Lecture 2 – Outline

• Maxwell’s Equations
• Electrostatics
  – Coulomb’s Law
  – Gauss’ Law
• Examples

Reading assignment
Prof. Kudeki’s ECE 329 Lecture Notes on Fields and Waves:
2) Static electric fields – Coulomb’s and Gauss’ laws
FROM THE BUSINESS WORLD

CASE STUDY

Wave propagation speed is very relevant for our globally interconnected world.
The underwater fiber-optic cable would trace the shortest path between Europe and Asia.
London-Tokyo via Northwest passage

$L_1 \approx 9,600 \text{ miles} = 9,600 \times 1,609 = 15,446,400 \text{ m}$

Assume dielectric constant of fiber optic glass core

$$\varepsilon_r = 2.25$$

Propagation along the axis of the fiber (exact $c$)

$$v_g = \frac{c}{\sqrt{\varepsilon_r}} = \frac{299,792,458}{\sqrt{2.25}} = 199,861,638.7 \frac{\text{m}}{\text{s}}$$

$$t_{LT1} = \frac{L}{v_g} = \frac{15,446,400}{199,861,638.7} = 0.0772855 \text{s}$$

With approximate $c$

$$t_{LT1} = \frac{L}{v_g} \approx \frac{15,446,400}{2 \times 10^8} = 0.077232 \text{s}$$

time is underestimated by $\approx 53 \mu\text{s}$
London-Tokyo via New York – San Francisco

\[ L_2 \approx 11,500 \text{ miles} = 11,500 \times 1,609 = 18,503,500 \text{ m} \]

Assume again

\[ v_g = 199,861,638.7 \frac{\text{m}}{\text{s}} \]

Propagation along the axis of the fiber

\[ t_{LT1} = \frac{L}{v_g} = \frac{15,446,400}{199,861,638.7} = 0.0772855 \text{ s} \]

\[ t_{LT2} = \frac{L}{v_g} \approx \frac{18,503,500}{199,861,638.7} = 0.0925816 \text{ s} \]

\[ \Delta t \approx 15.3 \text{ ms} \]
London-Tokyo via satellite?

Signals move faster in air/vacuum than in fiber optic glass, but a geosynchronous satellite is at an altitude

\[ H = 22,236 \text{ miles} = 35,786 \text{ km} \]

\[ \approx 4 \text{ times the height of Mt. Everest and } \approx 89\% \text{ length of the equator (40,075km)} \]

The speed of EM waves in air is \( v_g \approx 3 \times 10^8 \) instead of \( v_g \approx 2 \times 10^8 \) in glass but the link distance to be covered would be at least more than five times the shortest one between London and Tokyo via the Northwest passage.

Nonetheless, for purposes other than trading, the additional delay may be quite reasonable and business entities are indeed exploring an expansion of satellite links:


Static Regimes – Fields do not vary in time

ELECTROSTATICS

• The electric charges do not change position in time.
• Therefore, \( \rho, E \) and \( D \) are constant and there is no magnetic field \( H \), since there is no current density \( J \).

MAGNETOSTATICS

• The charge crossing a given cross-section (current) does not vary in time so that \( J, H \) and \( B \) are constant.
• Although charges are moving, the steady current maintains a constant charge density \( \rho \) in space and the electric field \( E \) is static.
Maxwell’s Equations

\[ \nabla \cdot \mathbf{D} = \rho_v, \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \]

\[ \nabla \cdot \mathbf{B} = 0, \]

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \]

\[ \mathbf{D} = \varepsilon \mathbf{E} \]

\[ \mathbf{B} = \mu \mathbf{H} \]
Static Electric Fields (Electrostatics)

Maxwell’s equations reduce to

\[ \nabla \cdot \mathbf{D} = \rho_v, \]
\[ \nabla \times \mathbf{E} = 0. \]

\[ \mathbf{D} = \varepsilon \mathbf{E} \]

The main goal of electrostatics is to find the electric field \( \mathbf{E} \) and the associated electric flux density \( \mathbf{D} \) induced by a specified distribution of charge.
Coulomb’s Law – An isolated charge $Q$ induces an electric field at points of distance $R$ given by

$$E = \hat{R} \frac{Q}{4\pi \varepsilon R^2} \quad (V/m)$$

Radial Symmetry - Field does not depend on direction but only on distance.
If the charge is at the origin and in vacuum

\[ E(r) = \frac{Q}{4\pi \varepsilon_0 r^2} \hat{r} \]

where \( \mathbf{r} \) is the position vector

\[ \varepsilon_0 = 8.854188 \times 10^{-12} \, \text{F/m} \]

\[ r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \]

The field exerts a force

\[ \mathbf{F} = qE(r) \]

on a test charge \( q \) at distance \( r \)
Charge away from the origin

\[ E(r) = \frac{Q_1}{4\pi \varepsilon_0 |r - r_1|^2} \frac{r - r_1}{|r - r_1|} = \frac{Q_1 (r - r_1)}{4\pi \varepsilon_0 |r - r_1|^3} \]
Difference between vectors
Multiple charges (superposition of effects)

\[ E(r) = \sum_{n} \frac{Q_n \ (r - r_n)}{4\pi \varepsilon_o |r - r_n|^3} \]
Example

Two point charges with $Q_1 = 2 \times 10^{-5}$ C and $Q_2 = -4 \times 10^{-5}$ C are located in free space at points with Cartesian coordinates $(1, 3, -1)$ and $(-3, 1, -2)$, respectively.

Find:

(a) the electric field at $(3, 1, -2)$ and
(b) the force on a charge $q = 8 \times 10^{-5}$ C, located at that point.

All distances are in meters.
\( \mathbf{E} = \frac{1}{4\pi \varepsilon_0} \left[ Q_1 \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^3} + Q_2 \frac{\mathbf{r} - \mathbf{r}_2}{|\mathbf{r} - \mathbf{r}_2|^3} \right] \)

**Location vectors**

\[
\begin{align*}
\mathbf{r}_1 &= \hat{x} + 3\hat{y} - \hat{z} \\
\mathbf{r}_2 &= -3\hat{x} + \hat{y} - 2\hat{z} \\
\mathbf{r} &= 3\hat{x} + \hat{y} - 2\hat{z}
\end{align*}
\]

**Distances from probe charge**

\[
\begin{align*}
|\mathbf{r} - \mathbf{r}_1|^3 &= (\sqrt{4 + 4 + 1})^3 = 27 \\
|\mathbf{r} - \mathbf{r}_2|^3 &= (6)^3 = 216
\end{align*}
\]

\[
\begin{align*}
\mathbf{E} &= \frac{1}{4\pi \varepsilon_0} \left[ 2 \frac{(2\hat{x} - 2\hat{y} - \hat{z})}{27} - 4 \frac{6\hat{x}}{216} \right] \times 10^{-5} = \\
&= \frac{1}{4\pi \varepsilon_0} \left[ \frac{(4\hat{x} - 4\hat{y} - 2\hat{z})}{27} - \frac{3\hat{x}}{27} \right] \times 10^{-5} = \\
&= \frac{\hat{x} - 4\hat{y} - 2\hat{z}}{108\pi \varepsilon_0} \times 10^{-5} \text{ (V/m)}
\end{align*}
\]
(b)

Field
\[ \mathbf{E} = \frac{\hat{x} - 4\hat{y} - 2\hat{z}}{108\pi \varepsilon_0} \times 10^{-5} \text{ (V/m)} \]

Probe charge
\[ q = 8 \times 10^{-5} \text{ C} \]

Force on \( q \)
\[ \mathbf{F} = q\mathbf{E} = 8 \times 10^{-5} \times \frac{\hat{x} - 4\hat{y} - 2\hat{z}}{108\pi \varepsilon_0} \times 10^{-5} \text{ (V/m)} \]
\[ \mathbf{F} = \frac{2\hat{x} - 8\hat{y} - 4\hat{z}}{27\pi \varepsilon_0} \times 10^{-10} \text{ (N)} \]
Example

Two identical charges are located on the $x$ axis at

\[ x_1 = 3 \text{ m} \]
\[ x_2 = 7 \text{ m} \]

At what point in space is the net electric field zero?
Example
Two identical charges are located on the $x$ axis at

$x_1 = 3 \text{ m}$
$x_2 = 7 \text{ m}$

At what point in space is the net electric field zero?

Not considering the zero field solution at infinity, the only possibility for the field to be zero is on a $(y, z)$ plane where all points are equidistant from the two charges. By placing a probe charge $q$ on this plane, one can see that the only location with zero field is $x = 5$.  

Assume identical positive charges
Enclose a charge in a spherical surface of radius \( r \)

with surface area \( S = 4\pi r^2 \)

\[
E = \hat{R} \frac{Q}{4\pi \epsilon R^2} \quad \text{(V/m)}
\]

\[
E_r = \frac{Q}{4\pi \epsilon_0 r^2}
\]

\[
\epsilon_0 E_r S = Q \quad \rightarrow \quad \epsilon_0 \oint_S \mathbf{E} \cdot d\mathbf{S} = Q
\]

flux of \( \mathbf{E} \) out of surface \( S \)
NOTE:

For a spherical surface surrounding a charge at its center, the actual field at the surface is radial and therefore normal.

In the general case, the flux considers the normal components of the field at any points of the surface, that is the projection of the surface field onto the direction of the unit vector normal to the surface element.
For an arbitrary volume $V$ enclosed by surface $S$, which encloses an arbitrary net charge distribution $Q_V$ we have Gauss' Law

$$\int_S \mathbf{D} \cdot d\mathbf{S} = Q_V$$

where $\mathbf{D} = \epsilon_0 \mathbf{E}$ is the displacement field.

Gauss’ Law provides an alternative to Coulomb’s Law for computing static fields.
NOTE: Coulomb’s Law is only valid in static conditions (fixed charges).

“Information” on charge position (i.e., the field caused by the charge) can only be transmitted at the speed of light.

Also, charge movements create currents generating magnetic fields, which must be considered to solve the electromagnetic field problem.

On the other hand, Gauss’s law is valid in all conditions, since the surface integral will only change when charges get to cross it, at whatever speed they might be travelling.
Example

Use Gauss’s law to obtain an expression for $\mathbf{E}$ due to an infinitely long line with uniform charge density $\rho = \lambda \frac{C}{m}$ that resides along the $z$ axis in free space.
Solution using Gauss’ Law

Since the charge density along the line is uniform, infinite in extent and residing along the $z$ axis, for symmetry $\mathbf{D}$ is in the radial direction and is independent of $z$ or rotation angle $\varphi$.

The total charge contained in the cylinder is

$$Q_V = \rho L = \lambda L \ [C]$$

Top and bottom surfaces do not contribute to the integral for the flux, only the lateral cylindrical surface:

$$\int_{z=0}^{L} \int_{\varphi=0}^{2\pi} \hat{r} \mathbf{D}_r \cdot \hat{r} \ d\varphi \ dz = \rho L = \lambda L$$
Solution using Gauss’ Law

\[ \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \hat{\mathbf{r}} D_r \cdot \hat{\mathbf{r}} \, d\phi \, dz = \rho \, L = \lambda \, L \]

\[ 2\pi \, r \, L \, D_r = \rho \, L = \lambda \, L \]

\[ \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}} \frac{D_r}{\varepsilon_0} = \hat{\mathbf{r}} \frac{\lambda}{2\pi \varepsilon_0 r} \]
Solution using Coulomb’s Law

Distribute discrete charges $Q$ along the $z$-axis, spaced by a distance $\Delta z = Q/\lambda$, at locations $z = n\Delta z$, where $n$ is an integer.

$$E(r) = \sum_{n=-\infty}^{\infty} \frac{Q}{4\pi \epsilon_0 |r - \hat{z}n\Delta z|^2} = \sum_{n=-\infty}^{\infty} \frac{\lambda \Delta z (r - \hat{z}n\Delta z)}{4\pi \epsilon_0 |r - \hat{z}n\Delta z|^3}$$

For $r \gg \Delta z$ (macroscopic limit)

$$E = \hat{r} \int_{-\infty}^{\infty} \frac{\lambda r}{4\pi \epsilon_0 (r^2 + z^2)^{3/2}} dz = \hat{r} \frac{\lambda}{2\pi \epsilon_0 r}$$

29 microscopic field

macroscopic field
Example

Consider a sphere of radius $r_0$ filled with a space-charge cloud of positive uniform density $\rho$. Find the electric field inside and outside the sphere. Assume free space throughout.
Solution

Total charge contained by the sphere $Q = \rho \frac{4}{3} \pi r_0^3$

Outside the sphere ($r > r_0$), the electric field is the same as the field of a point charge $Q$

$$E(r \geq r_0) = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{\rho}{3\varepsilon_0} \frac{r_0^3}{r^2}$$

(decay like $1/r^2$)

and on the surface of the sphere

$$E(r = r_0) = \frac{\rho}{3\varepsilon_0} r_0$$
Inside the sphere \((r < r_0)\), the enclosed charge grows with increasing \(r\)

\[
Q(r \leq r_0) = \frac{4\pi\rho}{3} r^3
\]

The field is obtained from

\[
\varepsilon_0 E(r \leq r_0) 4\pi r^2 = \frac{4\pi\rho}{3} r^3
\]

\[
E(r \leq r_0) = \frac{\rho}{3\varepsilon_0} r \quad \text{(linear growth)}
\]

As before, at the surface we have

\[
E(r = r_0) = \frac{\rho}{3\varepsilon_0} r_0
\]