical beam shape
  - Low divergence (<10° versus >30°)
- Better compatibility with wafer-scale manufacturing methods than edge-emitting lasers



Comparison of Laser Parameters

	Edge Emitting Laser	VCSEL
Beam Shape	Elliptical	Circular (easy to couple to optical fiber)
Modal Behavior	Always multiple longitudinal Can be single transverse	Always single longitudinal Multiple or single transverse
Threshold Current	5-500 mA	0.1-3 mA
Drive Current	10-2000 mA	0.5-10 mA
Spectral Properties	1-3 nm linewidth with multiple modes	<0.1 nm linewidth with single mode
Temperature Dependence	0.4 nm/°C	0.08 nm/°C
Output Power	1-1000 mW	0.5-5 mW

Reading Assignments:

- Section 8.6 of Chuang's book
- Section 10.6 of Chuang's book

- Sections 3.1-3.5 in Coldren, Corzine and Mašanović
- Chapter 6 in Coldren, Corzine and Mašanović (supplemental)
- Section 8.2 in Coldren, Corzine and Mašanović