ECE 329 – Fall 2022

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

Lecture 4

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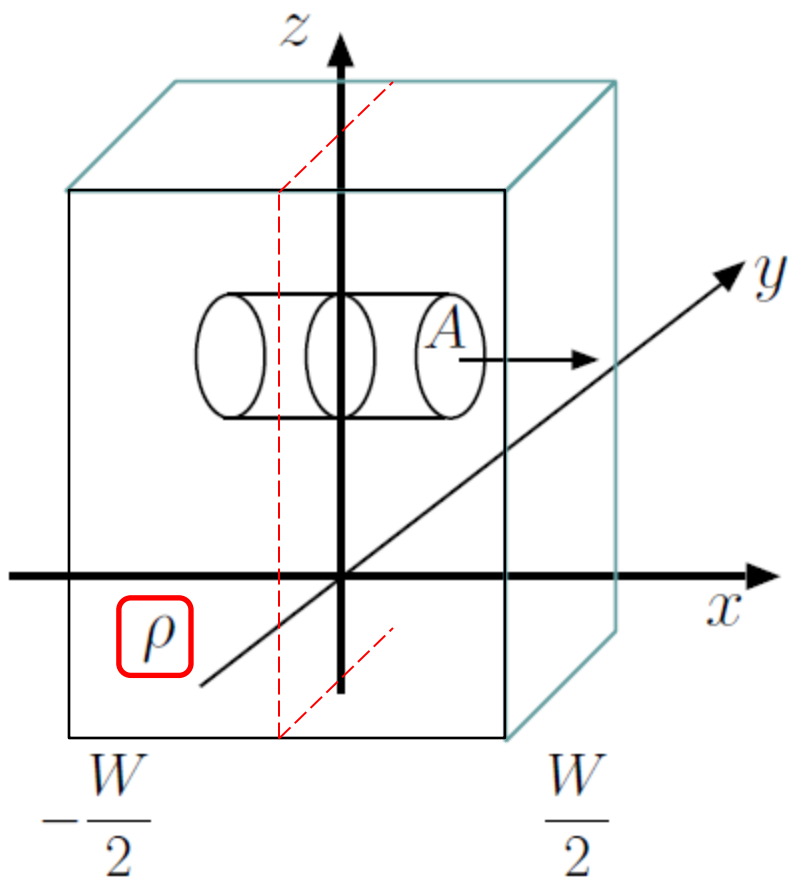
- **Complete examples on Field calculations**
- **Differential Operators in Electromagnetics**
  - **Divergence**
  - **Curl**

Example – Charged Slab

Now stretch the plane into an **infinite** slab of width W with charge density $\rho\,[\mathrm{C/m^3}]$. Determine the macroscopic electric field $\mathbf{E}$ inside and outside the slab.



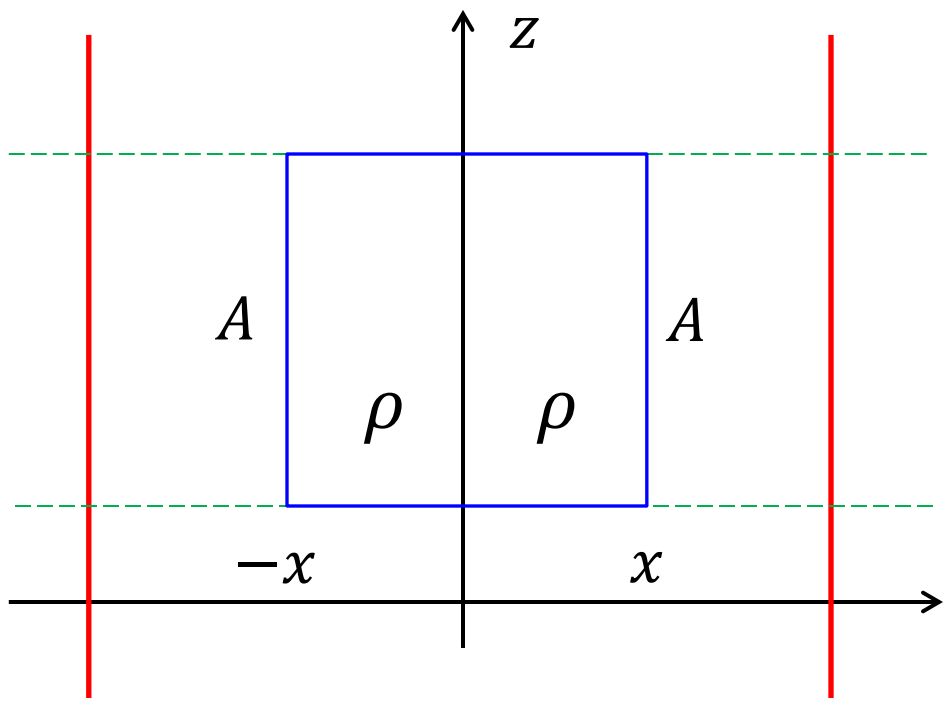
Again, from symmetry the field is of the form

$$\mathbf{E} = \hat{x}E_x(x)$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V$$

$$\epsilon_o E_x(x) A - \epsilon_o E_x(-x) A = \rho 2 x A$$

$$\boxed{\text{with } E_x(-x) = -E_x(x)}$$



Inside the slab

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V$$

$$\epsilon_o E_x(x) A - \epsilon_o E_x(-x) A = \rho 2 x A$$

$$\boxed{\text{with } E_x(-x) = -E_x(x)}$$

$$E_x(x) = \frac{\rho x}{\epsilon_o} \qquad \text{for } 0 < x < \frac{W}{2}$$

x **is positive**
Field is positive

$$E_x(x) = \frac{\rho x}{\epsilon_o} \qquad \text{for } -\frac{W}{2} < x < 0$$

x **is negative**
Field is negative

Outside the slab

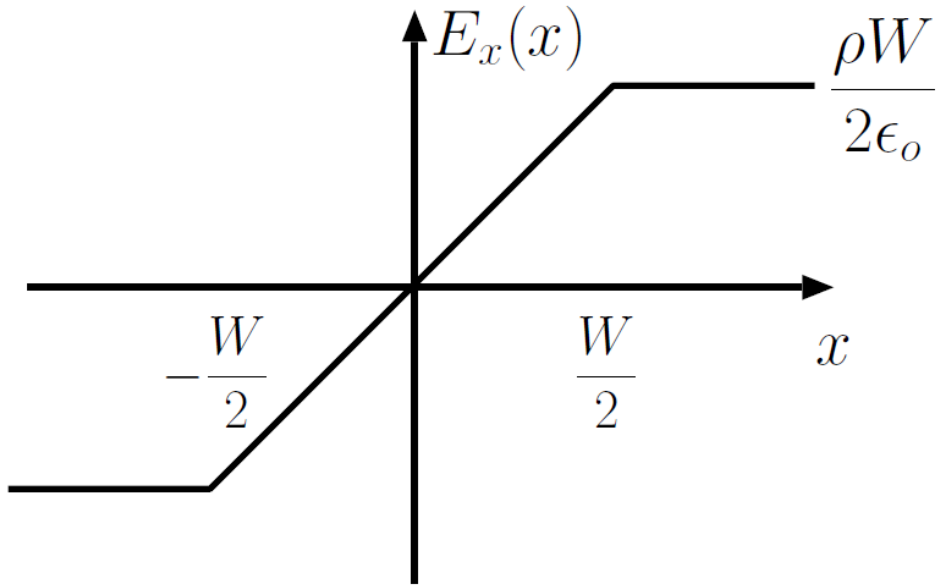
$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V$$

$$\epsilon_o E_x(x) A - \epsilon_o E_x(-x) A = AW\rho$$

$$\boxed{\text{with } E_x(-x) = -E_x(x)}$$

$$E_x(x) = -\frac{\rho\, W}{2\epsilon_o} \qquad \text{for} \qquad x < -\frac{W}{2}$$

$$E_x(x) = \frac{\rho\, W}{2\epsilon_o} \qquad \text{for} \qquad x > \frac{W}{2}$$



$$\mathbf{E} = \hat{x}E_x(x) = \begin{cases} -\hat{x}\frac{\rho W}{2\epsilon_o} & \text{for } x < -\frac{W}{2} \\ \hat{x}\frac{\rho x}{\epsilon_o} & \text{for } -\frac{W}{2} < x < \frac{W}{2} \\ \hat{x}\frac{\rho W}{2\epsilon_o} & \text{for } x > \frac{W}{2} \end{cases}$$

These elementary results can be used as building blocks for more complicated situations

- Coulomb field

$$\mathbf{E} = \hat{r}\frac{Q}{4\pi\epsilon_o r^2} \text{ of a point charge } Q,$$

- Field

$$\mathbf{E} = \hat{r}\frac{\lambda}{2\pi\epsilon_o r} \text{ of constant line density } \lambda,$$

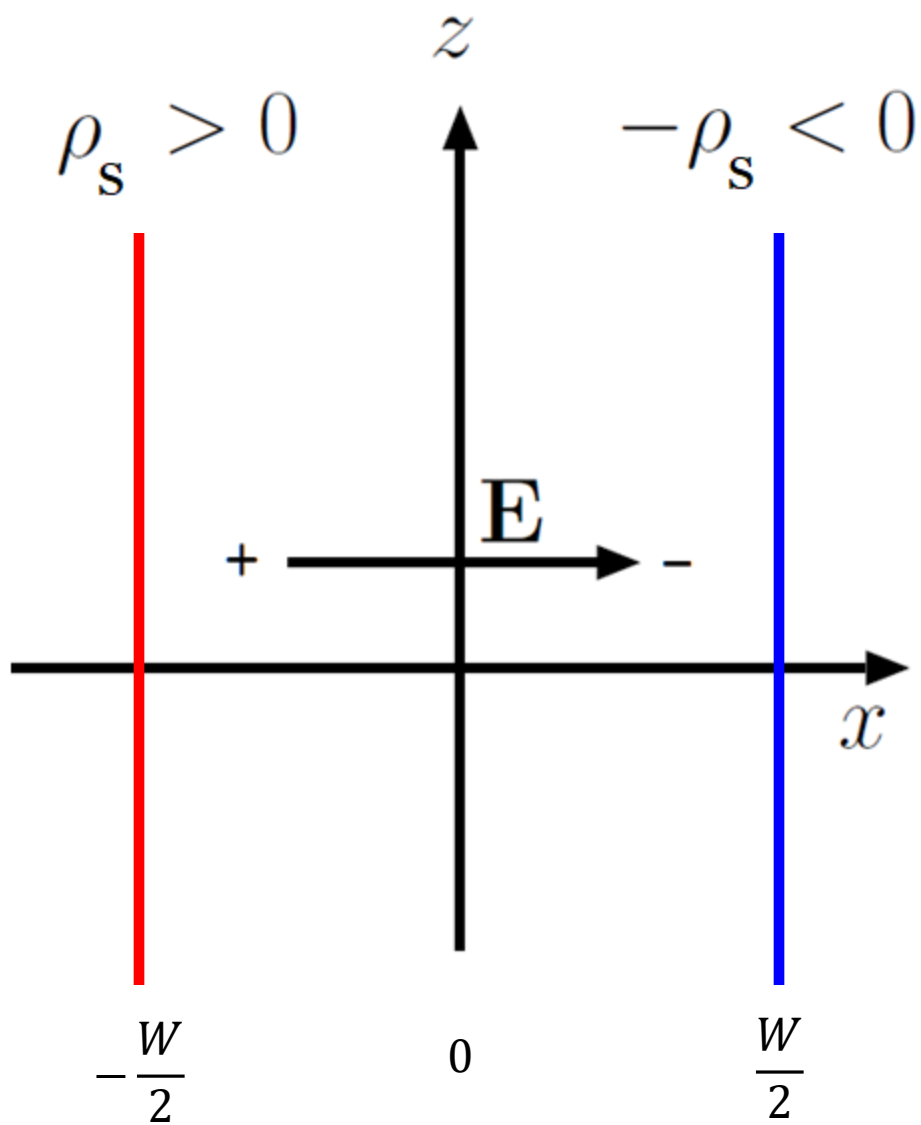
- Field

$$\mathbf{E} = \hat{x}\frac{\rho_s}{2\epsilon_o}\mathrm{sgn}(x) \text{ of constant surface density } \rho_s,$$

- Field

$$\mathbf{E} = \hat{x}\frac{\rho x}{\epsilon_o} \text{ of constant volume density } \rho$$

Example – Two planar surface charges



$$\mathbf{E}_+ = \hat{x}\frac{\rho_s}{2\epsilon_o}\mathrm{sgn}(x + \frac{W}{2})$$

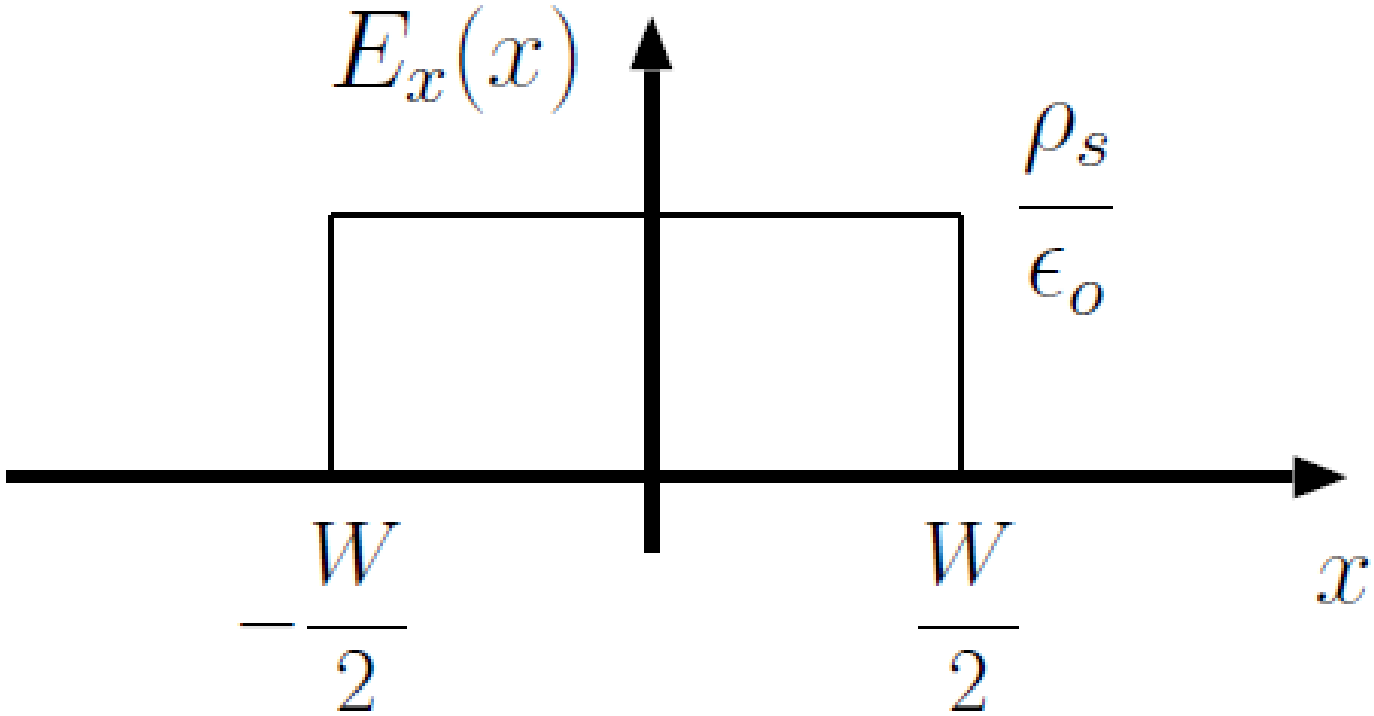
$$\mathbf{E}_- = -\hat{x}\frac{\rho_s}{2\epsilon_o}\mathrm{sgn}(x - \frac{W}{2})$$

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \hat{x}\frac{\rho_s}{\epsilon_o}\mathrm{rect}(\frac{x}{W})$$

$$= \begin{cases} \hat{x}\dfrac{\rho_s}{\epsilon_o} & -\frac{W}{2} < x < \frac{W}{2} \\ 0 & \text{otherwise} \end{cases}$$

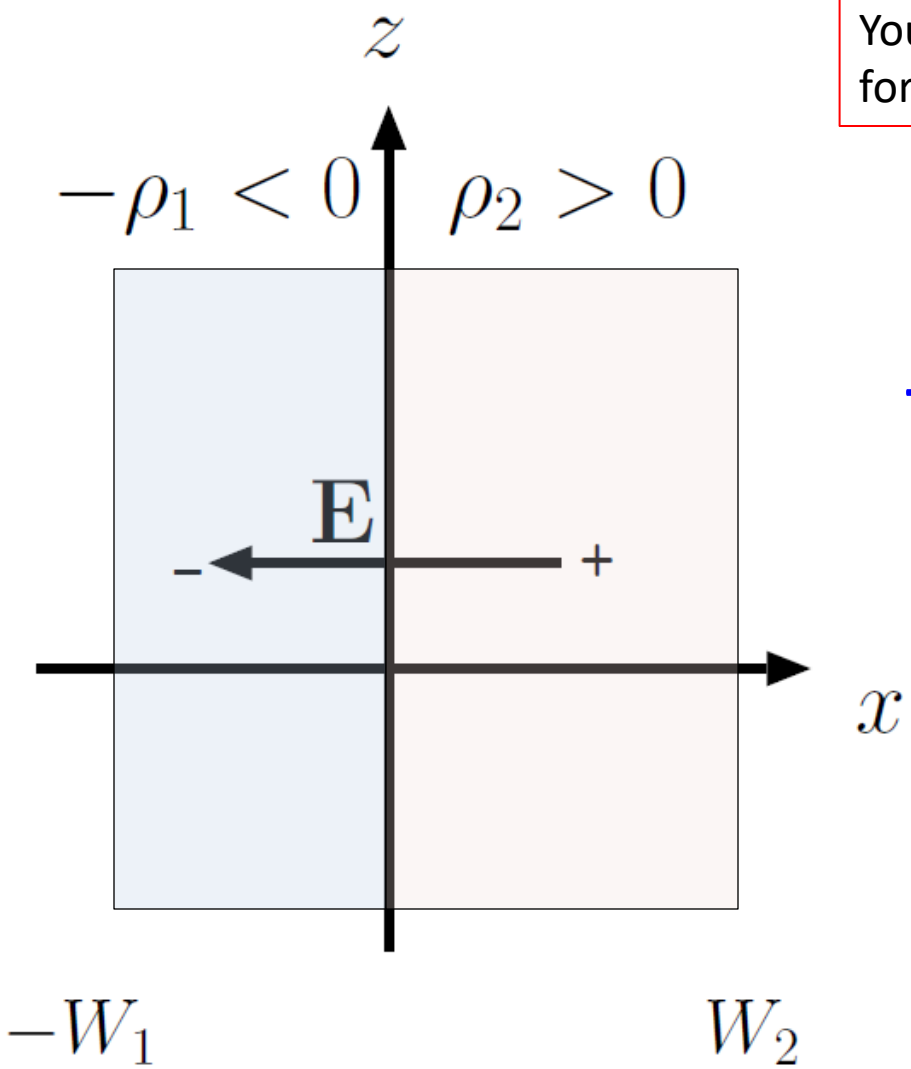
simple model of a parallel plate capacitor

Example – Two planar surface charges



$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- \quad = \hat{x}\frac{\rho_s}{\epsilon_o}\mathrm{rect}(\frac{x}{W}) \quad = \begin{cases} \hat{x}\,\frac{\rho_s}{\epsilon_o} & -\frac{W}{2} < x < \frac{W}{2} \\ 0 & \text{otherwise} \end{cases}$$

Example – Two Slabs with opposite volumetric charge

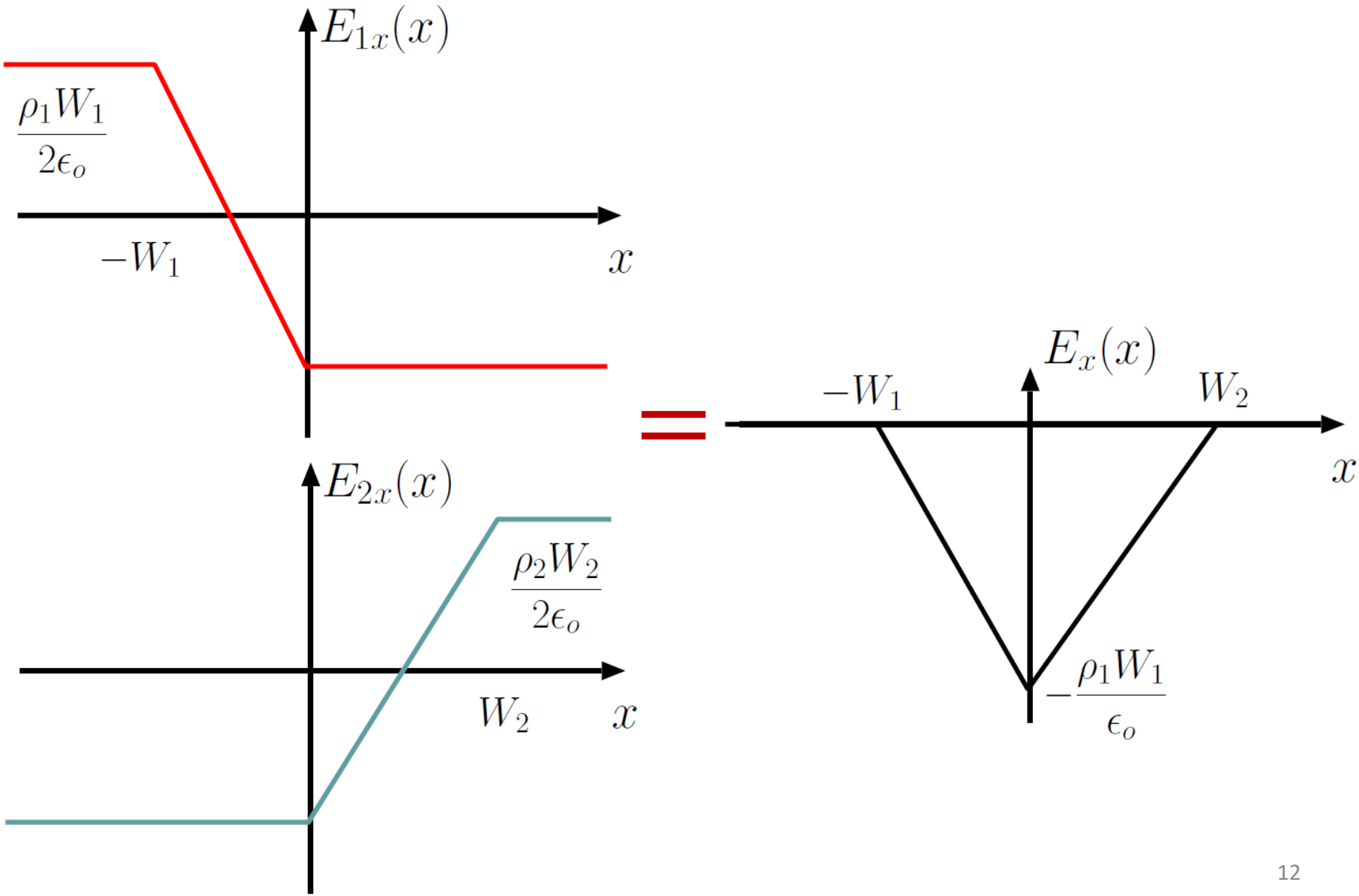


You will see in ECE 340 that this is a simple model for the active region of a p-n junction diode

The system is charge neutral when

$$W_1\rho_1 = W_2\rho_2$$

Graphically



Mathematically

$$\mathbf{E}_1 = \begin{cases} \hat{x}\frac{\rho_1 W_1}{2\epsilon_o}, & \text{for } x < -W_1 \\ -\hat{x}\frac{\rho_1(x+\frac{W_1}{2})}{\epsilon_o}, & \text{for } -W_1 < x < 0 \\ -\hat{x}\frac{\rho_1 W_1}{2\epsilon_o}, & \text{for } x > 0 \end{cases}$$

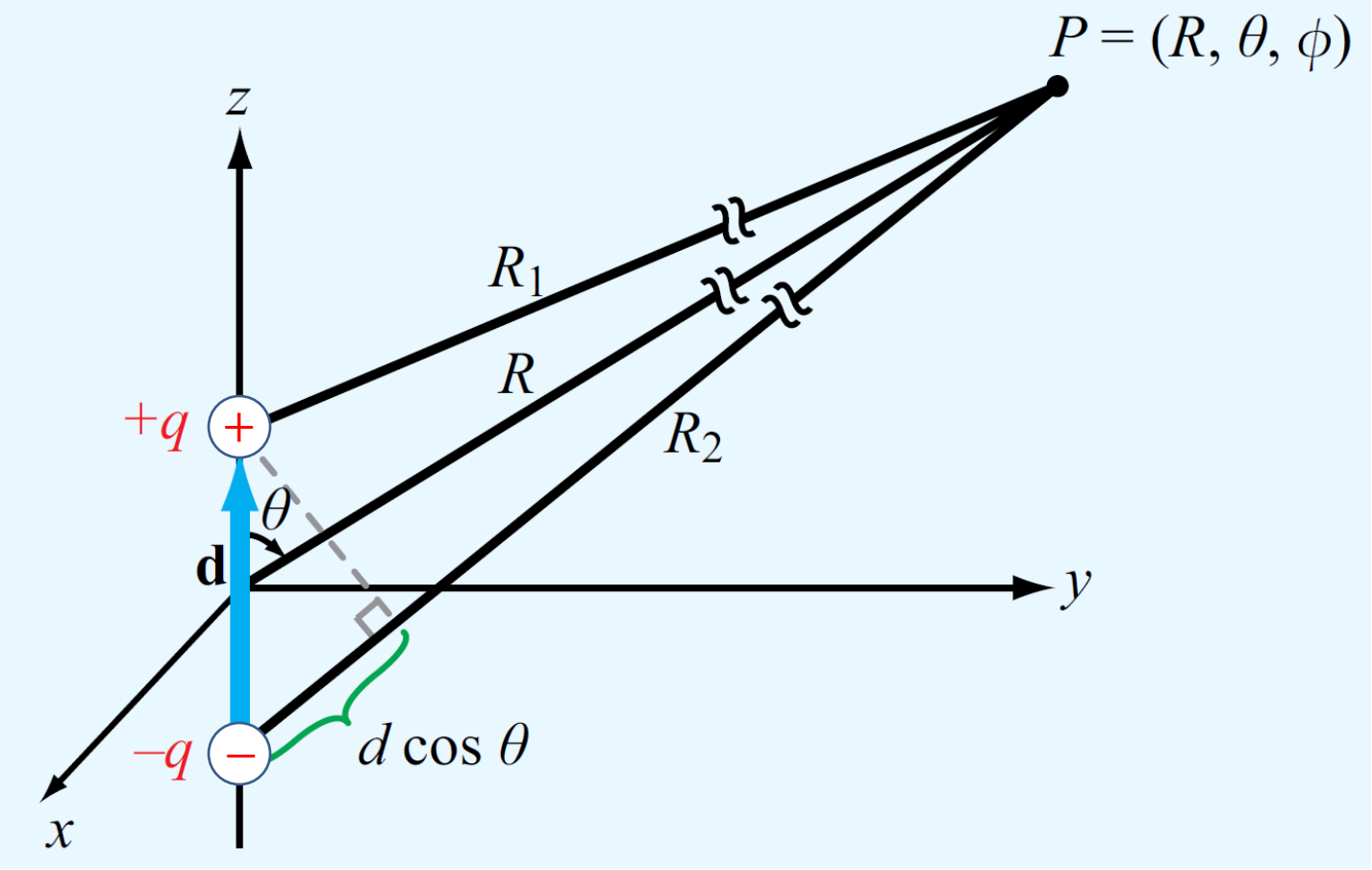
$$\mathbf{E}_2 = \begin{cases} -\hat{x}\frac{\rho_2 W_2}{2\epsilon_o}, & \text{for } x < 0 \\ \hat{x}\frac{\rho_2(x-\frac{W_2}{2})}{\epsilon_o}, & \text{for } 0 < x < W_2 \\ \hat{x}\frac{\rho_2 W_2}{2\epsilon_o}, & \text{for } x > W_2. \end{cases}$$

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \begin{cases} -\hat{x}\frac{\rho_1(x+W_1)}{\epsilon_o}, & \text{for } -W_1 < x < 0 \\ \hat{x}\frac{\rho_2(x-W_2)}{\epsilon_o}, & \text{for } 0 < x < W_2 \\ 0, & \text{otherwise.} \end{cases}$$

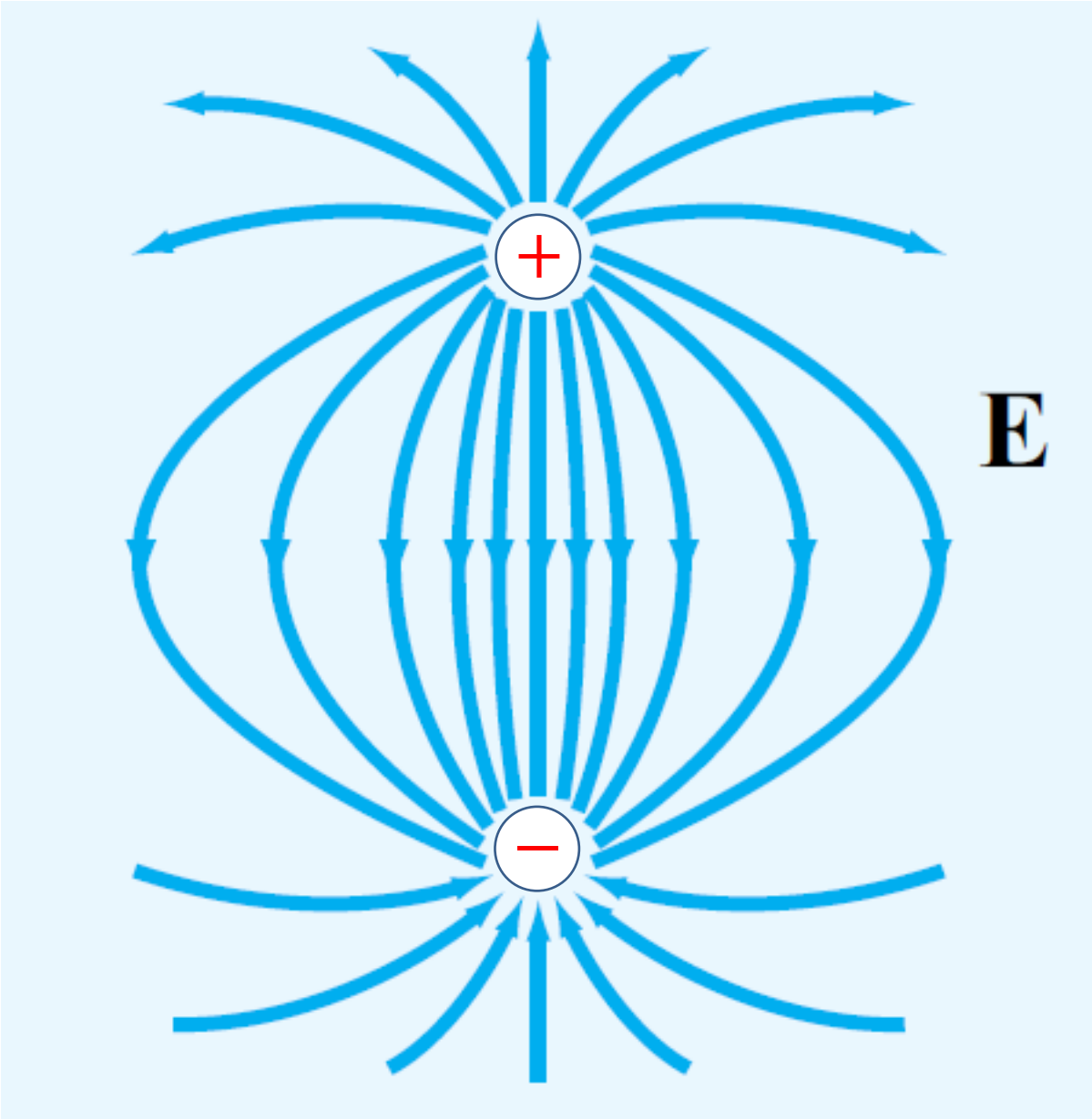
Fields cancel out outside the region $\boxed{-W_1 < x < W_2}$

Example – Electric Dipole

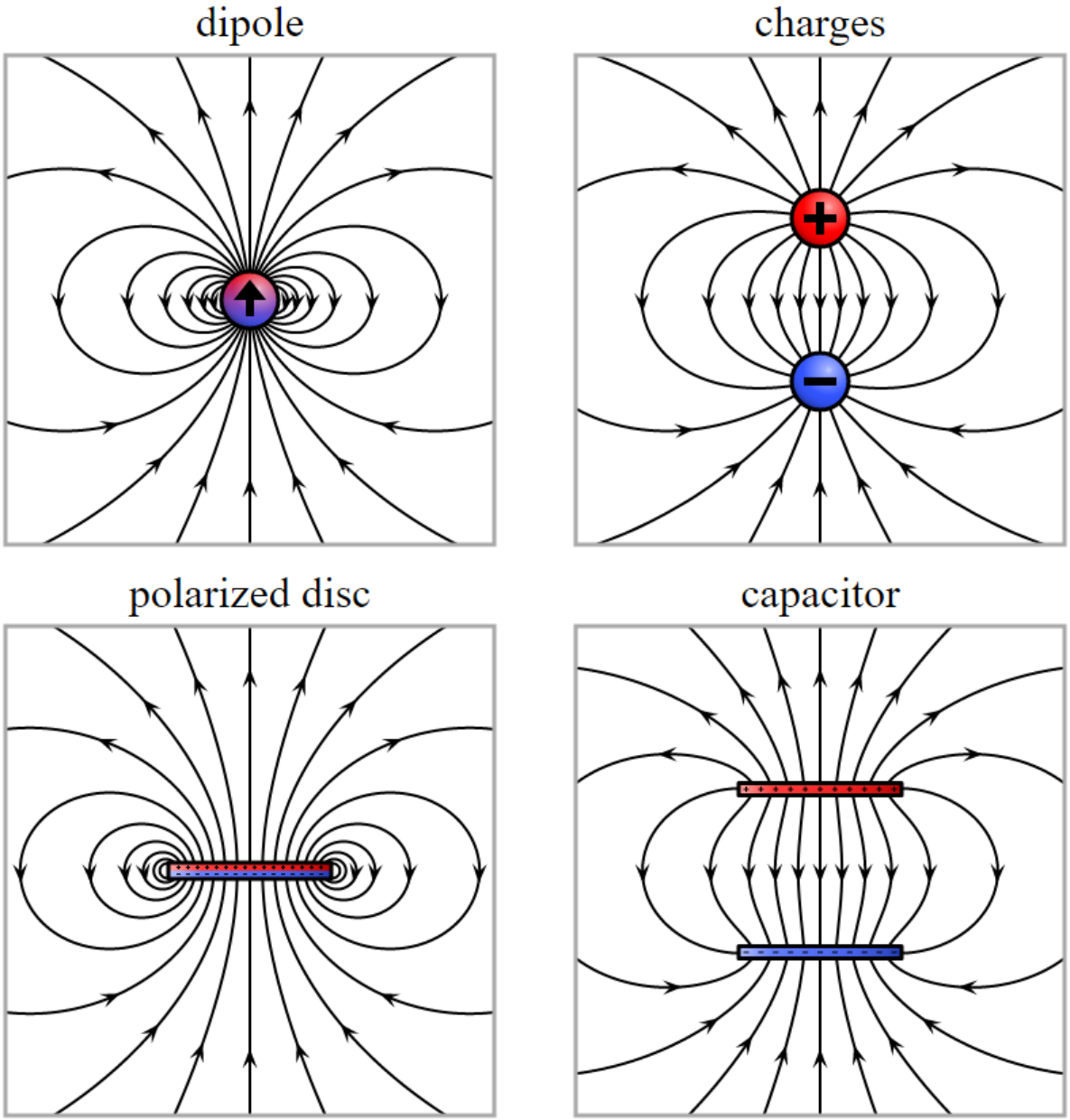
An ***electric dipole*** consists of two point charges of equal magnitude but opposite polarity, separated by a distance *d.*



Electric field pattern of the Electric Dipole

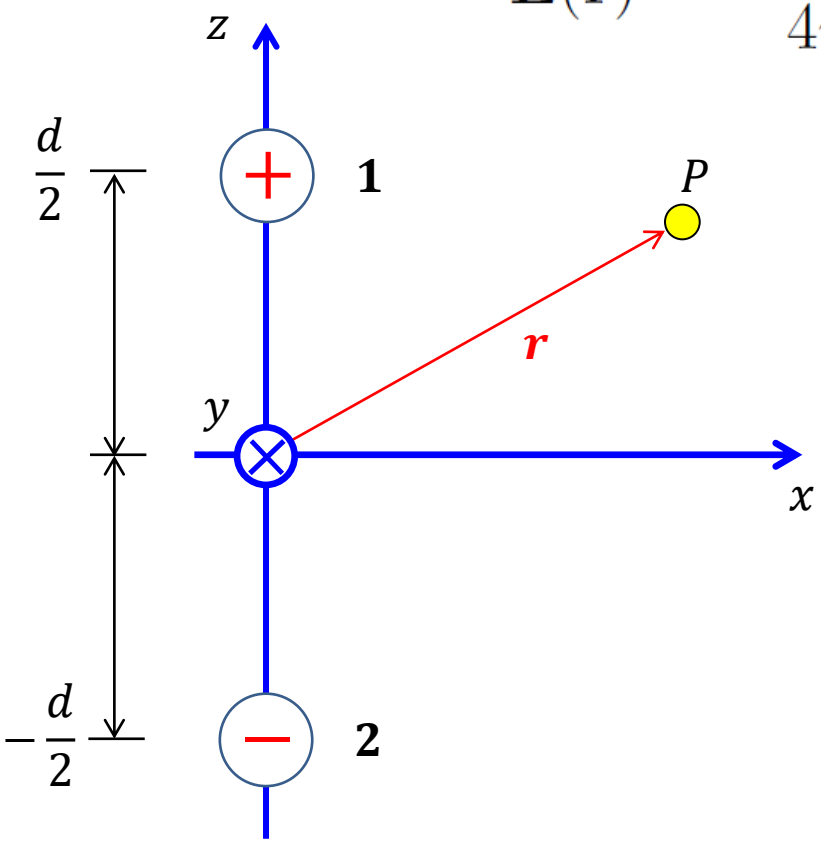


Examples of configurations acting as dipoles



Electric Field of the Electric Dipole from Coulomb's Law

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1(\mathbf{r}-\mathbf{r}_1)}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_1|^3} + \frac{Q_2(\mathbf{r}-\mathbf{r}_2)}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}_2|^3}$$

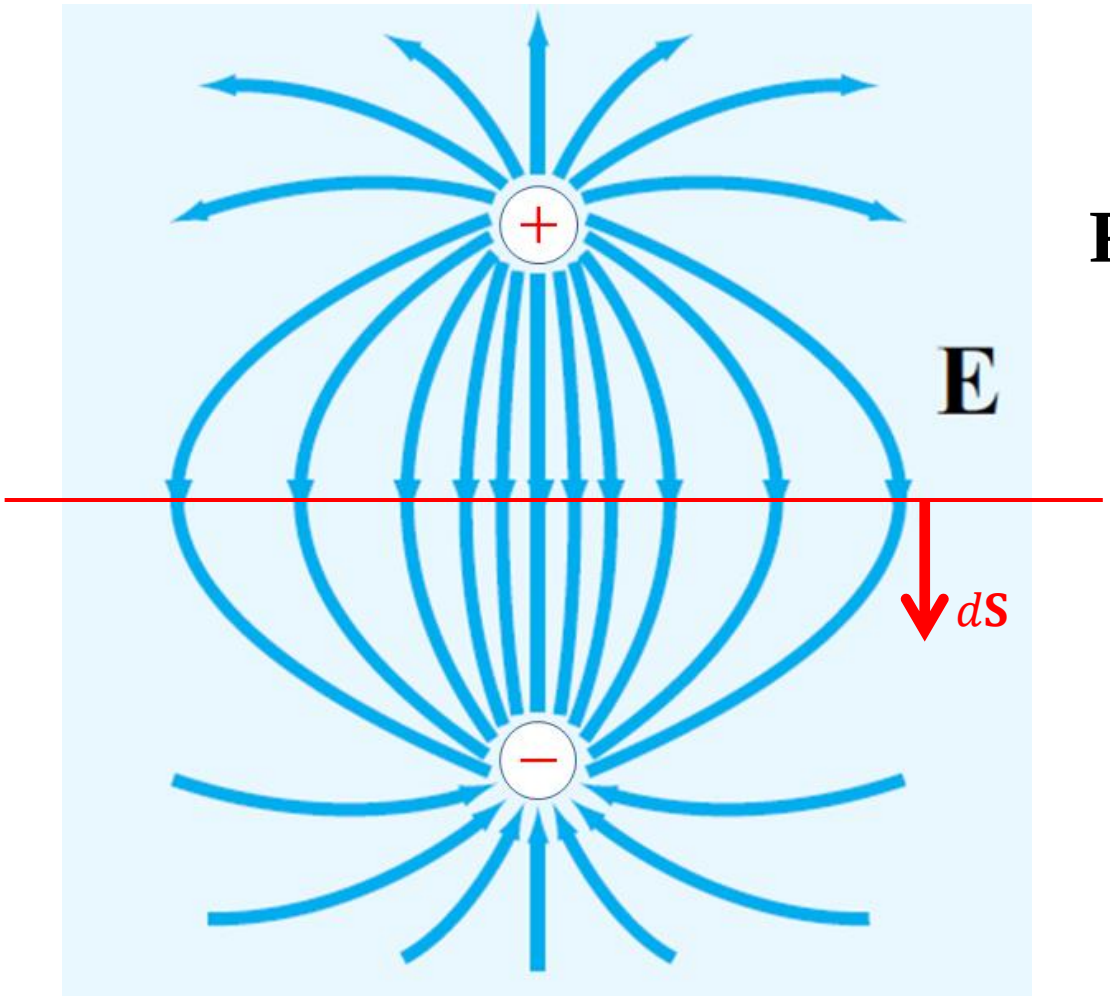


$$\boxed{\mathbf{E}_1(\mathbf{r}) = \frac{Q(\mathbf{r}-\frac{d}{2}\hat{z})}{4\pi\epsilon_o|\mathbf{r}-\frac{d}{2}\hat{z}|^3}}$$

$$\boxed{\mathbf{E}_2(\mathbf{r}) = \frac{-Q(\mathbf{r}+\frac{d}{2}\hat{z})}{4\pi\epsilon_o|\mathbf{r}+\frac{d}{2}\hat{z}|^3}}$$

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{Q(\mathbf{r}-\frac{d}{2}\hat{z})}{4\pi\epsilon_o|\mathbf{r}-\frac{d}{2}\hat{z}|^3} + \frac{-Q(\mathbf{r}+\frac{d}{2}\hat{z})}{4\pi\epsilon_o|\mathbf{r}+\frac{d}{2}\hat{z}|^3}}$$

Find the electric flux across the $\{x\text{-}y\}$ plane

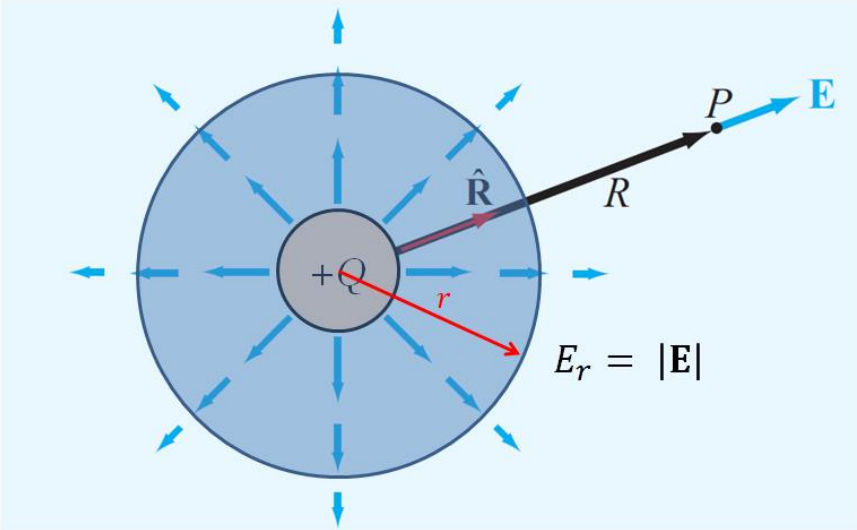


$$\textbf{Flux} = \int_{xy} \mathbf{E} \cdot d\mathbf{S}$$

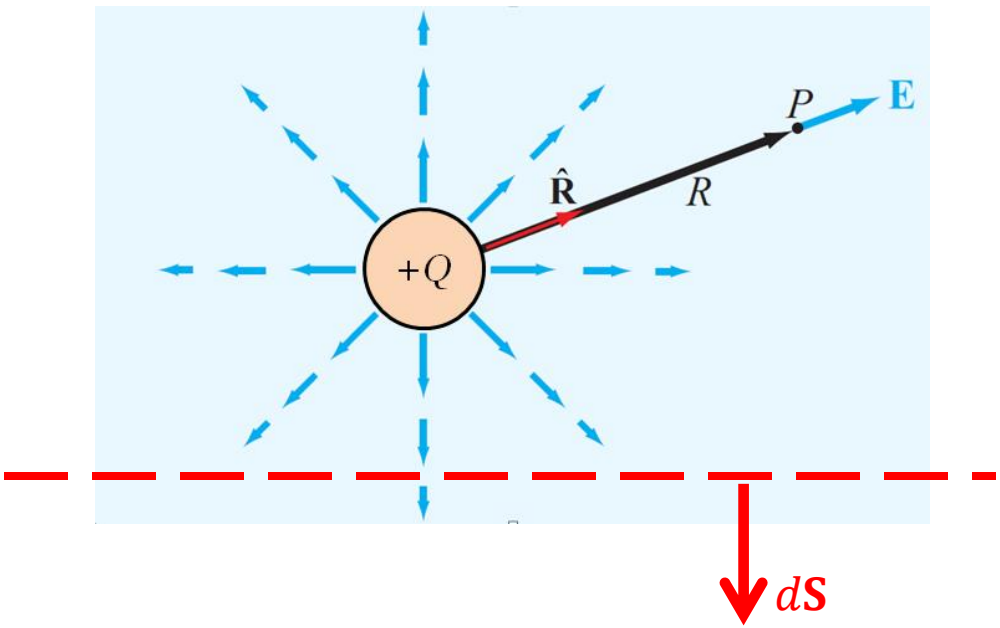
$$d\mathbf{S} = -\hat{z}dxdy$$

The total flux is the sum of the fluxes for the individual particles

Gauss' Law gives the flux for a closed surface enclosing the charge



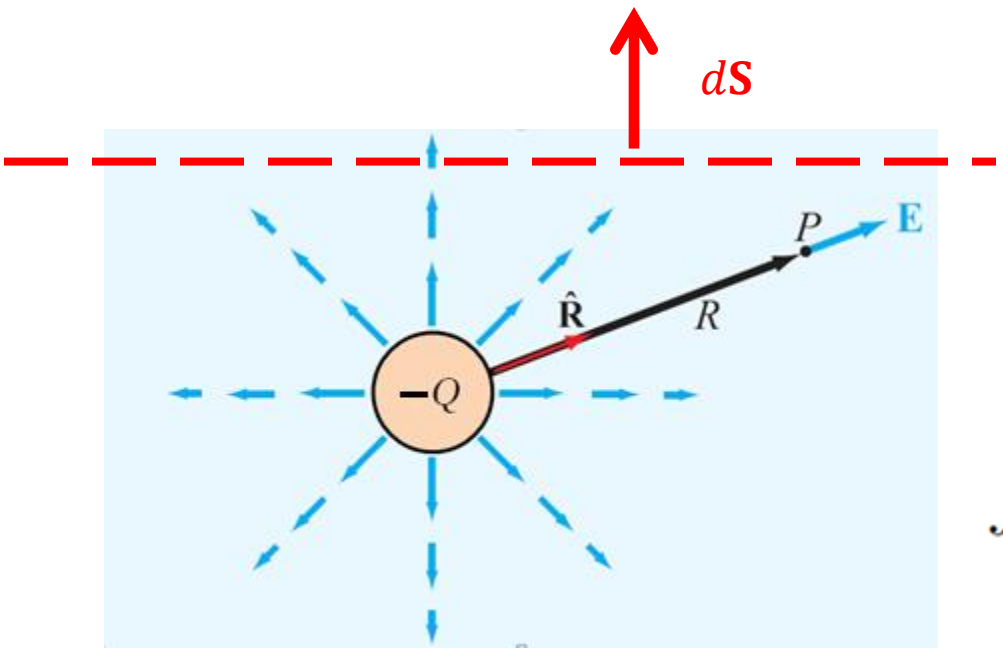
$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_o}$$



For our problem we can infer that half the flux crosses the $\{x\text{-}y\}$ plane and write

$$\int_{xy} \mathbf{E} \cdot (-\hat{z} dx dy) = \frac{Q}{2\epsilon_o}$$

Similarly, for the negative charge below the $\{x\text{-}y\}$ plane



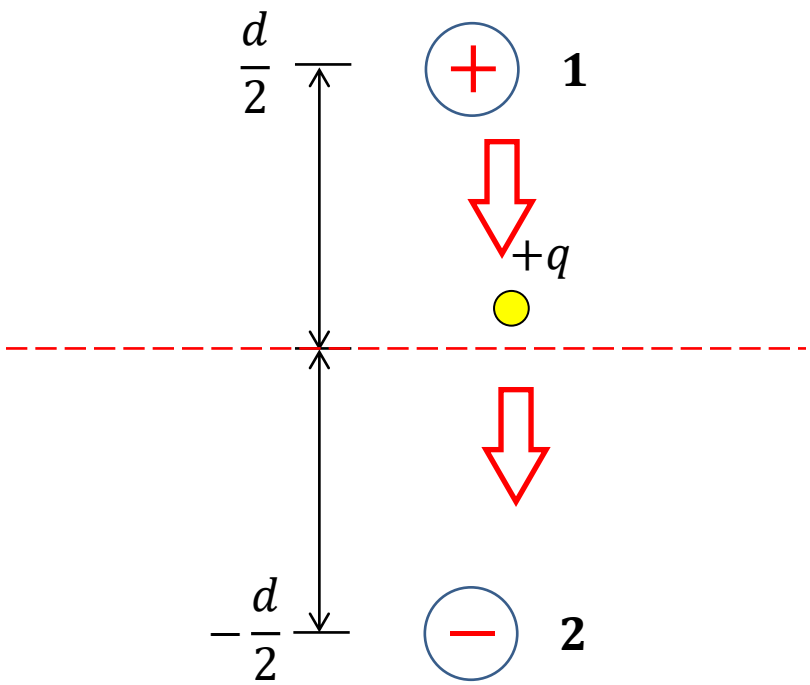
$$\oint_S \mathbf{E}_2 \cdot d\mathbf{S} = \frac{-Q}{\epsilon_o}$$

$$\int_{xy} \mathbf{E}_2 \cdot \underbrace{(\hat{z}dxdy)}_{\text{upward } d\mathbf{S}} = -\frac{Q}{2\epsilon_o}$$

Since we are looking for the total flux crossing the plane downward, also for the negative charge we get the flux

$$\int_{xy} \mathbf{E}_2 \cdot \underbrace{(-\hat{z}dxdy)}_{\text{downward } d\mathbf{S}} = \frac{Q}{2\epsilon_o}$$

In fact, if you place a positive probe charge in the middle of the dipole, the positive charge repels it downward and the negative charge attracts it downward



The total flux is obtained by superposition

$$\int_{xy} \mathbf{E} \cdot (-\hat{z} dx dy) = \frac{Q}{2\epsilon_o} + \frac{Q}{2\epsilon_o} = \frac{Q}{\epsilon_o}$$

Without taking shortcuts, one could simply solve the integral directly, which requires a change of variables (a lot more work!)

$$dxdy \longrightarrow rdrd\phi \quad \textbf{with} \quad r \equiv \sqrt{x^2+y^2}$$

$$\int_{xy} \mathbf{E}(x,y,0)\cdot(-\hat{z}dxdy) \;=$$

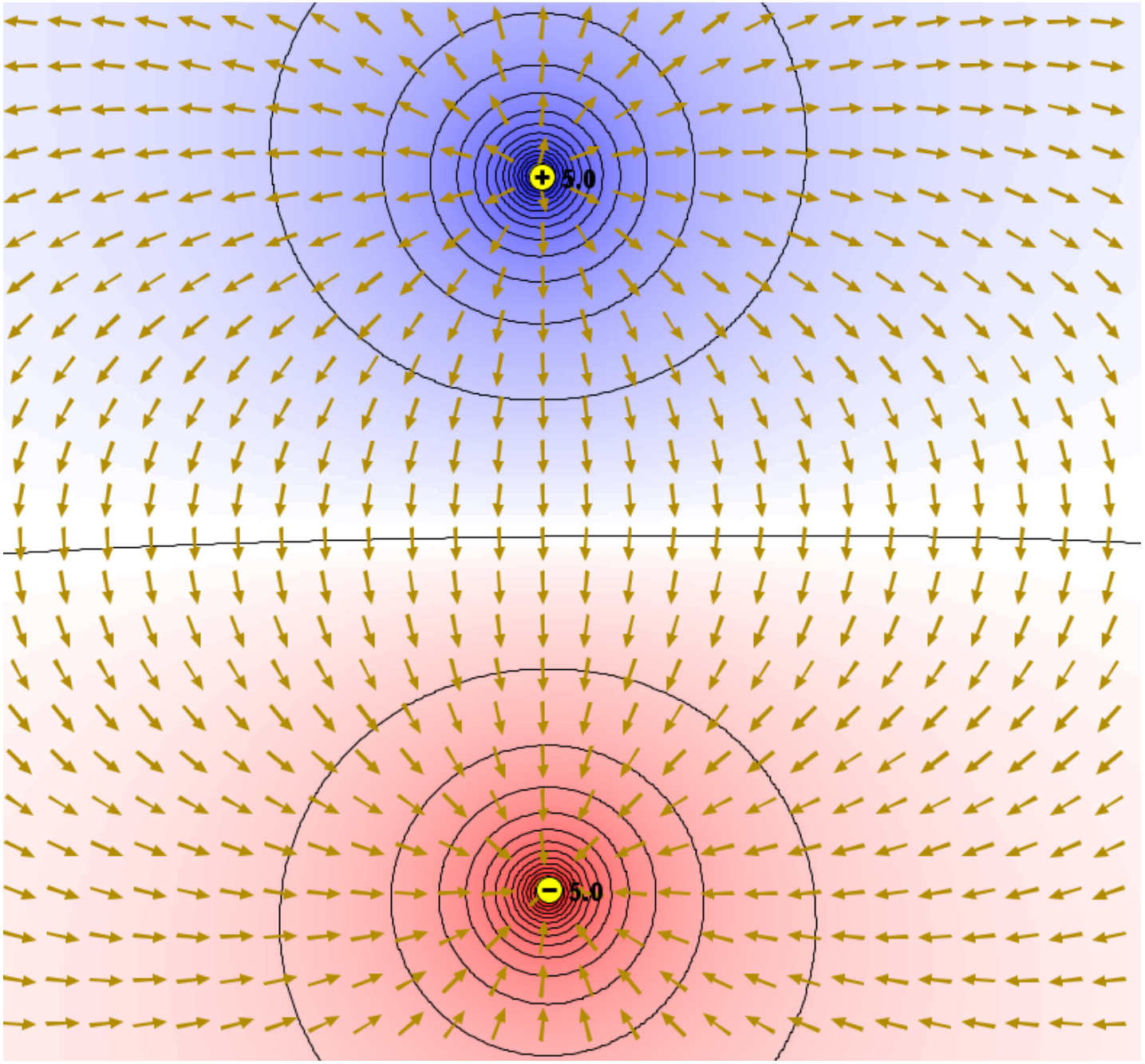
$$= \int_{xy} \frac{Q}{4\pi\epsilon_o}\left[\frac{(x,y,-\frac{d}{2})}{|(x,y,-\frac{d}{2})|^3} - \frac{(x,y,\frac{d}{2})}{|(x,y,\frac{d}{2})|^3}\right]\cdot(-\hat{z}dxdy)$$

$$= \frac{Q}{4\pi\epsilon_o}\int_{xy}\frac{d}{|(x,y,-\frac{d}{2})|^3}dxdy$$

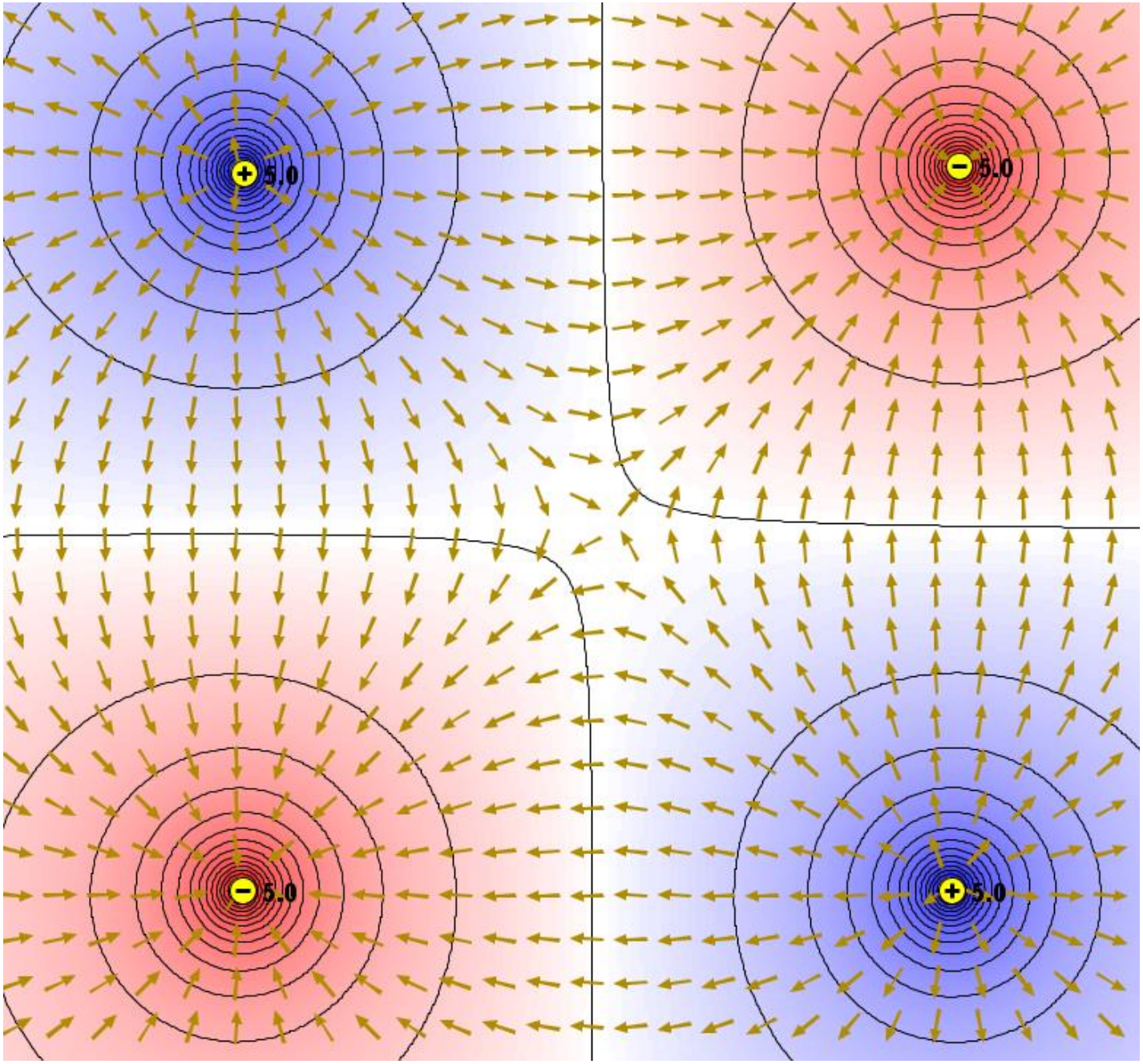
change variables and integrate over φ which just gives a factor of 2π and eliminates 2π from the denominator

$$= \frac{Qd}{2\epsilon_o}\int_{r=0}^{\infty}\frac{r}{(r^2+(\frac{d}{2})^2)^{3/2}}dr \;=\; \frac{Q}{\epsilon_o}$$

dipole potential and field



A quadrupole



Quadrupole ion trap for chromatography

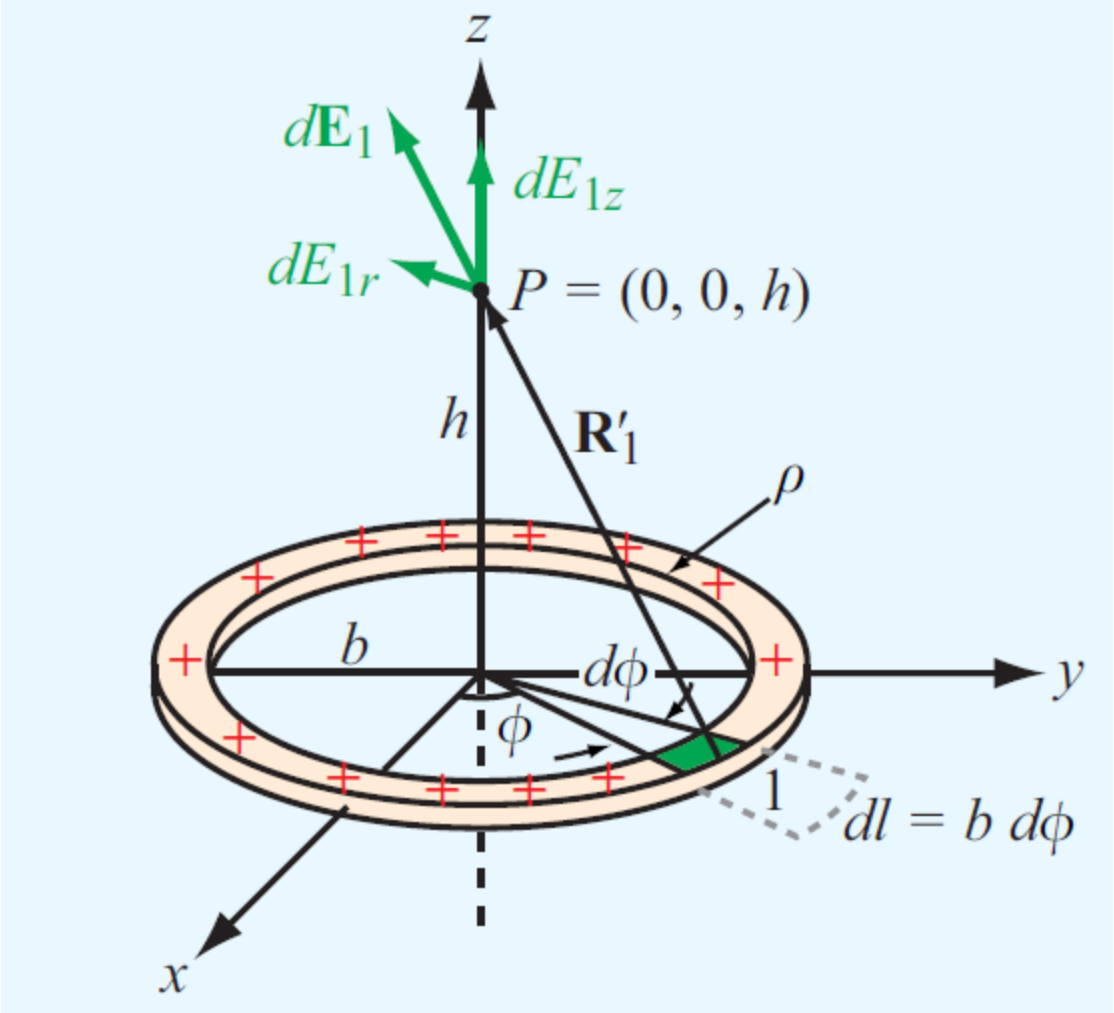
http://tools.thermofisher.com/content/sfs/brochures/WP-10427-GC-LC-MS-Workhorses-Quadrupoles-Ion-Traps-WP10427-EN.pdf

Example – Electric field of a Ring of Charge

The following example has been added for you to have extra practice material but we do not have time to explain it in class. Enjoy.

Example – Electric field of a Ring of Charge

We look for the Electric Field on points along the axis of the ring



segment length

$$dl = bd\varphi$$

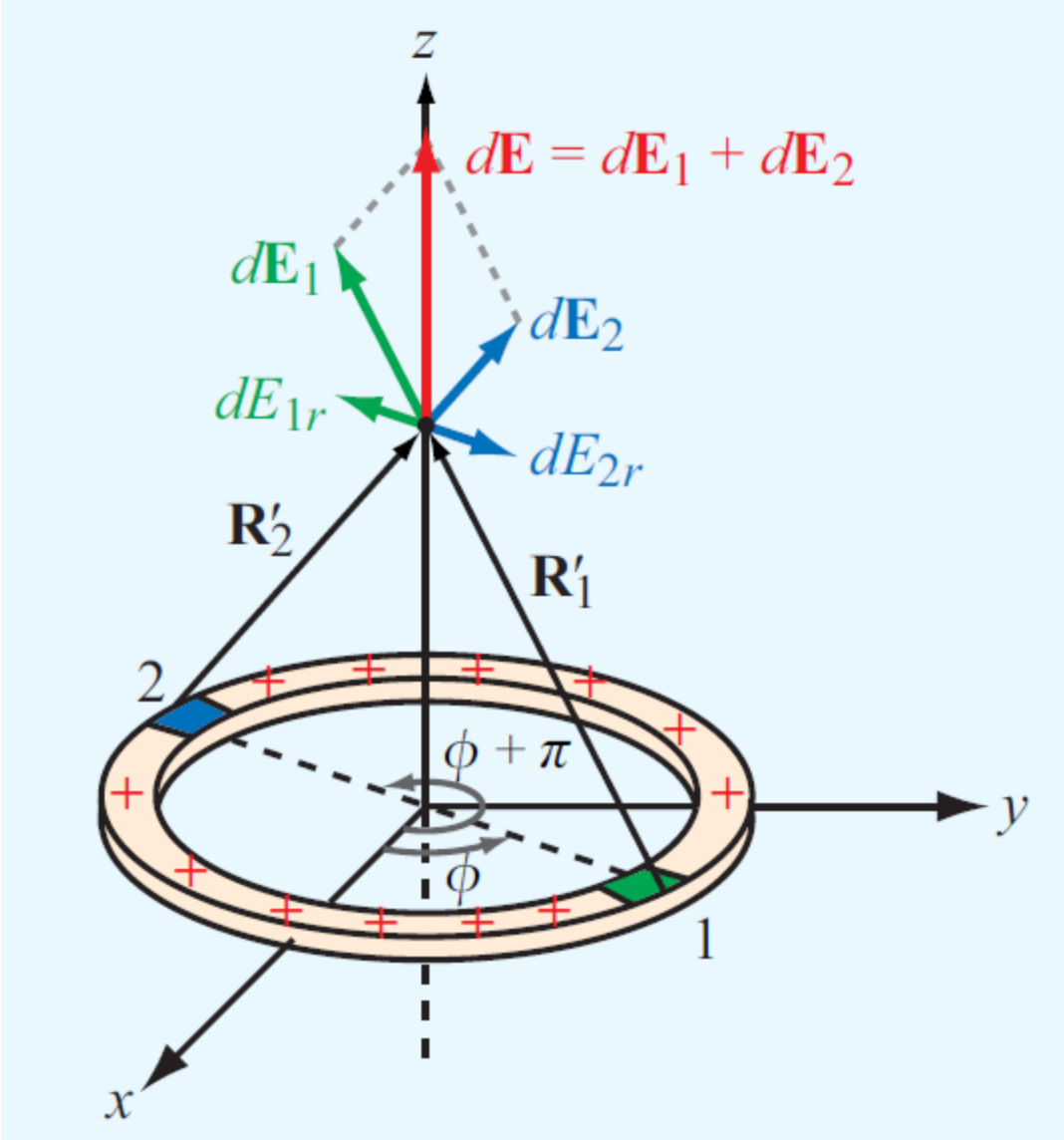
b = **radius**

charge in segment

$$dq = \rho dl = \rho b d\varphi$$

The field $d\mathbf{E}_1$ due to infinitesimal segment 1

Example – Electric field of a Ring of Charge



$$\mathbf{R}'_1 = -\hat{\mathbf{r}}b + \hat{\mathbf{z}}h$$

$$R'_1 = |\mathbf{R}'_1| = \sqrt{b^2 + h^2}$$

$$\hat{\mathbf{R}}'_1 = \frac{\mathbf{R}'_1}{|\mathbf{R}'_1|} = \frac{-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h}{\sqrt{b^2 + h^2}}$$

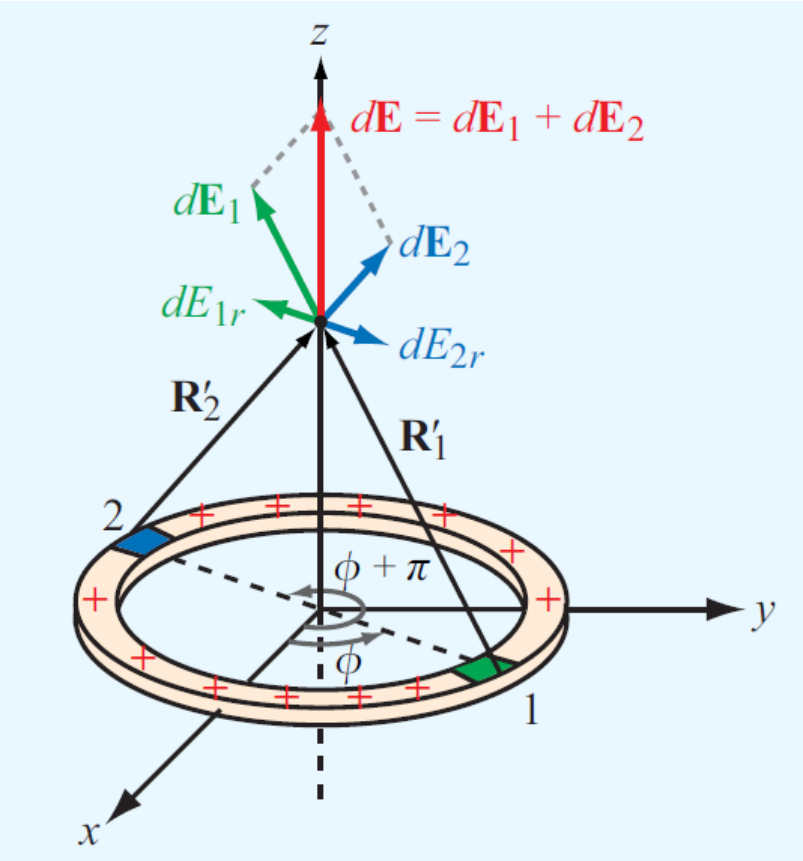
The fields $d\mathbf{E}_1$ and $d\mathbf{E}_2$ due to infinitesimal segments at diametrically opposite locations

Example – Electric field of a Ring of Charge

$$d\mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\,\hat{\mathbf{R}}_1'\,\frac{\rho\,dl}{R_1'^{\,2}} = \frac{\rho\,b}{4\pi\epsilon_0}\,\frac{(-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2+h^2)^{3/2}}\,d\phi$$

$$d\mathbf{E}_2 = \frac{1}{4\pi\epsilon_0}\,\hat{\mathbf{R}}_1'\,\frac{\rho\,dl}{R_1'^{\,2}} = \frac{\rho\,b}{4\pi\epsilon_0}\,\frac{(\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2+h^2)^{3/2}}\,d\phi$$

Coulomb's Law in differential form



The radial components are opposite and cancel out. The axial components add up.

$$d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2$$

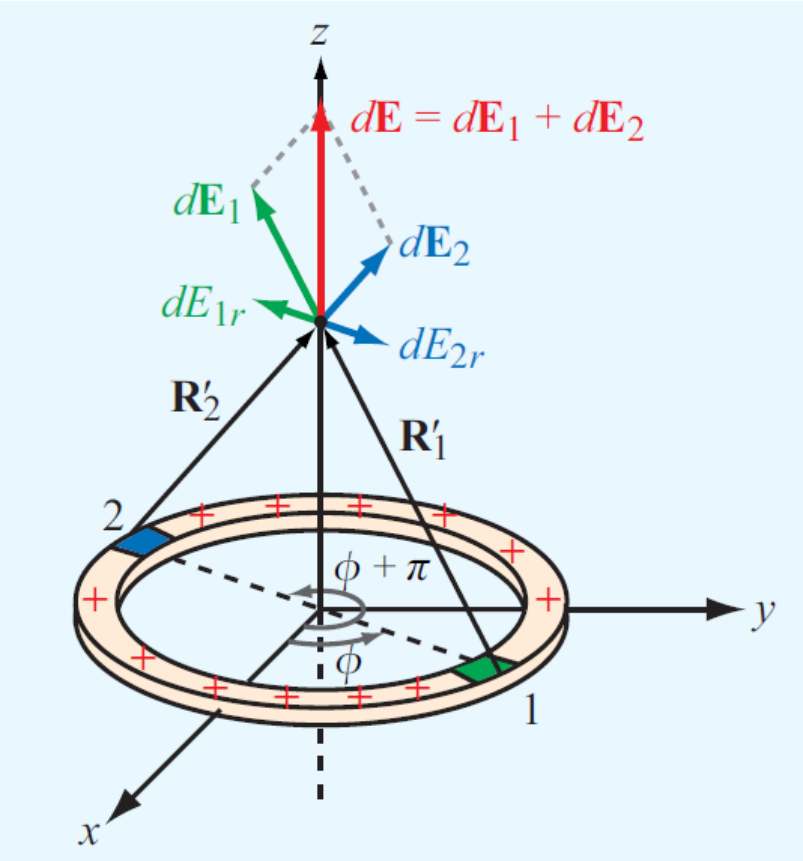
$$= \hat{\mathbf{z}}\,\frac{\rho\,bh}{2\pi\epsilon_0}\,\frac{d\phi}{(b^2+h^2)^{3/2}}$$

Example – Electric field of a Ring of Charge

The total field is obtained by integration in the range $0 \le \phi \le \pi$

total charge on the ring

$$Q = 2\pi b\rho$$



$$\mathbf{E} = \hat{\mathbf{z}}\,\frac{\rho\ bh}{2\pi\epsilon_0(b^2+h^2)^{3/2}}\int_0^{\pi} d\phi$$

$$= \hat{\mathbf{z}}\,\frac{\rho\ bh}{2\epsilon_0(b^2+h^2)^{3/2}}\ \boxed{\times\frac{2\pi}{2\pi}}$$

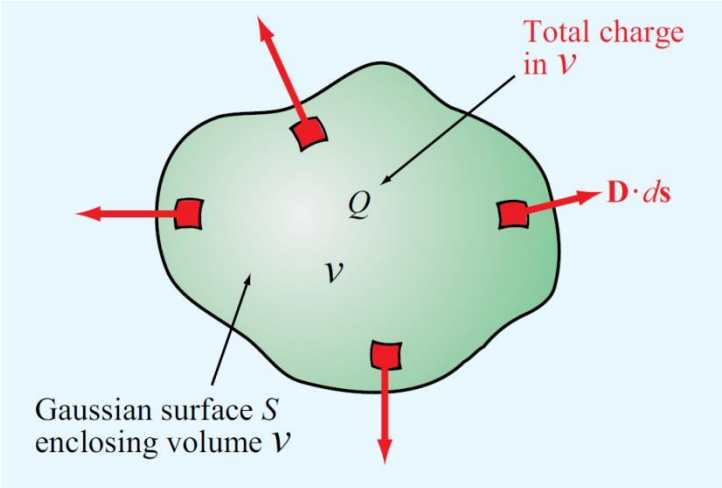
then multiply

$$= \hat{\mathbf{z}}\,\frac{h}{4\pi\epsilon_0(b^2+h^2)^{3/2}}\ Q,$$

Divergence of a Vector Field

We have introduced earlier the "integral" form of Gauss' Law

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V$$

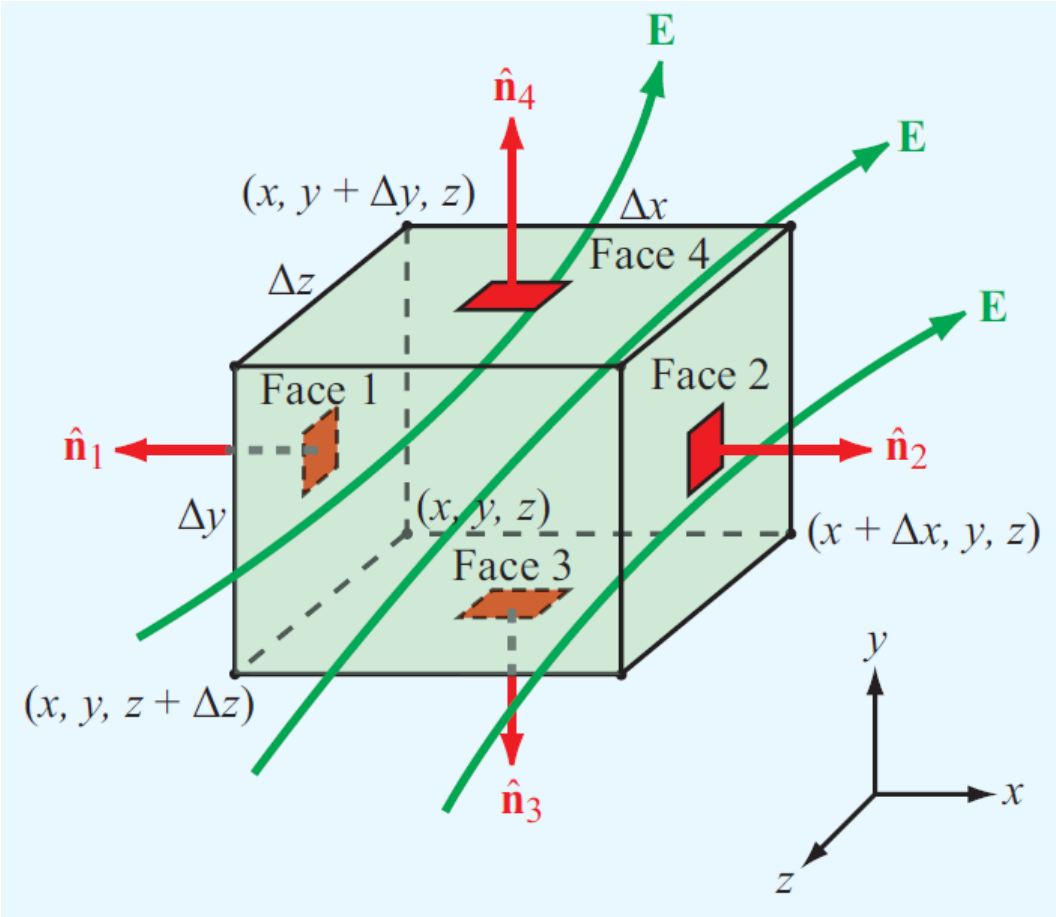


or $$\oint_S \mathbf{E} \cdot d\mathbf{S} = Q_V / \epsilon_0$$

where Q_V represents the integral of the charge contained in the volume delimited by the closed surface

$$Q_V = \int_V \rho \, dV$$

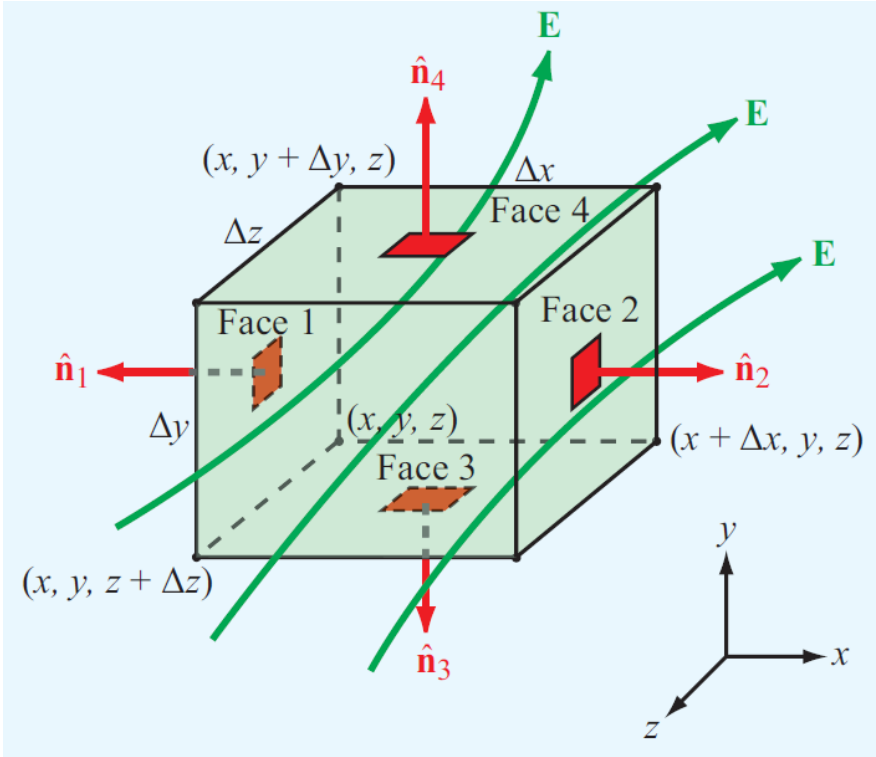
Now, consider instead a very small element of volume



NOTE: There are also Faces 5 and 6 for the flux but they are omitted in the drawing to avoid excessive clutter

$$\Delta V = \Delta x \Delta y \Delta z$$

$$\int_{\Delta V} \rho \, dV \approx \rho(x, y, z) \, \Delta x \Delta y \Delta z$$



For **Face 1** with area $\Delta y \Delta z$ and unit vector $\hat{\mathbf{n}}_1 = -\hat{\mathbf{x}}$

the outward flux F_1 is

$$F_1 = \int_{\text{Face 1}} \mathbf{D} \cdot \hat{\mathbf{n}}_1 \, ds = \int_{\text{Face 1}} \epsilon_0 \, \mathbf{E} \cdot \hat{\mathbf{n}}_1 \, ds$$

$$= \int_{\text{Face 1}} \epsilon_0 \, (\hat{\mathbf{x}} E_x + \hat{\mathbf{y}} E_y + \hat{\mathbf{z}} E_z) \cdot (-\hat{\mathbf{x}}) \, dy \, dz \approx -\epsilon_0 \, E_x(1) \, \Delta y \, \Delta z$$

Similarly, for **Face 2** with area $\Delta y \Delta z$ and unit vector $\hat{\mathbf{n}}_2 = \hat{\mathbf{x}}$

$$F_2 = \epsilon_0 \, E_x(2) \, \Delta y \, \Delta z$$

Since the two faces are separated by an infinitesimal distance Δx, ignoring terms $(\Delta x)^2$ and higher, we can also expand

$$E_x(2) = E_x(1) + \frac{\partial E_x}{\partial x} \Delta x$$

By combining the two equations above we obtain

$$F_2 = \epsilon_0 \left[E_x(1) + \frac{\partial E_x}{\partial x} \Delta x \right] \Delta y \, \Delta z$$

and with the previous result $F_1 = -\epsilon_0 \, E_x(1) \, \Delta y \, \Delta z$

$$\Rightarrow \quad \boxed{F_1 + F_2 = \epsilon_0 \frac{\partial E_x}{\partial x} \Delta x \, \Delta y \, \Delta z}$$

For the other two pairs of opposite faces

$$F_3 + F_4 = \epsilon_0 \frac{\partial E_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

$$F_5 + F_6 = \epsilon_0 \frac{\partial E_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

The total outward flux is the sum of these results

$$\oint_S \epsilon_0 \, \mathbf{E} \cdot d\mathbf{s} = F_1 + F_2 + F_3 + F_4 + F_5 + F_6$$

$$= \epsilon_0 \underbrace{\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right)}_{\text{divergence} \quad \nabla \cdot \mathbf{E}} \underbrace{\Delta x \, \Delta y \, \Delta z}_{\Delta V}$$

Remember that we have assumed a "very small" volume so, from the previous result, the divergence can be defined as a limit for $\Delta V \to 0$

$$\nabla \cdot \mathbf{E} \equiv \lim_{\Delta V \to 0} \frac{\oint_S \mathbf{E} \cdot d\mathbf{s}}{\Delta V} = \frac{\rho}{\epsilon_0}$$

This is Gauss' Law in differential form

If the net flux out of the surface S is positive, it is as if the volume ΔV contains a "***source***" of field lines.

If the net flux out of the surface S is negative, it is as if the volume ΔV contains a "***sink***" of field lines.

For a uniform field, the same amount of flux enters and leaves the volume, therefore, the divergence is zero. Divergence is always zero for fields not "generated" by sources (**divergence-free** or **divergence-less** fields).

We can extend the differential results to finite volumes and state the **Divergence Theorem**:

$$\int_V \nabla \cdot \mathbf{E}\ dV = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

As long as a relation like $\mathbf{D} = \epsilon \mathbf{E}$ holds, the same formulations used for the electric field $\mathbf{E}$ are also valid for the displacement vector $\mathbf{D}$, differing only by a multiplication constant ϵ.

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV \qquad \nabla \cdot \mathbf{D} = \rho$$

$$\int_V \nabla \cdot \mathbf{D}\, dV = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

Example – Determine the *Divergence* for the vector field given and calculate it at point P.

$$\mathbf{E} = \hat{\mathbf{x}}3x^2 + \hat{\mathbf{y}}2z + \hat{\mathbf{z}}x^2z$$

$$\boxed{P = (2, -2, 0)}$$

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\
&= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(x^2z) \\
&= 6x + 0 + x^2 \\
&= x^2 + 6x.
\end{aligned}$$

$$\boxed{\text{At } (2, -2, 0),\ \nabla \cdot \mathbf{E}\Big|_{(2,-2,0)} = 16.}$$

Exercise in class

Given

$$\mathbf{A} = e^{-2y}(\hat{\mathbf{x}}\sin 2x + \hat{\mathbf{y}}\cos 2x)$$

find the divergence $\nabla \cdot \mathbf{A}$

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find the divergence $\nabla \cdot \mathbf{A}$

$$A_x = e^{-2y}\sin 2x$$
$$A_y = e^{-2y}\cos 2x$$
$$A_z = 0$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
$$= e^{-2y}\,2\cos 2x - 2e^{-2y}\cos 2x + 0 = 0$$

Exercise in class

If $\mathbf{E} = \hat{\mathbf{r}}\ A_0\, r$ in spherical coordinates, where A_0 is a constant, calculate the flux of $\mathbf{E}$ through a spherical surface of radius R centered at the origin.

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If $\mathbf{E} = \hat{\mathbf{r}}\ A_0\, r$ in spherical coordinates, where A_0 is a constant, calculate the flux of $\mathbf{E}$ through a spherical surface of radius R centered at the origin.

For $r = R$ the field is $\mathbf{E} = \hat{\mathbf{r}}\ A_0\, R$ and it is uniform and normal to the sphere. So the flux is simply equal to the magnitude of the field multiplied by the surface area of the sphere.

$$\iint_S \mathbf{E} \cdot d\mathbf{s} = 4\pi R^2 \times A_0 R = 4\pi A_0 R^3$$

Statement to remember

The <u>Divergence</u> differential operator is always applied to a <u>vector</u> and the result is a <u>scalar</u>.

For comparison, the <u>Gradient</u> operates on <u>scalars</u> and the result is a <u>vector</u>.

Curl of a Vector Field

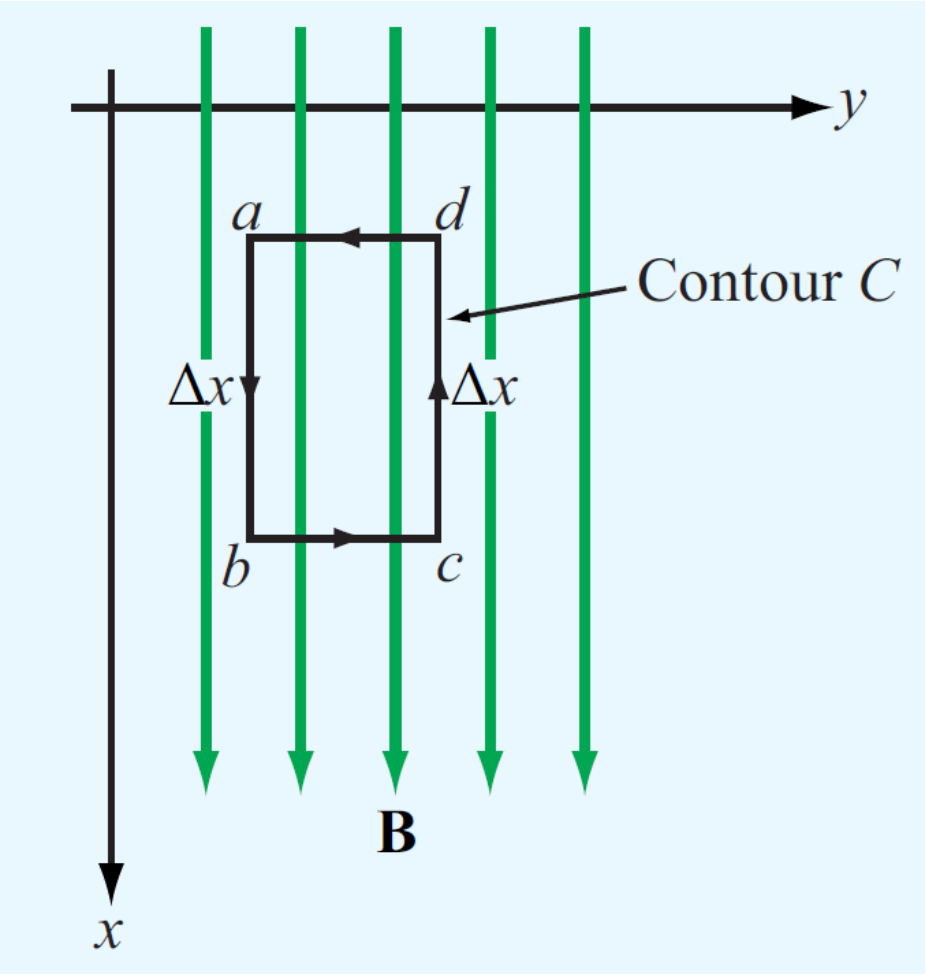
The curl of a vector field $\mathbf{B}$ describes its "rotational" property, or <u>circulation</u> which is defined as the line integral of $\mathbf{B}$ around a closed contour C

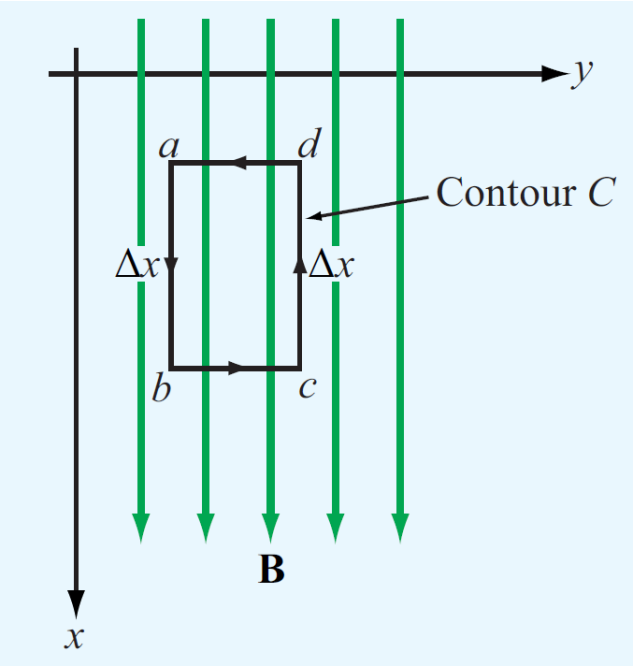
$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

As an introduction, a couple of simple examples follow to illustrate the physical meaning of circulation.

Example - Consider a contour $abcd$ in a uniform field

$$\mathbf{B} = \hat{\mathbf{x}} B_0$$





$$\Delta x = b - a$$
$$= c - d$$

Circulation around the $abcd$ contour is given by:

$$\text{Circulation} = \int_a^b \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{x}}\, dx + \int_b^c \cancel{\hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{y}}\, dy} \quad \boxed{\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0}$$

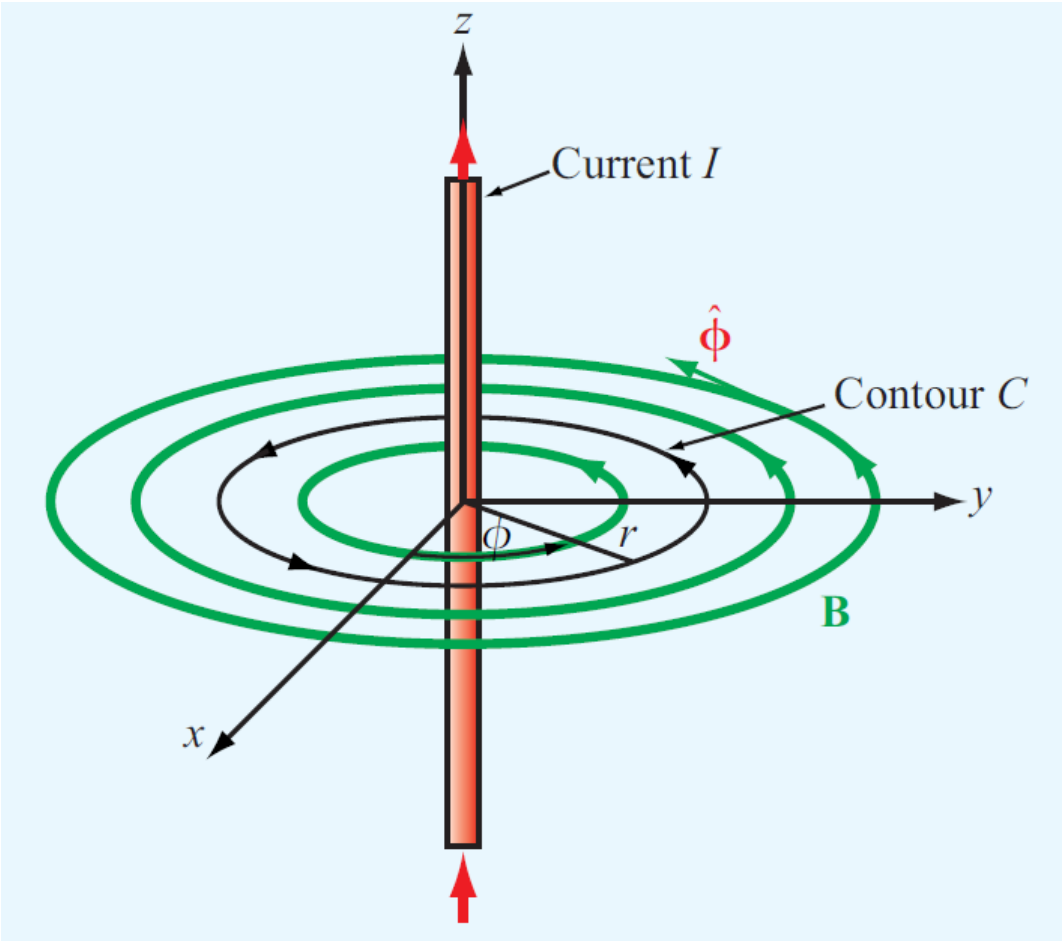
$$+ \int_c^d \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{x}}\, dx + \int_d^a \cancel{\hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{y}}\, dy} \quad \boxed{\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0}$$

$$= B_0\, \Delta x - B_0\, \Delta x = 0$$

Conclusion:

Circulation of a uniform field is zero

Example – Magnetic flux density $\mathbf{B}$ due to an infinite wire carrying a dc current I, surrounded by free space.



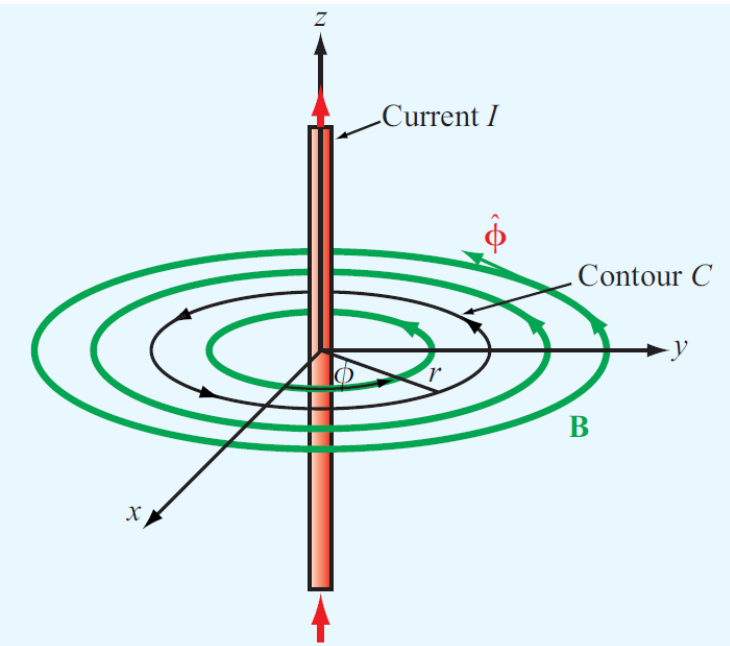
$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}$$

(azimuthal field)

$\mathbf{B}$ is directed along the azimuthal unit vector $\hat{\boldsymbol{\phi}}$.

Differential length vector on contour is $d\mathbf{l} = \hat{\boldsymbol{\phi}} r \, d\phi$

The circulation of $\mathbf{B}$ in a circular contour of radius r is



$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

$$= \int_0^{2\pi} \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \cdot \hat{\boldsymbol{\phi}} r \, d\phi = \mu_0 I$$

The circulation is not zero in this case.

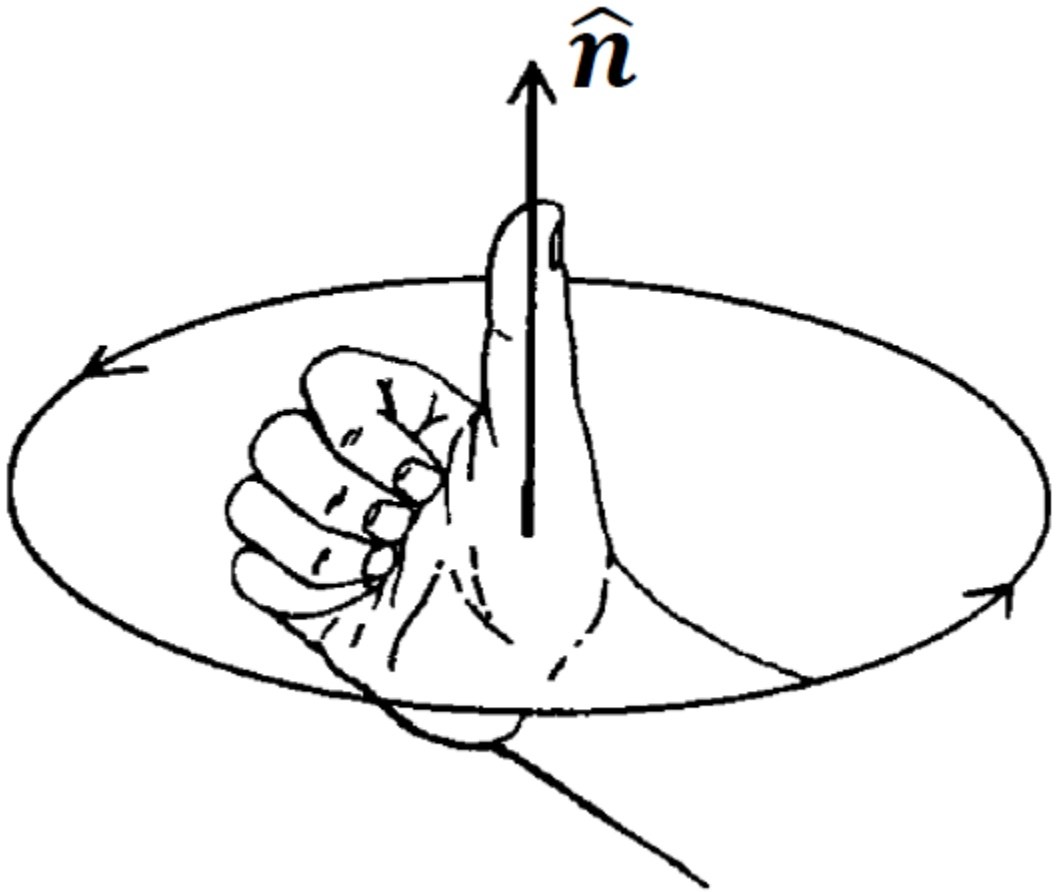
However, a contour on the x-z or y-z plane would result in zero circulation since there would be no azimuthal component with $d\mathbf{l}$ always perpendicular to $\hat{\boldsymbol{\phi}}$.

The curl of a vector field $\mathbf{B}$ (denoted with $\nabla \times \mathbf{B}$) is a differential operator defined mathematically as

$$\nabla \times \mathbf{B} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{\mathbf{n}} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{\max}$$

It represents the circulation of the vector $\mathbf{B}$ per unit area, with the area of the contour Δs oriented such that the circulation is maximum.

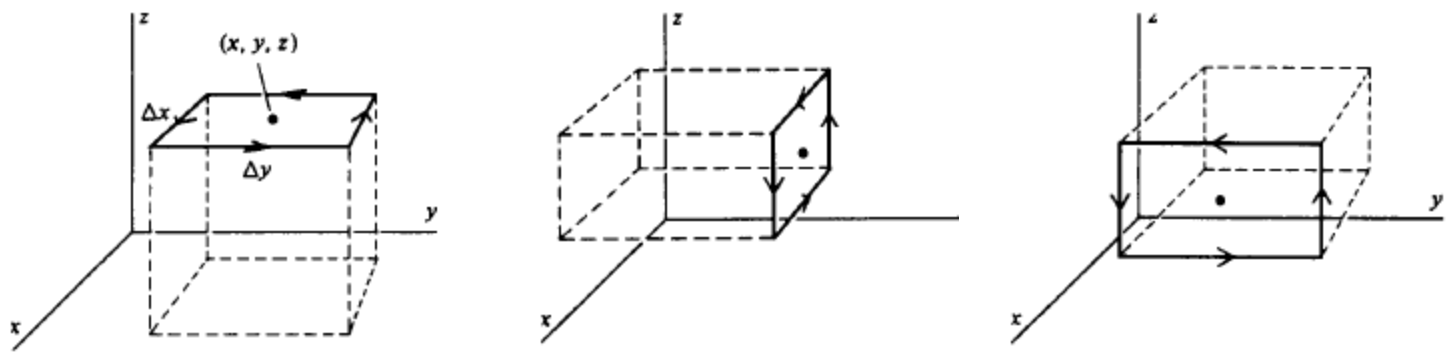
The curl of a vector is also a vector, with direction $\hat{n}$ which is the normal to the area of the contour Δs defined according to the right-hand rule: the thumb points along $\hat{n}$ and the fingers follow the contour.



For a vector in Cartesian coordinates

$$\mathbf{B} = \hat{\mathbf{x}} B_x + \hat{\mathbf{y}} B_y + \hat{\mathbf{z}} B_z$$

the curl involves line integrations along paths that define a differential 3D box



We are not going to retrace the lengthy and tedious derivation, but just state the results (which are of great practical utility).

The general result has the form

$$\nabla \times \mathbf{B} = \hat{\mathbf{x}}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

This result can be described conveniently by the determinant of a matrix

$$\nabla \times \mathbf{B} = \det \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

What can we say about the curl of an electrostatic field? Recall the curl mathematical definition

$$\nabla \times \mathbf{E} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{\mathbf{n}} \oint_C \mathbf{E} \cdot d\mathbf{l} \right]_{\max}$$

For any arbitrary closed path, the line integral of an electrostatic field always vanishes

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

which implies also

$$\boxed{\nabla \times \mathbf{E} = 0}$$

Example – Find the curl of the vector field

$$\mathbf{E} = \hat{x}\cos y + \hat{y}1$$

Solution

$$\nabla \times \mathbf{E} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos y & 1 & 0 \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}1\right) - \hat{y}\left(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}\cos y\right) + \hat{z}\left(\frac{\partial}{\partial x}1 - \frac{\partial}{\partial y}\cos y\right)$$

$$= \hat{x}0 - \hat{y}0 + \hat{z}(0 + \sin y)$$

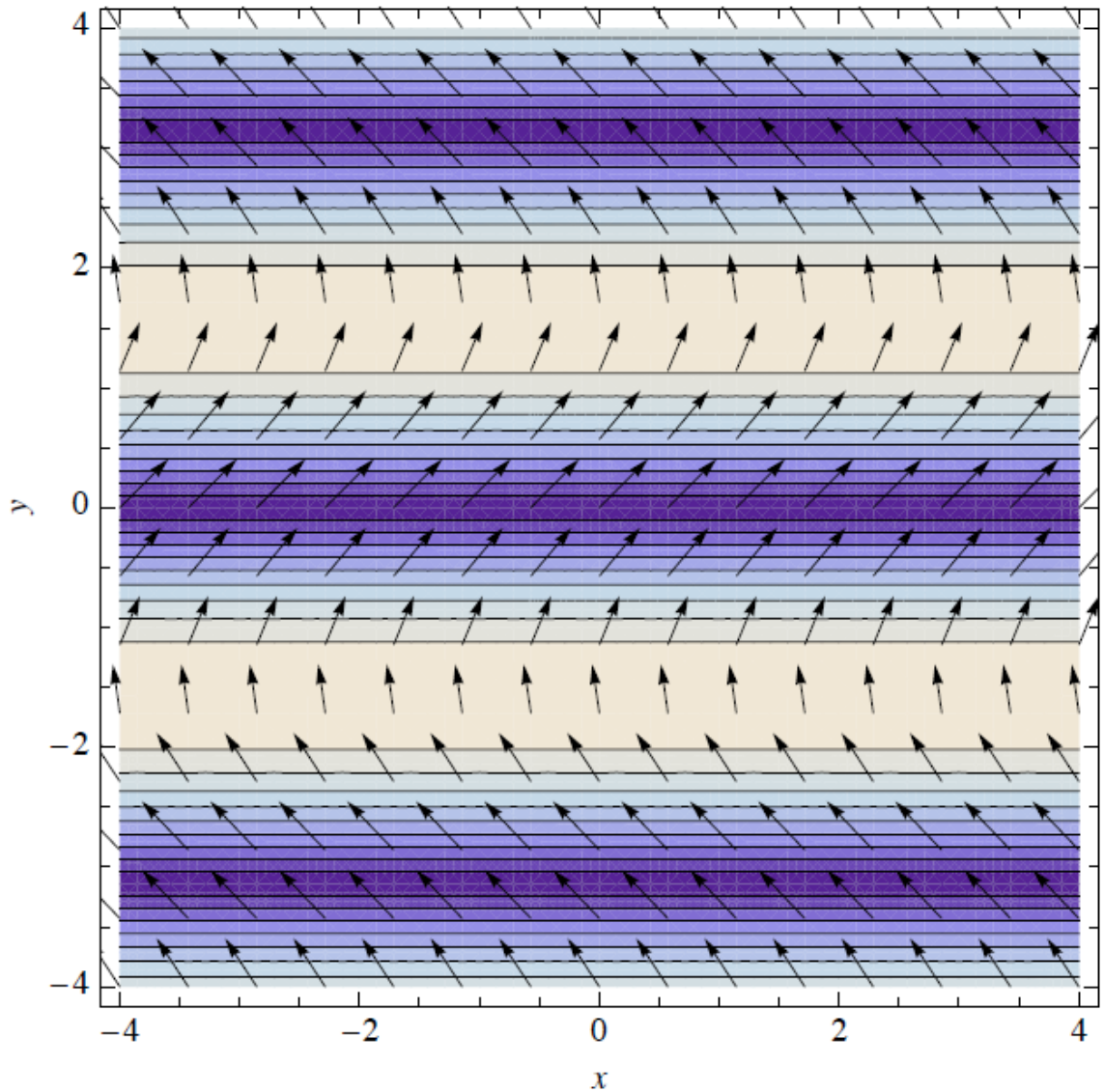
$$\boxed{\nabla \times \mathbf{E} = \hat{z}\sin y}$$

solution is a vector field

$$\mathbf{E} = \hat{x}\cos y + \hat{y}1$$

color map

$$|\nabla \times \mathbf{E}| = |\sin y|$$



Statement to remember

The <u>Curl</u> differential operator is applied to a <u>vector</u> and the result is another <u>vector</u>.

The notation $\nabla \times \mathrm{B}$ <u>does not</u> mean that the curl is the cross-product between ∇ and B!

You can see in Maxwell's Equations:

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{V}},$$

Divergence of a Vector = Scalar

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

Curl of a Vector = Vector

$$\nabla \cdot \mathbf{B} = 0,$$

Divergence of a Vector = Scalar

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Curl of a Vector = Vector

Important vector identities involving the Curl:

$$(1) \quad \nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$$(2) \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$(3) \quad \nabla \times (\nabla V) = 0$$