ECE 329 – Fall 2022

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

Lecture 5

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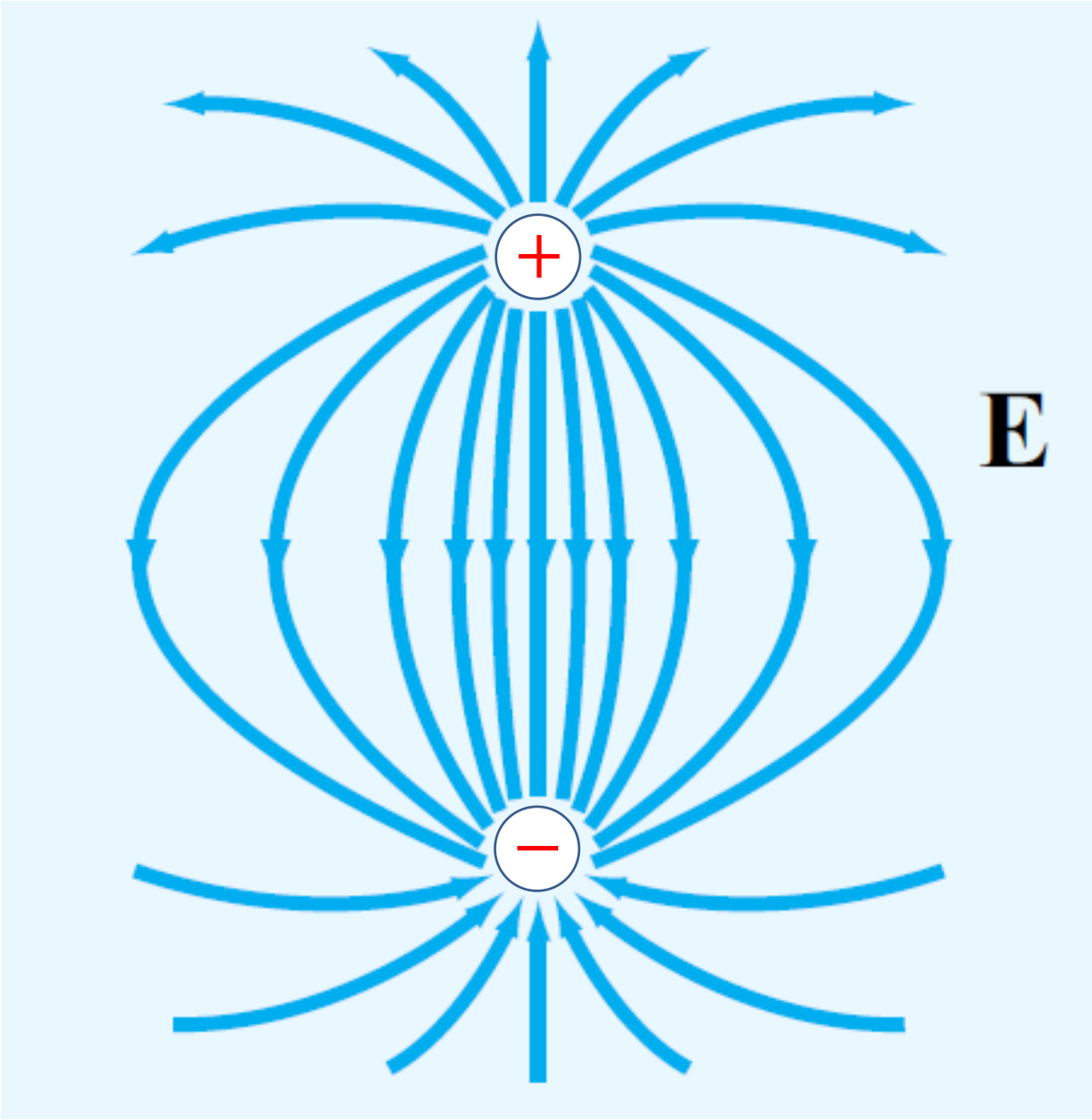
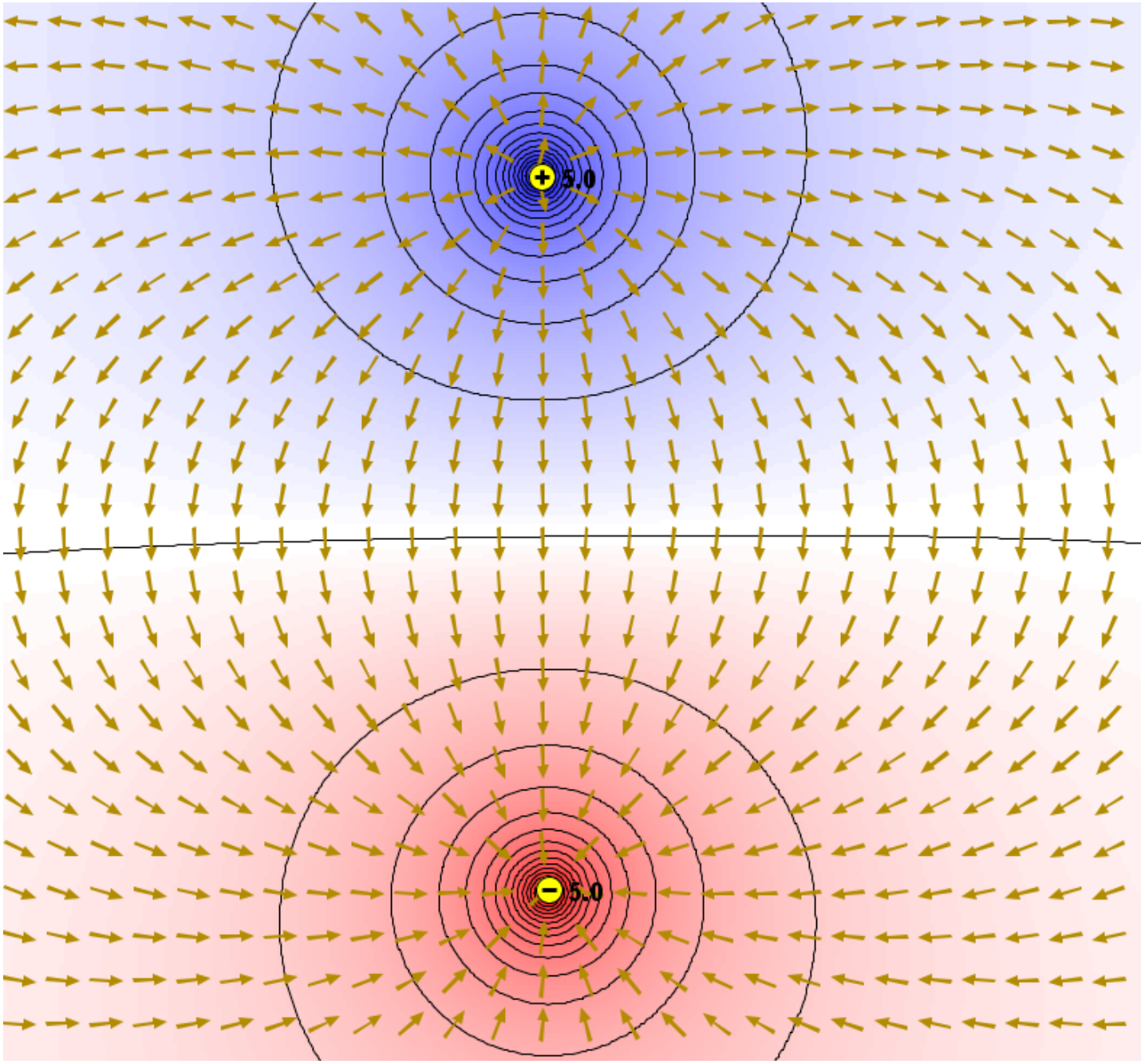
- **Differential Operators in Electromagnetics**
  - **Divergence**
  - **Curl**
- **Electrostatic Potential**

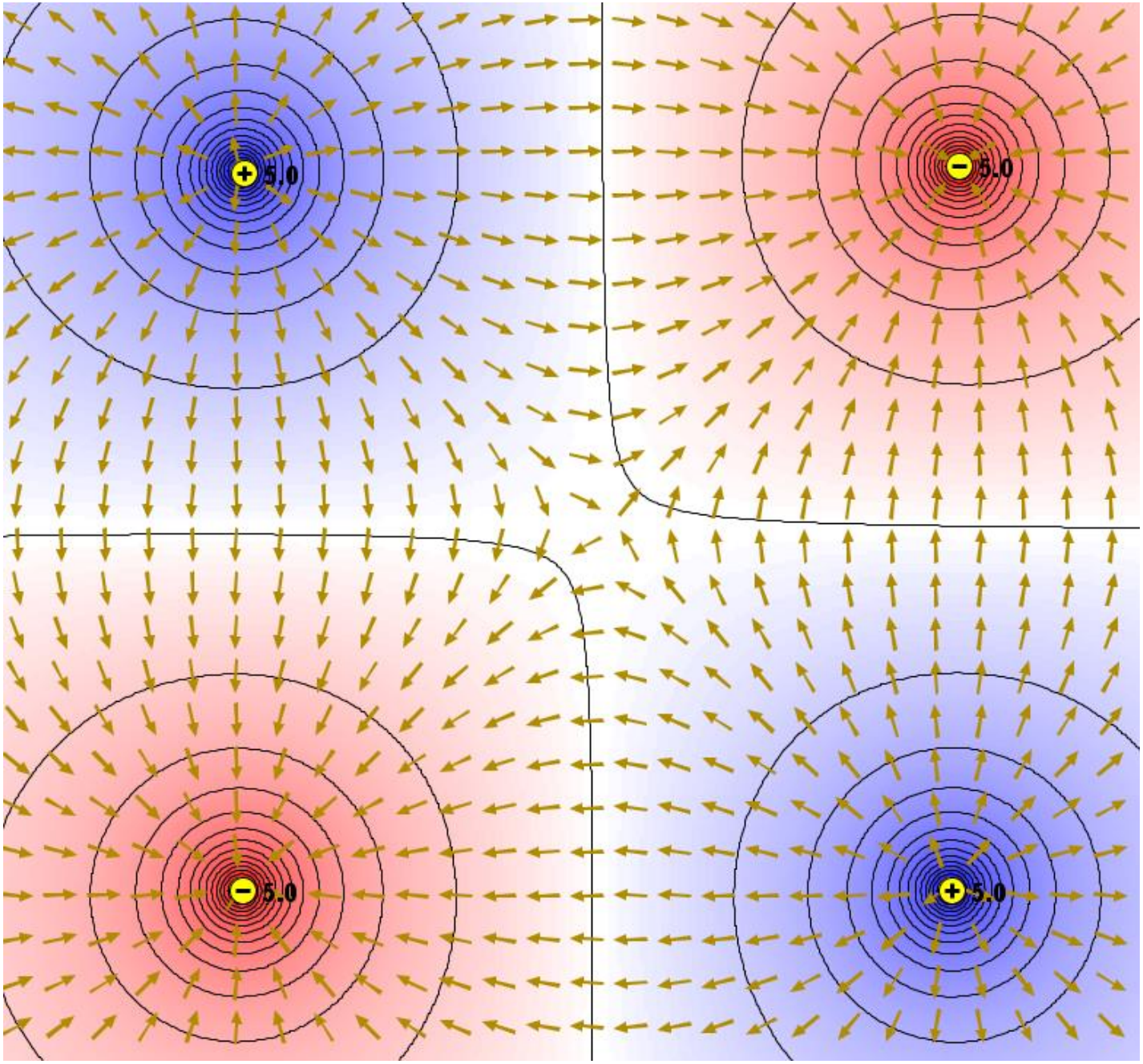
Electric field pattern of the Electric Dipole



dipole potential and field



A quadrupole



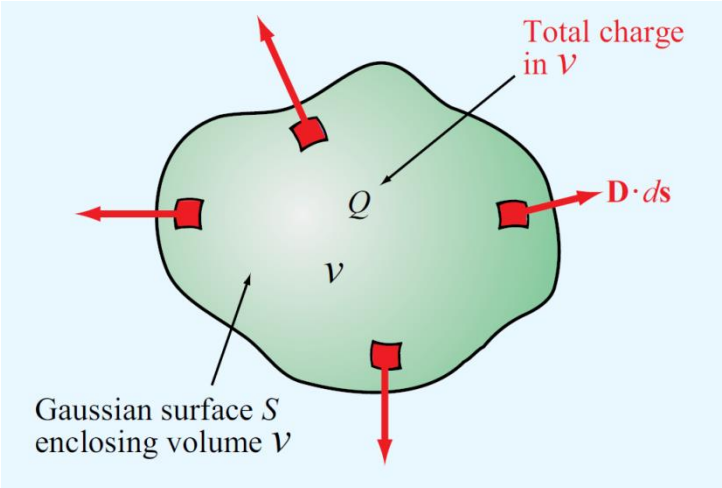
Quadrupole ion trap for chromatography

http://tools.thermofisher.com/content/sfs/brochures/WP-10427-GC-LC-MS-Workhorses-Quadrupoles-Ion-Traps-WP10427-EN.pdf

Divergence of a Vector Field

We have introduced earlier the "integral" form of Gauss' Law

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V$$

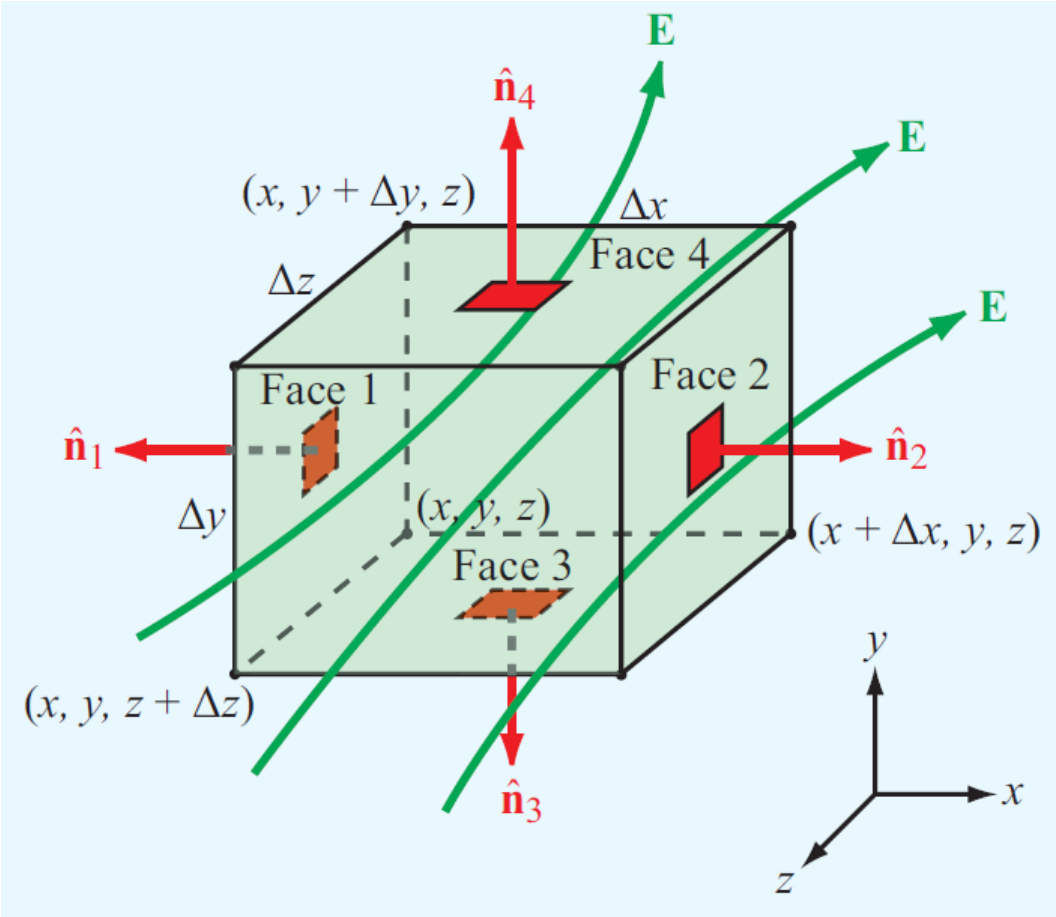


or $$\oint_S \mathbf{E} \cdot d\mathbf{S} = Q_V / \epsilon_0$$

where Q_V represents the integral of the charge contained in the volume delimited by the closed surface

$$Q_V = \int_V \rho \, dV$$

Now, consider instead a very small element of volume



NOTE: There are also Faces 5 and 6 for the flux but they are omitted in the drawing to avoid excessive clutter

$$\Delta V = \Delta x \Delta y \Delta z$$

$$\int_{\Delta V} \rho \, dV \approx \rho(x, y, z) \, \Delta x \Delta y \Delta z$$

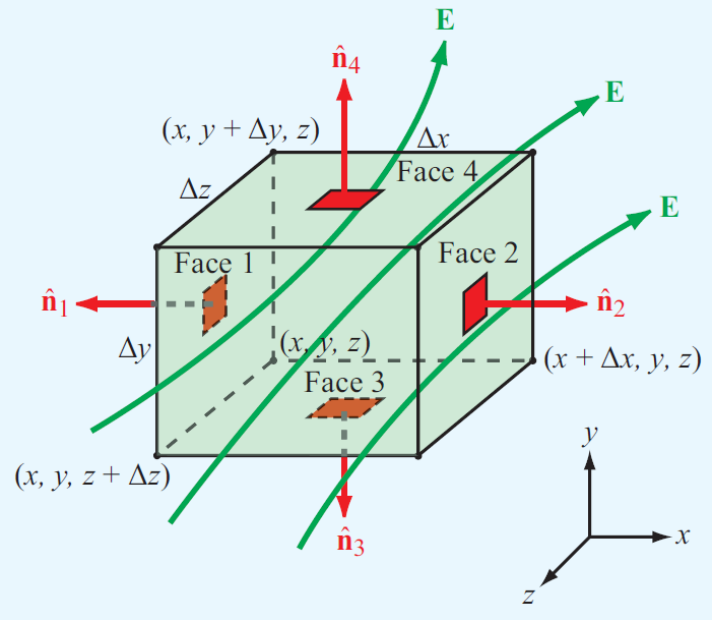
Divergence is the differential version of the flux

$$\nabla \cdot \mathbf{E} \equiv \lim_{\Delta V \to 0} \frac{\oint_S \mathbf{E} \cdot d\mathbf{s}}{\Delta V} = \frac{\rho}{\epsilon_0}$$

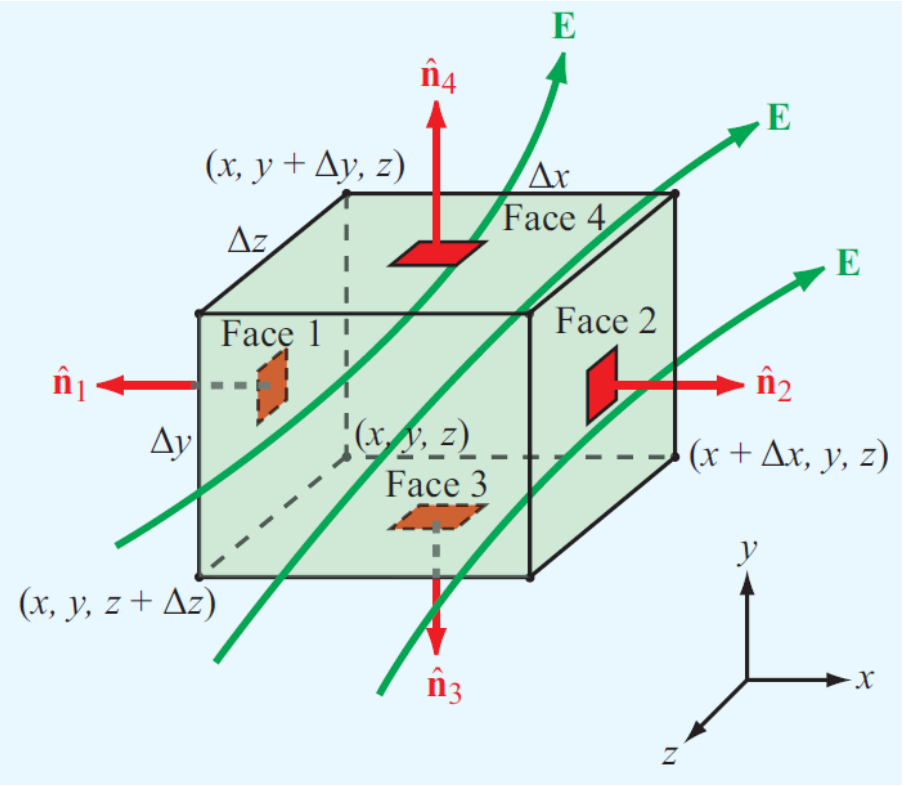
This is Gauss' Law in differential form

One can show that:

$$\nabla \cdot \mathbf{E} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right)$$



Derivation
(for those interested)



For **Face 1** with area $\Delta y \Delta z$ and unit vector $\hat{\mathbf{n}}_1 = -\hat{\mathbf{x}}$

the outward flux F_1 is

$$F_1 = \int_{\text{Face 1}} \mathbf{D} \cdot \hat{\mathbf{n}}_1 \, ds = \int_{\text{Face 1}} \epsilon_0 \, \mathbf{E} \cdot \hat{\mathbf{n}}_1 \, ds$$

$$= \int_{\text{Face 1}} \epsilon_0 \, (\hat{\mathbf{x}} E_x + \hat{\mathbf{y}} E_y + \hat{\mathbf{z}} E_z) \cdot (-\hat{\mathbf{x}}) \, dy \, dz \approx -\epsilon_0 \, E_x(1) \, \Delta y \, \Delta z$$

Similarly, for **Face 2** with area $\Delta y \Delta z$ and unit vector $\hat{\mathbf{n}}_2 = \hat{\mathbf{x}}$

$$F_2 = \epsilon_0 \, E_x(2) \, \Delta y \, \Delta z$$

Since the two faces are separated by an infinitesimal distance Δx, ignoring terms $(\Delta x)^2$ and higher, we can also expand

$$E_x(2) = E_x(1) + \frac{\partial E_x}{\partial x} \Delta x$$

By combining the two equations above we obtain

$$F_2 = \epsilon_0 \left[E_x(1) + \frac{\partial E_x}{\partial x} \Delta x \right] \Delta y \, \Delta z$$

and with the previous result $F_1 = -\epsilon_0 \, E_x(1) \, \Delta y \, \Delta z$

$$\Rightarrow \quad \boxed{F_1 + F_2 = \epsilon_0 \frac{\partial E_x}{\partial x} \Delta x \, \Delta y \, \Delta z}$$

For the other two pairs of opposite faces

$$F_3 + F_4 = \epsilon_0 \frac{\partial E_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

$$F_5 + F_6 = \epsilon_0 \frac{\partial E_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

The total outward flux is the sum of these results

$$\oint_S \epsilon_0 \, \mathbf{E} \cdot d\mathbf{s} = F_1 + F_2 + F_3 + F_4 + F_5 + F_6$$

$$= \epsilon_0 \underbrace{\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right)}_{\text{divergence} \quad \nabla \cdot \mathbf{E}} \underbrace{\Delta x \, \Delta y \, \Delta z}_{\Delta V}$$

End of derivation

Remember that we have assumed a "very small" volume so, from the previous result, the divergence can be defined as a limit for $\Delta V \to 0$

$$\nabla \cdot \mathbf{E} \equiv \lim_{\Delta V \to 0} \frac{\oint_S \mathbf{E} \cdot d\mathbf{s}}{\Delta V} = \frac{\rho}{\epsilon_0}$$

This is Gauss' Law in differential form

If the net flux out of the surface S is positive, it is as if the volume ΔV contains a "***source***" of field lines.

If the net flux out of the surface S is negative, it is as if the volume ΔV contains a "***sink***" of field lines.

For a uniform field, the same amount of flux enters and leaves the volume, therefore, the divergence is zero. Divergence is always zero for fields not "generated" by sources (**divergence-free** or **divergence-less** fields).

We can extend the differential results to finite volumes and state the **Divergence Theorem**:

$$\int_V \nabla \cdot \mathbf{E}\ dV = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

As long as a relation like $\mathbf{D} = \epsilon \mathbf{E}$ holds, the same formulations used for the electric field $\mathbf{E}$ are also valid for the displacement vector $\mathbf{D}$, differing only by a multiplication constant ϵ.

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV \qquad \nabla \cdot \mathbf{D} = \rho$$

$$\int_V \nabla \cdot \mathbf{D}\, dV = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

Example – Determine the *Divergence* for the vector field given and calculate it at point P.

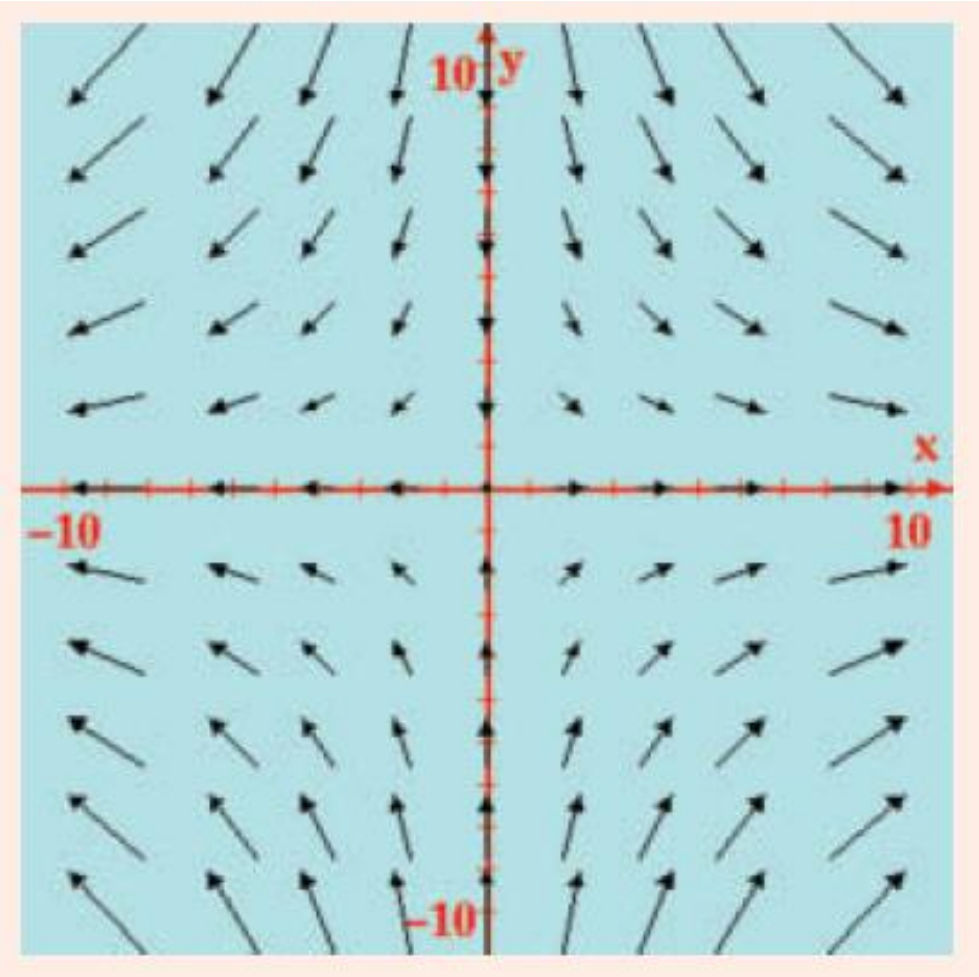
$$\mathbf{E} = \hat{\mathbf{x}}3x^2 + \hat{\mathbf{y}}2z + \hat{\mathbf{z}}x^2z$$

$$\boxed{P = (2, -2, 0)}$$

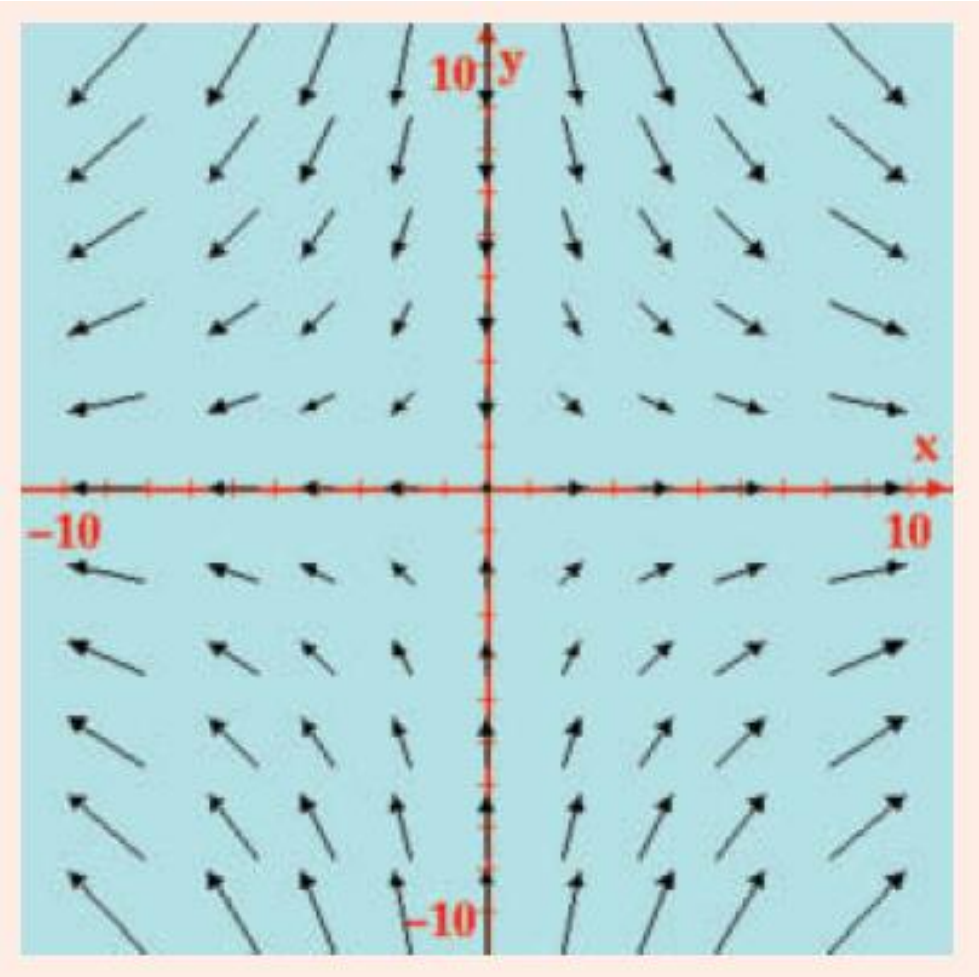
$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\
&= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(x^2z) \\
&= 6x + 0 + x^2 \\
&= x^2 + 6x.
\end{aligned}$$

$$\boxed{\text{At } (2, -2, 0),\ \nabla \cdot \mathbf{E}\Big|_{(2,-2,0)} = 16.}$$

Exercise in class – The arrow representation in the Figure gives the distribution of a vector field $\mathbf{A}$ in the x-y plane. Could you guess at a glance what the divergence is everywhere?



Exercise in class – The arrow representation in the Figure gives the distribution of a vector field $\mathbf{A}$ in the x-y plane. Could you guess at a glance what the divergence is everywhere?



The graph suggests that at the boundaries the input flux balances out the output flux. Draw also a square around the center point and see how field arrows point in and out.

So, we expect the divergence to be zero. In fact, this is a plot of the field $\mathbf{A} = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$ and the divergence is indeed calculated to be $\nabla \cdot \mathbf{A} = 0$ everywhere.

Statement to remember

The <u>Divergence</u> differential operator is always applied to a <u>vector</u> and the result is a <u>scalar</u>.

For comparison, the <u>Gradient</u> operates on <u>scalars</u> and the result is a <u>vector</u>.

Curl of a Vector Field

The curl of a vector field $\mathbf{B}$ (denoted with $\nabla \times \mathbf{B}$) is a differential operator defined mathematically as

$$\nabla \times \mathbf{B} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{\mathbf{n}} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{\max}$$

It represents the circulation of the vector $\mathbf{B}$ per unit area, with the area of the contour Δs oriented such that the circulation is maximum.

Curl of a Vector Field

Result in Cartesian coordinates

$$\nabla \times \mathbf{B} = \hat{\mathbf{x}}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

This result can be described conveniently by the determinant of a matrix

$$\nabla \times \mathbf{B} = \det \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

Curl of a Vector Field

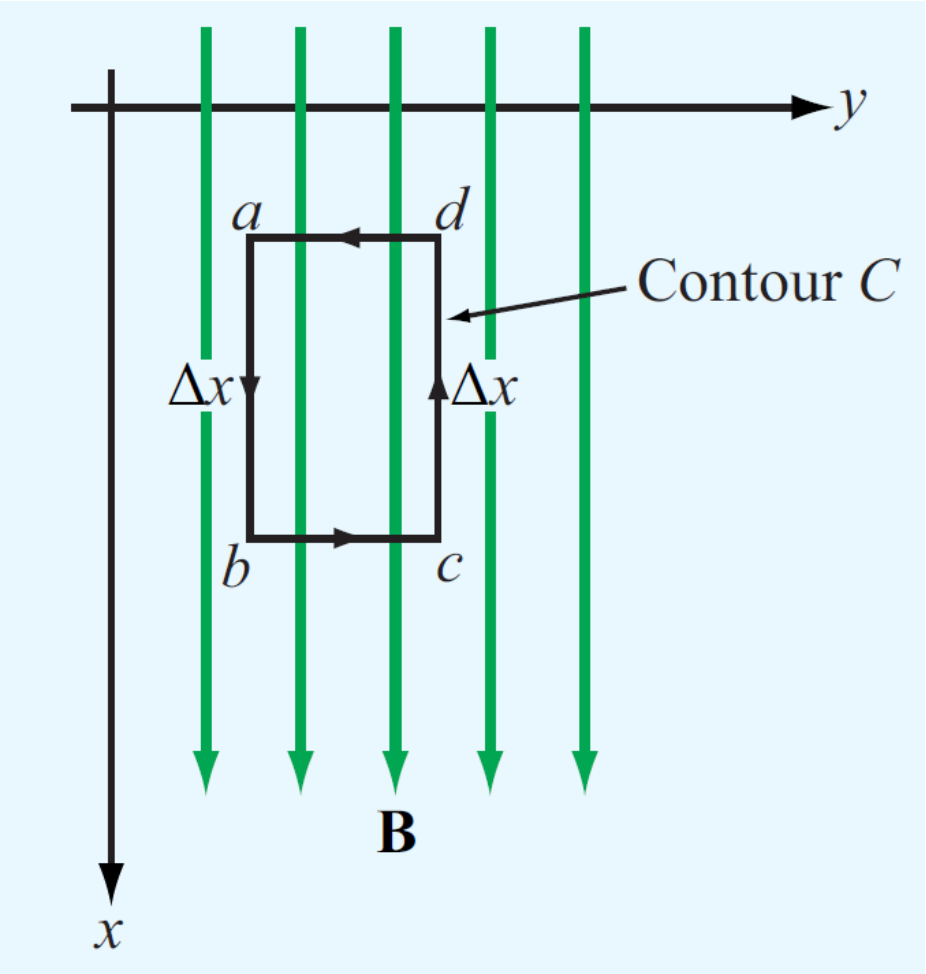
The curl of a vector field $\mathbf{B}$ describes its "rotational" property, or <u>circulation</u> which is defined as the line integral of $\mathbf{B}$ around a closed contour C

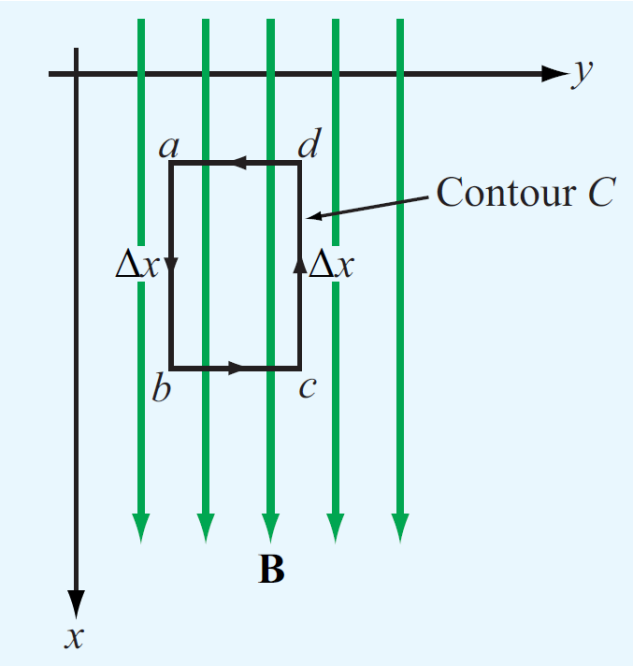
$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

As an introduction, a couple of simple examples follow to illustrate the physical meaning of circulation.

Example - Consider a contour $abcd$ in a uniform field

$$\mathbf{B} = \hat{\mathbf{x}} B_0$$





$$\Delta x = b - a$$
$$= c - d$$

Circulation around the $abcd$ contour is given by:

$$\text{Circulation} = \int_a^b \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{x}}\, dx + \int_b^c \cancel{\hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{y}}\, dy} \quad \boxed{\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0}$$

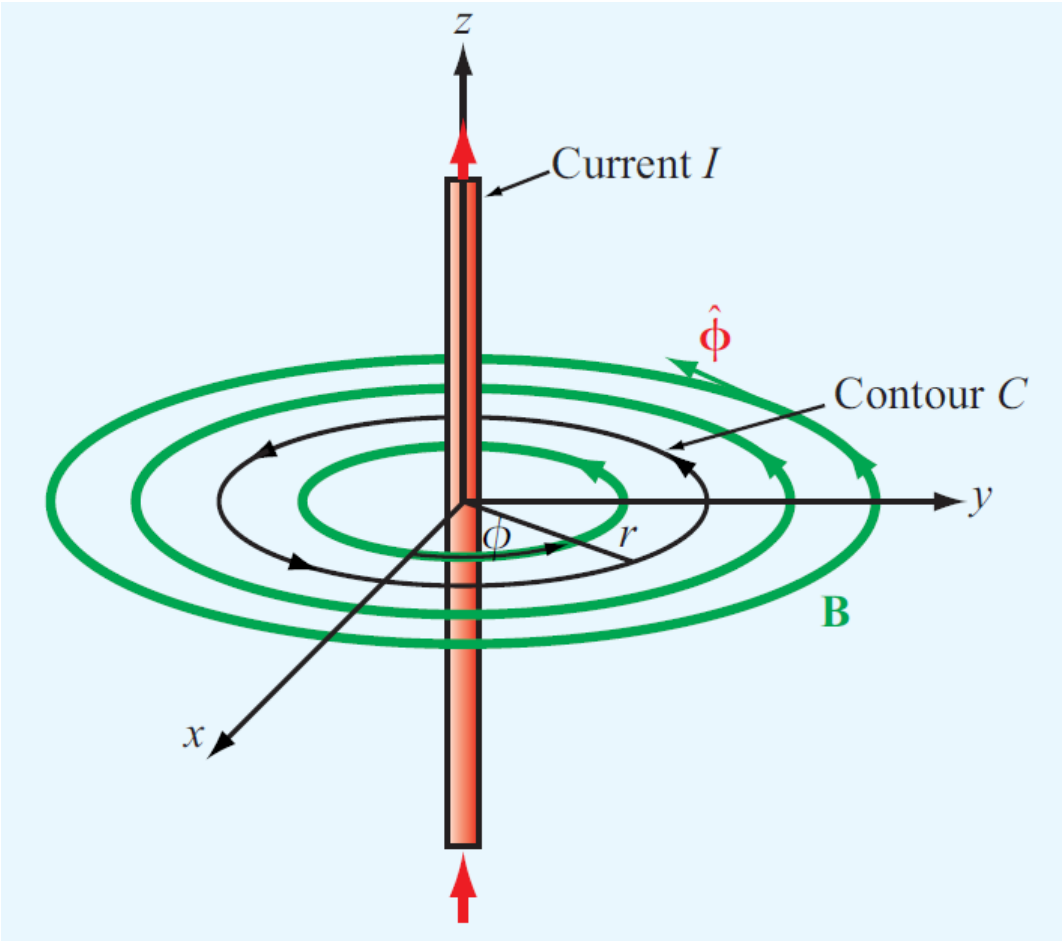
$$+ \int_c^d \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{x}}\, dx + \int_d^a \cancel{\hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{y}}\, dy} \quad \boxed{\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0}$$

$$= B_0\, \Delta x - B_0\, \Delta x = 0$$

Conclusion:

Circulation of a uniform field is zero

Example – Magnetic flux density $\mathbf{B}$ due to an infinite wire carrying a dc current I, surrounded by free space.



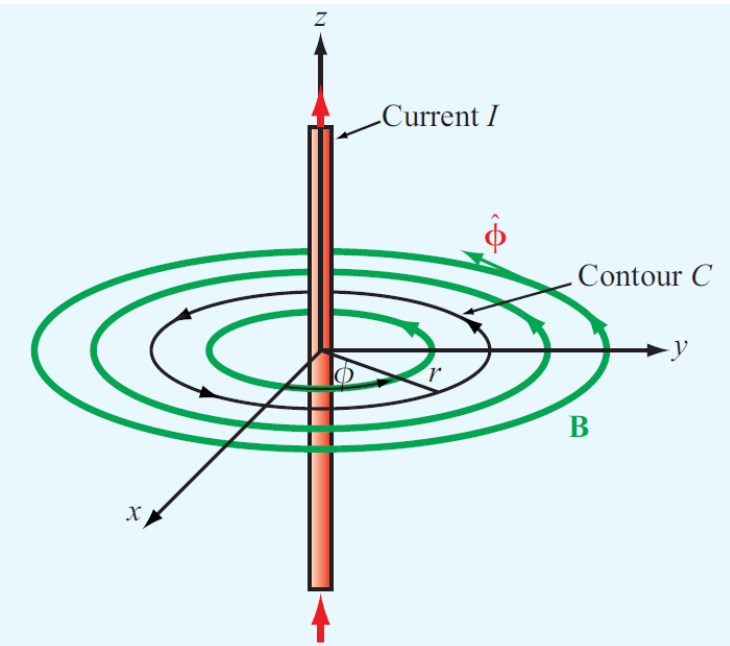
$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}$$

(azimuthal field)

$\mathbf{B}$ is directed along the azimuthal unit vector $\hat{\boldsymbol{\phi}}$.

Differential length vector on contour is $d\mathbf{l} = \hat{\boldsymbol{\phi}} r \, d\phi$

The circulation of B in a circular contour of radius r is



$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

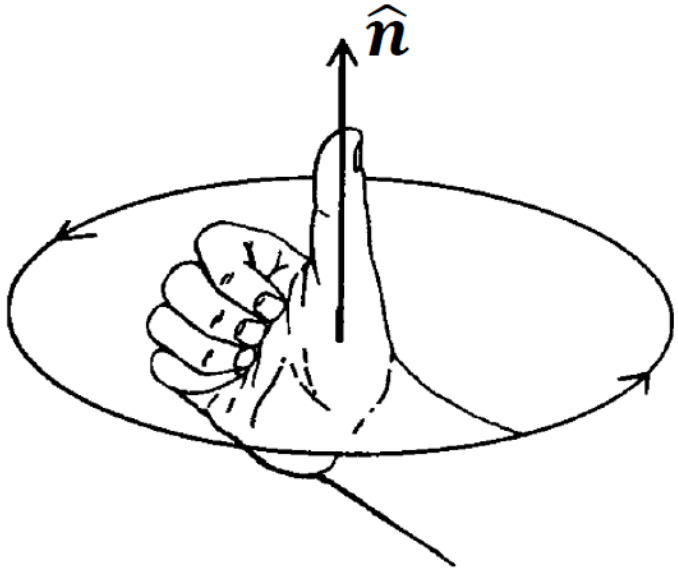
$$= \int_0^{2\pi} \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \cdot \hat{\boldsymbol{\phi}} r \, d\phi = \mu_0 I$$

The circulation is not zero in this case.

However, a contour on the x-z or y-z plane would result in zero circulation since there would be no azimuthal component with $d\mathbf{l}$ always perpendicular to $\hat{\boldsymbol{\phi}}$.

$$\nabla \times \mathbf{B} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{\mathbf{n}} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{\max}$$

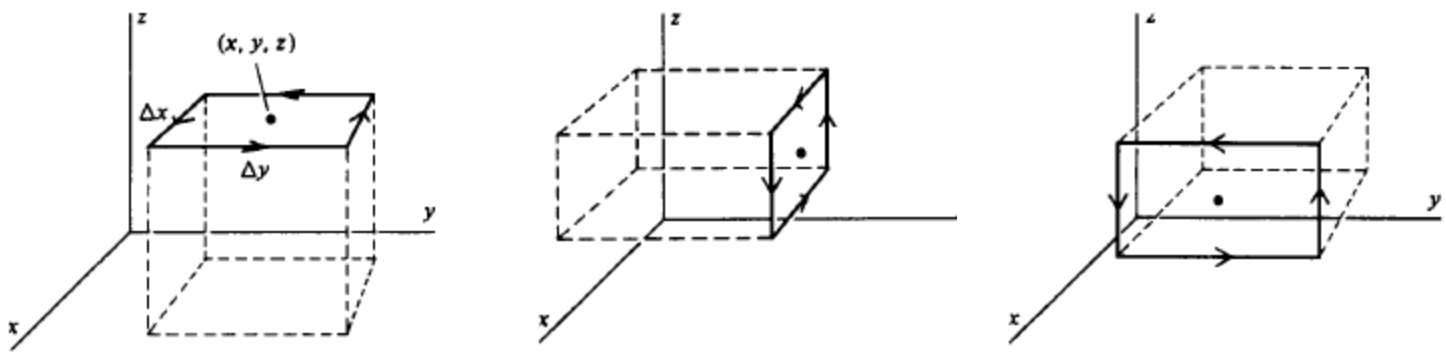
The curl of a vector is also a vector, with direction $\hat{n}$ which is the normal to the area of the contour Δs defined according to the right-hand rule: the thumb points along $\hat{n}$ and the fingers follow the contour.



For a vector in Cartesian coordinates

$$\mathbf{B} = \hat{\mathbf{x}} B_x + \hat{\mathbf{y}} B_y + \hat{\mathbf{z}} B_z$$

the curl involves line integrations along paths that define a differential 3D box



We are not going to retrace the lengthy and tedious derivation but just state the results (which are of great practical utility).

Again:

$$\nabla \times \mathbf{B} = \hat{\mathbf{x}}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$\nabla \times \mathbf{B} = \det \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

Statement to remember

The <u>Curl</u> differential operator is applied to a <u>vector</u> and the result is another <u>vector</u>.

The notation $\nabla \times \mathrm{B}$ <u>does not</u> mean that the curl is the cross-product between ∇ and B!

You can see in Maxwell's Equations:

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{V}},$$

Divergence of a Vector = Scalar

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

Curl of a Vector = Vector

$$\nabla \cdot \mathbf{B} = 0,$$

Divergence of a Vector = Scalar

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Curl of a Vector = Vector

Important vector identities involving the Curl:

$$(1) \quad \nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$$(2) \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$(3) \quad \nabla \times (\nabla V) = 0$$

What can we say about the curl of an electrostatic field? Recall the curl mathematical definition

$$\nabla \times \mathbf{E} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{\mathbf{n}} \oint_C \mathbf{E} \cdot d\mathbf{l} \right]_{\max}$$

For any arbitrary closed path, the line integral of an electrostatic field always vanishes

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

which implies also

$$\boxed{\nabla \times \mathbf{E} = 0}$$

Curl-free vector field

Recall the previous vector identity

$$(3) \quad \nabla \times (\nabla V) = 0$$

A vector field which is the result of the gradient of a scalar function must have zero curl.

The electrostatic field is obtained from the gradient of the electrostatic potential

$$\mathbf{E} = -\nabla V$$

where

$$\nabla V \equiv \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

therefore, we expect

$$\boxed{\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = 0}$$

This can be easily verified

$$\nabla \times (\nabla V) = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix}$$

The components of the curl are

$$\hat{x}\left[\cancel{\frac{\partial}{\partial y}\frac{\partial V}{\partial z}} - \cancel{\frac{\partial}{\partial z}\frac{\partial V}{\partial y}}\right] = 0$$

$$\hat{y}\left[\cancel{\frac{\partial}{\partial z}\frac{\partial V}{\partial x}} - \cancel{\frac{\partial}{\partial x}\frac{\partial V}{\partial z}}\right] = 0$$

$$\hat{z}\left[\cancel{\frac{\partial}{\partial x}\frac{\partial V}{\partial y}} - \cancel{\frac{\partial}{\partial y}\frac{\partial V}{\partial x}}\right] = 0$$

The units of electric field are [Volts/meter]. **Since**

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

then the units of electrostatic potential are [Volts]

The relationship

$$\mathbf{E} = -\nabla V$$

indicates that the electrostatic field points from regions of high potential to regions of low potential. The negative sign originates from having assigned negative polarity to electrons.

If you think of the electrostatic potential as an equivalent of mechanical potential energy, positive charges go "downhill" and negative charges go "uphill".

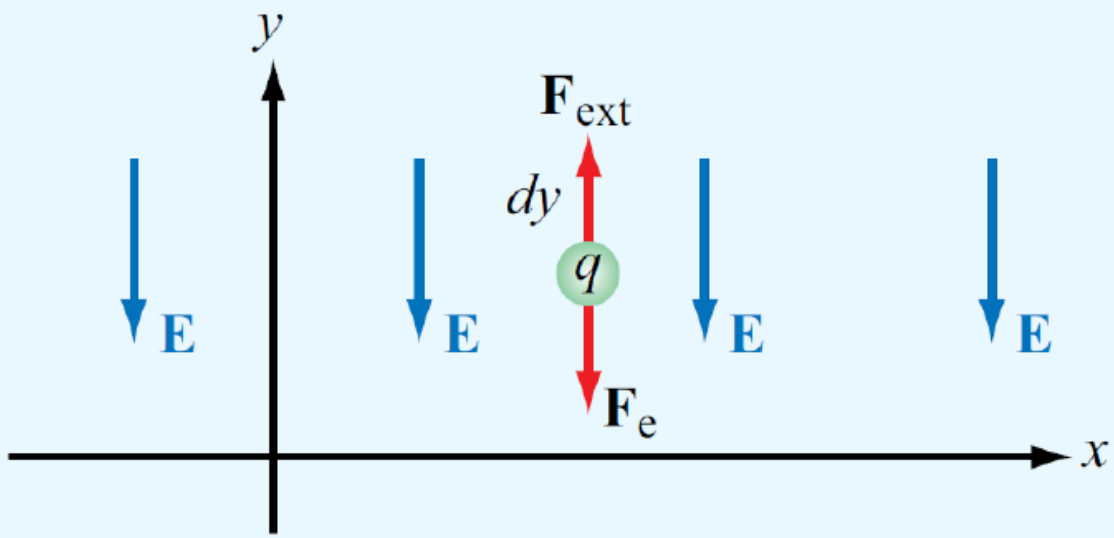
The terms "voltage" or "voltage potential" are often used as synonymous of "electric potential".

The potential difference between two points represents the amount of work (or potential energy) required to move a unit charge from one point to another.

Consider a positive charge q in a uniform electric field $\mathbf{E} = -\hat{\mathbf{y}}E$, exerting a force $\mathbf{F}_\mathrm{e} = q\mathbf{E}$ along $-y$. To move the charge along $+y$, we need an external force $\mathbf{F}_\mathrm{ext}$.

To move the charge q without acceleration (constant speed) the net force must be zero, that is:

$$\mathbf{F}_\mathrm{ext} = -\mathbf{F}_\mathrm{e} = -q\mathbf{E}$$



Work done in moving a charge q a distance dy against the electric field $\mathbf{E}$ is $dW = qE\,dy$.

The work done to move by a differential distance is

$$dW = \mathbf{F}_{\text{ext}} \cdot d\mathbf{l} = -q\mathbf{E} \cdot \hat{\mathbf{y}}\, dy = qE\, dy \qquad \text{(J)}$$

The differential potential energy per unit charge is

$$dV = \frac{dW}{q} = -\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J/C or V)}$$

The potential difference necessary to move between two points

$$\int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$$

In summary, for a given electrostatic potential, the electrostatic field is simply found by taking the negative of the gradient

$$\mathbf{E} = -\nabla V$$

For a given electrostatic field, the potential difference between two points can be found by performing a vector line integral

$$\int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$$

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$$

The line integral of the electrostatic field $\mathbf{E}$ around any closed contour, regardless of the path taken, is zero

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{(electrostatics)}$$

This is a consequence of the electrostatic field being curl-free (or "irrotational") .

Any field with this property is called <u>conservative</u>.

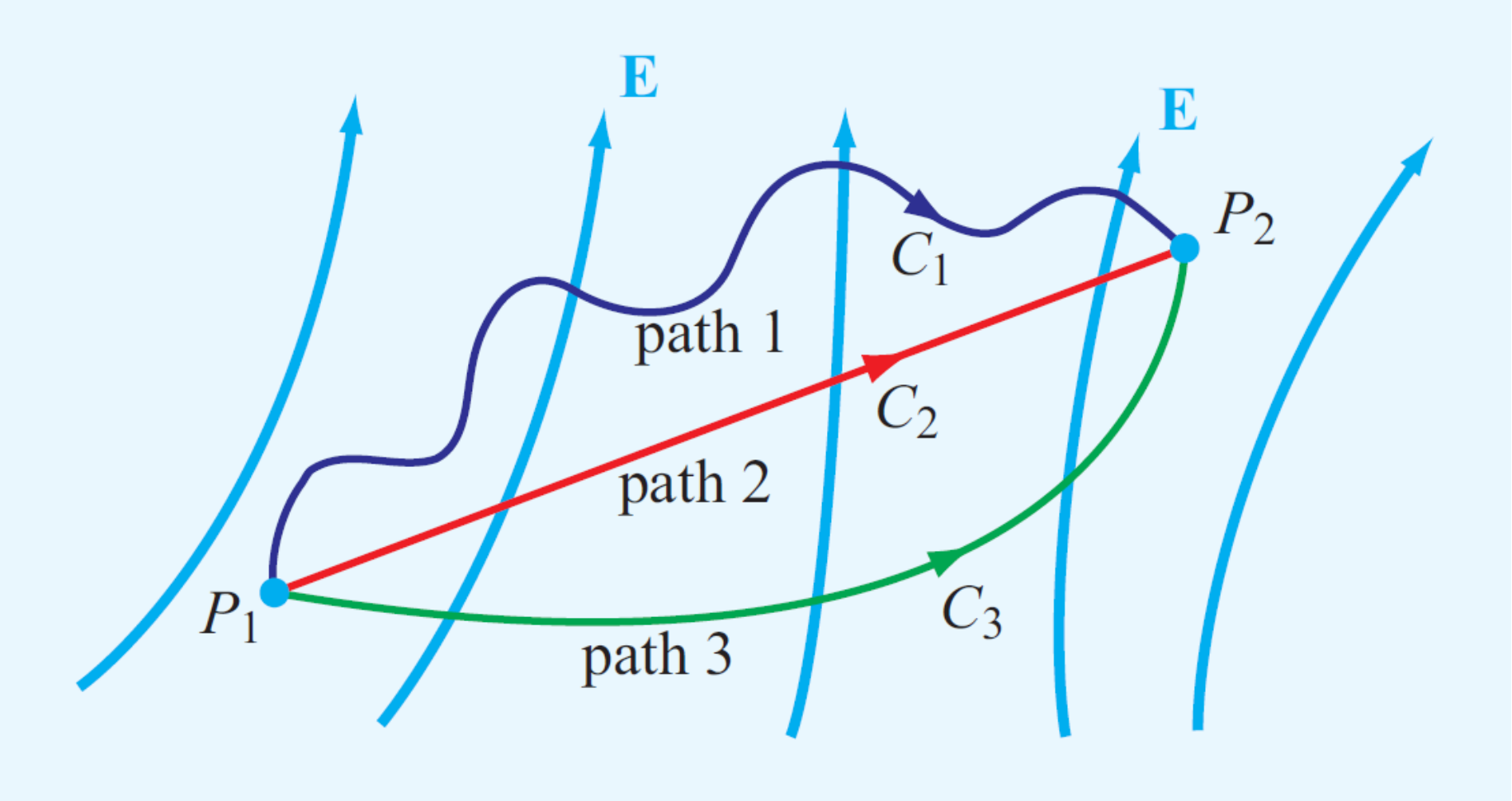
As a matter of fact, inspection of Maxwell's equations tells us that the electrostatic field is conservative.

In static conditions: $\partial / \partial t = 0$

$$\nabla \times \vec{E}(t) = \cancel{-\frac{\partial \vec{B}(t)}{\partial t}} = 0$$

Curl-free condition

In a conservative field, the potential difference between two points is the same irrespective of the path taken to calculate the line integral of the field.



Like the mechanical potential energy, the electrostatic potential is defined with respect to an arbitrary reference, so we talk about "potential difference" rather than an absolute potential.

For EM fields, a common choice is to select a very far away point (infinity) where the reference potential can be safely set to zero. The origin of coordinates may also be a convenient choice. In a circuit, we usually select a "ground" node.

In these situations the potential can be treated as an absolute value, in practice, since $V_1 = 0$ and

$$V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l} \quad (\mathrm{V})$$

reference at ∞

$$V = -\int_{0}^{P} \mathbf{E} \cdot d\mathbf{l} \quad (\mathrm{V})$$

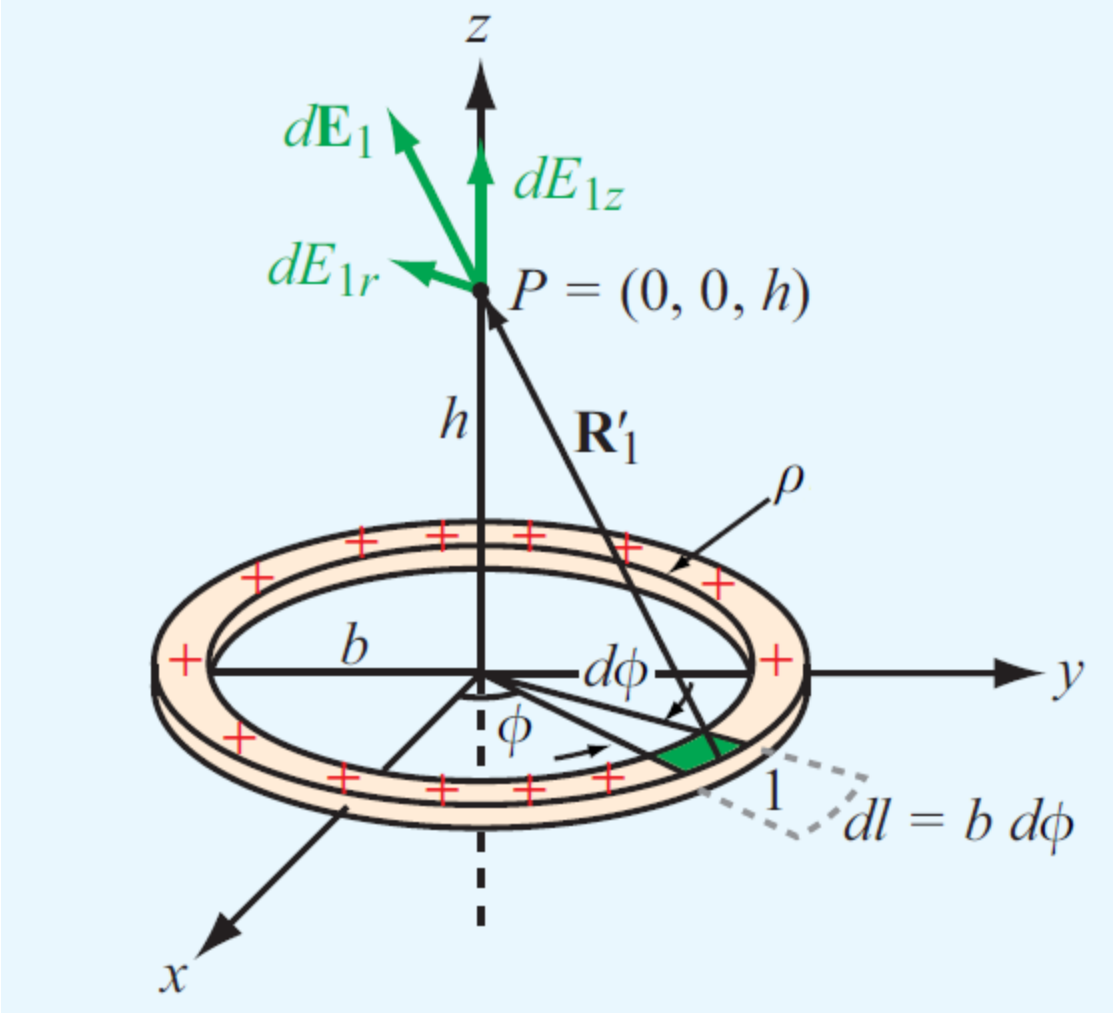
reference at (0,0,0)

Some extra examples which we may not be able to cover in class

The following example have been added for you to have extra practice material but most likely we do not have time for them in class.

Example – Electric field of a Ring of Charge

We look for the Electric Field on points along the axis of the ring



segment length

$$dl = bd\varphi$$

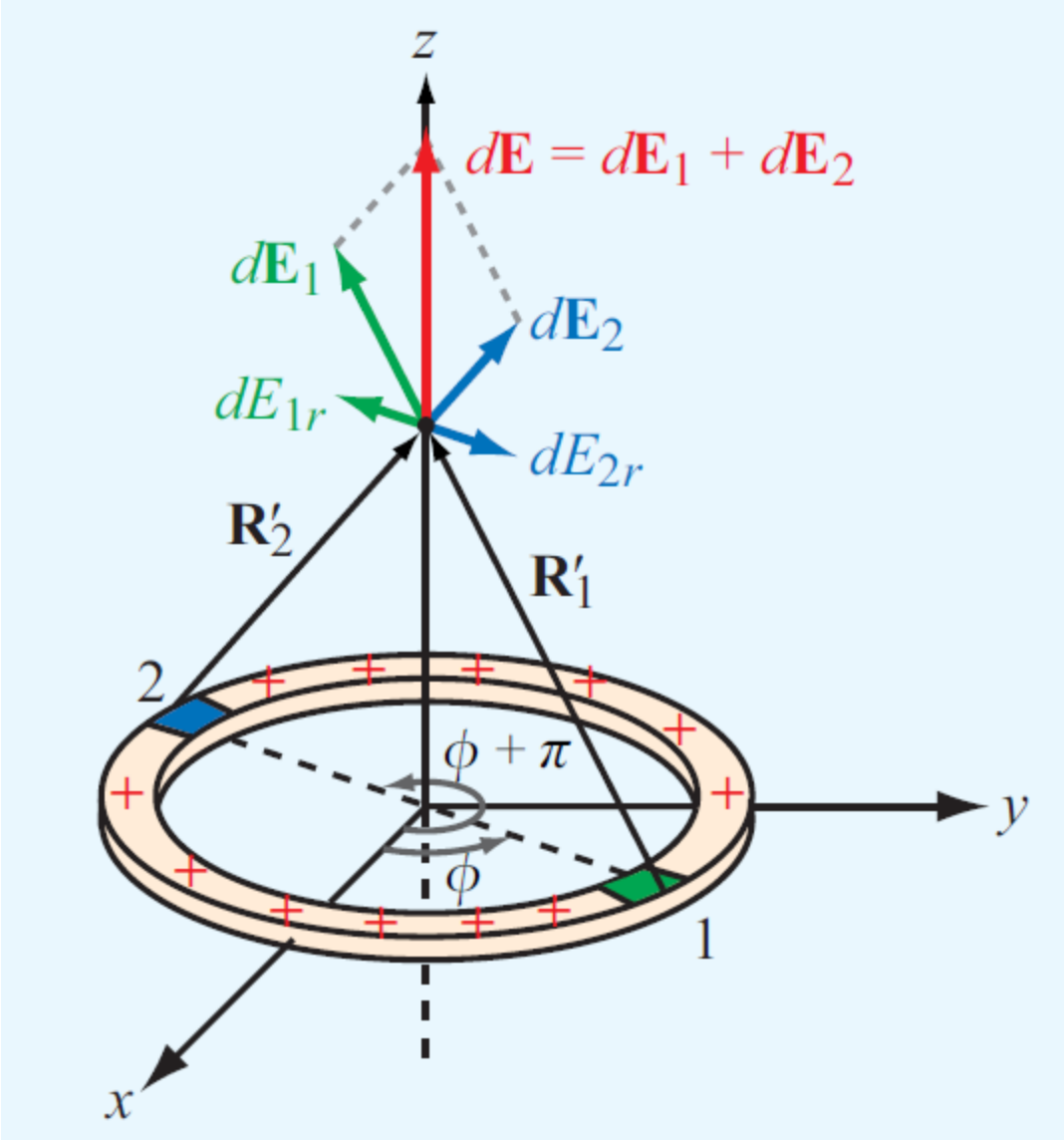
b = **radius**

charge in segment

$$dq = \rho dl = \rho b d\varphi$$

The field $d\mathbf{E}_1$ due to infinitesimal segment 1

Example – Electric field of a Ring of Charge



$$\mathbf{R}'_1 = -\hat{\mathbf{r}}b + \hat{\mathbf{z}}h$$

$$R'_1 = |\mathbf{R}'_1| = \sqrt{b^2 + h^2}$$

$$\hat{\mathbf{R}}'_1 = \frac{\mathbf{R}'_1}{|\mathbf{R}'_1|} = \frac{-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h}{\sqrt{b^2 + h^2}}$$

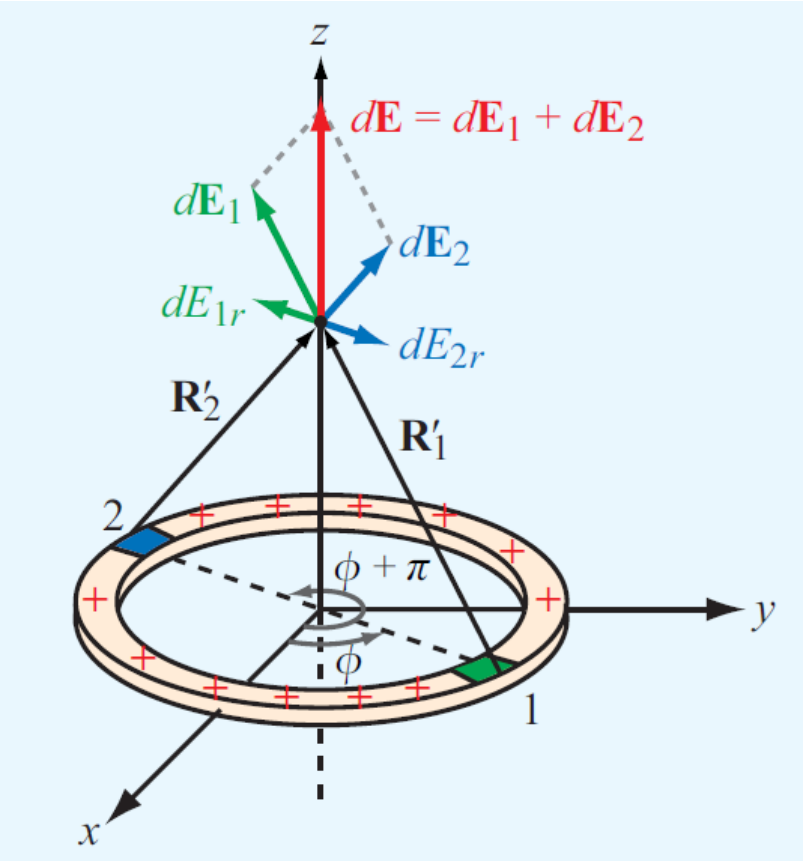
The fields $d\mathbf{E}_1$ and $d\mathbf{E}_2$ due to infinitesimal segments at diametrically opposite locations

Example – Electric field of a Ring of Charge

$$d\mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\,\hat{\mathbf{R}}_1'\,\frac{\rho\,dl}{R_1'^{\,2}} = \frac{\rho\,b}{4\pi\epsilon_0}\,\frac{(-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2+h^2)^{3/2}}\,d\phi$$

$$d\mathbf{E}_2 = \frac{1}{4\pi\epsilon_0}\,\hat{\mathbf{R}}_1'\,\frac{\rho\,dl}{R_1'^{\,2}} = \frac{\rho\,b}{4\pi\epsilon_0}\,\frac{(\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2+h^2)^{3/2}}\,d\phi$$

Coulomb's Law in differential form



The radial components are opposite and cancel out. The axial components add up.

$$d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2$$

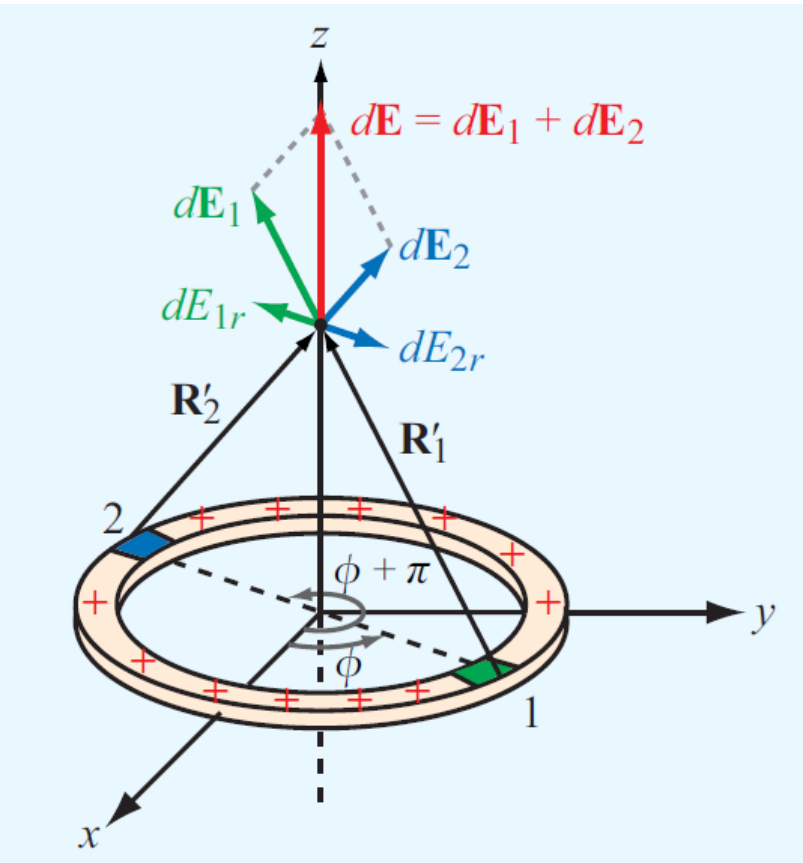
$$= \hat{\mathbf{z}}\,\frac{\rho\,bh}{2\pi\epsilon_0}\,\frac{d\phi}{(b^2+h^2)^{3/2}}$$

Example – Electric field of a Ring of Charge

The total field is obtained by integration in the range $0 \le \phi \le \pi$

total charge on the ring

$$Q = 2\pi b\rho$$



$$\mathbf{E} = \hat{\mathbf{z}}\,\frac{\rho\ bh}{2\pi\epsilon_0(b^2+h^2)^{3/2}}\int_0^{\pi} d\phi$$

$$= \hat{\mathbf{z}}\,\frac{\rho\ bh}{2\epsilon_0(b^2+h^2)^{3/2}}\ \boxed{\times\frac{2\pi}{2\pi}}$$

then multiply

$$= \hat{\mathbf{z}}\,\frac{h}{4\pi\epsilon_0(b^2+h^2)^{3/2}}\ Q,$$

Exercise in class

Given

$$\mathbf{A} = e^{-2y}(\hat{\mathbf{x}}\sin 2x + \hat{\mathbf{y}}\cos 2x)$$

find the divergence $\nabla \cdot \mathbf{A}$

$$A_x = e^{-2y}\sin 2x$$
$$A_y = e^{-2y}\cos 2x$$
$$A_z = 0$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
$$= e^{-2y}\,2\cos 2x - 2e^{-2y}\cos 2x + 0 = 0$$

Exercise in class

If $\mathbf{E} = \hat{\mathbf{r}}\ A\,r$ in spherical coordinates, where A is a constant, calculate the flux of $\mathbf{E}$ through a spherical surface of radius R centered at the origin.

For $r = R$ the field is $\mathbf{E} = \hat{\mathbf{r}}\ A\,R$ and it is uniform and normal to the sphere. So the flux is simply equal to the magnitude of the field multiplied by the surface area of the sphere.

$$\iint_S \mathbf{E} \cdot d\mathbf{s} = 4\pi R^2 \times AR = 4\pi A R^3$$

Example – Find the curl of the vector field (in the class notes)

$$\mathbf{E} = \hat{x}\cos y + \hat{y}1$$

Solution

$$\nabla \times \mathbf{E} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos y & 1 & 0 \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}1\right) - \hat{y}\left(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}\cos y\right) + \hat{z}\left(\frac{\partial}{\partial x}1 - \frac{\partial}{\partial y}\cos y\right)$$

$$= \hat{x}0 - \hat{y}0 + \hat{z}(0 + \sin y)$$

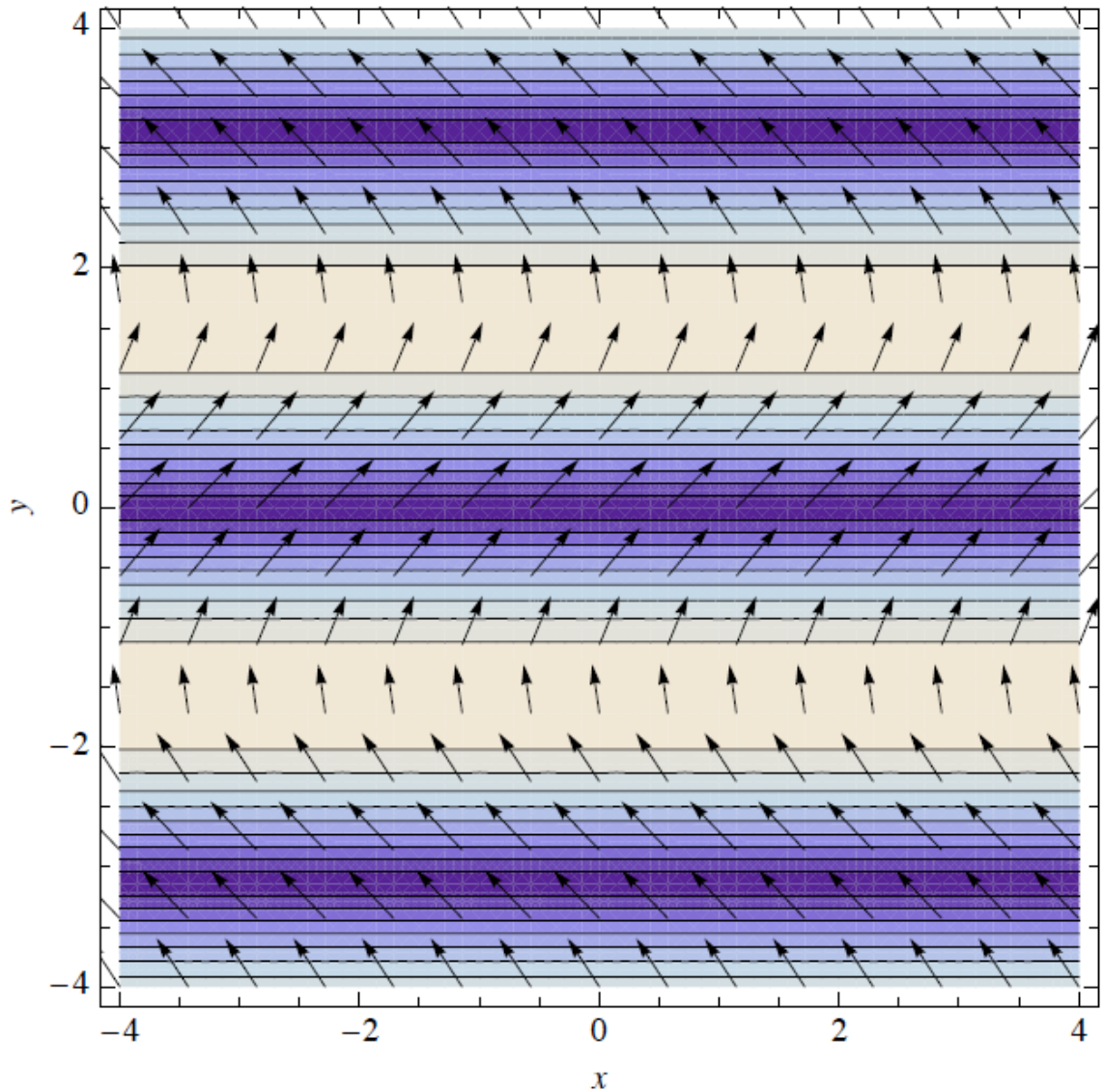
$$\boxed{\nabla \times \mathbf{E} = \hat{z}\sin y}$$

solution is a vector field

$$\mathbf{E} = \hat{x}\cos y + \hat{y}1$$

color map

$$|\nabla \times \mathbf{E}| = |\sin y|$$



Example (in the class notes) – We found earlier that the electrostatic field of a line charge λ along the *z*-axis is radial

$$\mathbf{E}(x, y, z) = \hat{r}\frac{\lambda}{2\pi\epsilon_o r}$$

Verify that it is curl-free.

We can express this result in terms of cartesian coordinates, since

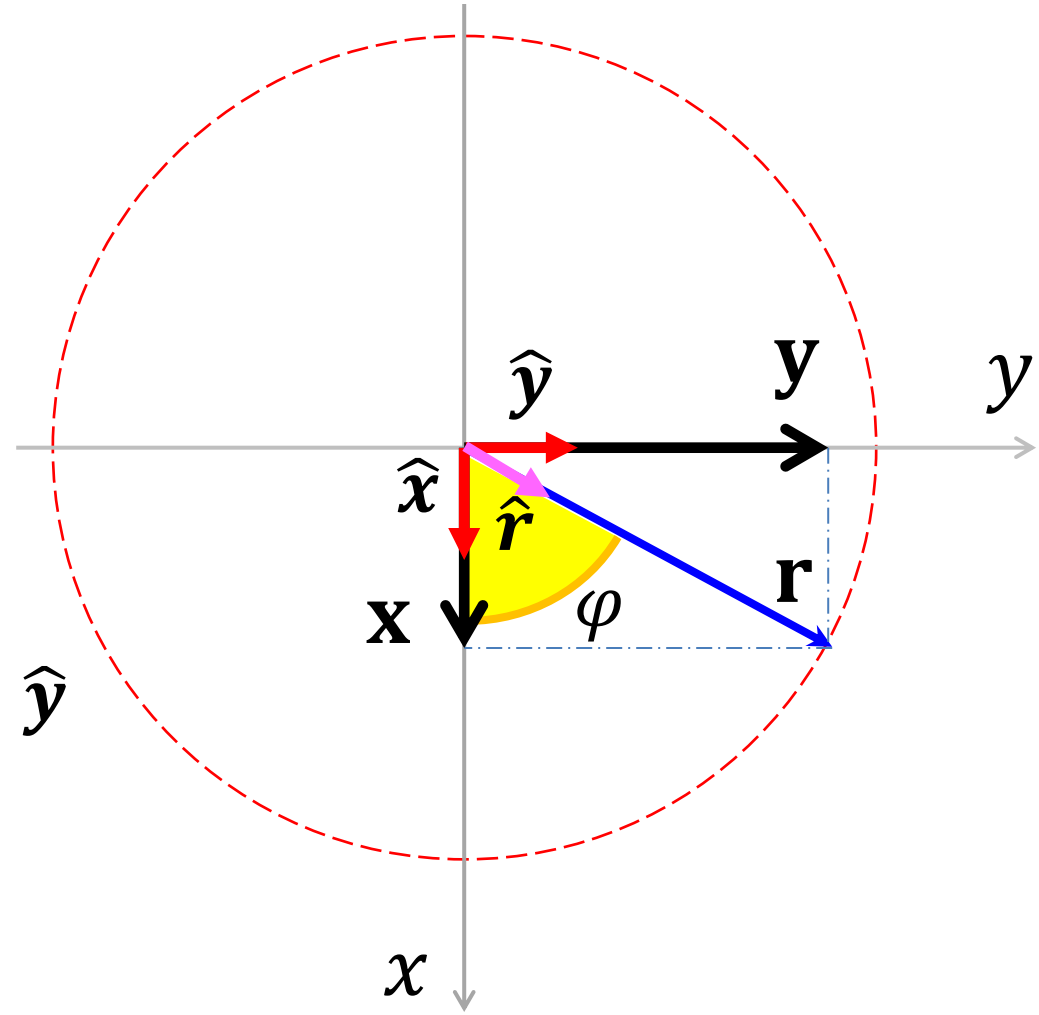
$$r^2 = x^2 + y^2 \qquad \hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi = (\frac{x}{r}, \frac{y}{r}, 0)$$

The electric field has components

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_o}(\frac{x}{r^2}, \frac{y}{r^2}, 0)$$

Derivation

$$\mathbf{r} = r\,\hat{\boldsymbol{r}}$$
$$\mathbf{x} = x\,\hat{\boldsymbol{x}}$$
$$\mathbf{y} = y\,\hat{\boldsymbol{y}}$$



$$\begin{aligned}\mathbf{r} &= r\,\hat{\boldsymbol{r}} \\ &= r\cos\varphi\;\hat{\boldsymbol{x}} + r\sin\varphi\;\hat{\boldsymbol{y}} \\ &= x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}}\end{aligned}$$

$$\hat{\boldsymbol{r}} = \frac{x}{r}\,\hat{\boldsymbol{x}} + \frac{y}{r}\,\hat{\boldsymbol{y}}$$

$$\mathbf{E}(x,y,z) = \hat{\boldsymbol{r}}\,\frac{\lambda}{2\pi\epsilon_0 r} = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{x}{r^2}\,\hat{\boldsymbol{x}} + \frac{y}{r^2}\,\hat{\boldsymbol{y}}\right)$$

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_o}\left(\frac{x}{r^2}, \frac{y}{r^2}, 0\right)$$

The curl of the electric field is found as

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \frac{\lambda}{2\pi\epsilon_o} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^2} & \frac{y}{r^2} & 0 \end{vmatrix}$$

$$= \frac{\lambda}{2\pi\epsilon_o}\hat{z}\left(\underbrace{\frac{\partial}{\partial x}\frac{y}{r^2} - \frac{\partial}{\partial y}\frac{x}{r^2}}\right)$$

$$\frac{\partial}{\partial x}\frac{y}{r^2} - \frac{\partial}{\partial y}\frac{x}{r^2} = y\frac{\partial}{\partial x}\frac{1}{r^2} - x\frac{\partial}{\partial y}\frac{1}{r^2} = y\frac{-2x}{r^4} - x\frac{-2y}{r^4} = 0.$$

we have $\nabla \times \mathbf{E} = 0$ **and the electric field is curl-free.**

Calculation of the partial derivatives

$$r^2 = x^2 + y^2 = f(x,y)$$

Recall that: $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$$\frac{\partial}{\partial x}\frac{y}{r^2} = y\frac{\partial}{\partial x}\frac{1}{f(x,y)} = -y\frac{1}{f^2(x,y)}\frac{\partial}{\partial x}\left(x^2+y^2\right) = -y\frac{1}{r^4}2x$$

$$\frac{\partial}{\partial y}\frac{x}{r^2} = x\frac{\partial}{\partial y}\frac{1}{f(x,y)} = -x\frac{1}{f^2(x,y)}\frac{\partial}{\partial y}\left(x^2+y^2\right) = -x\frac{1}{r^4}2y$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\frac{y}{r^2} = \frac{\partial}{\partial y}\frac{x}{r^2}$$

One could also use directly the curl in cylindrical coordinates, which can be found on formula tables in various textbooks. It would be straightforward to verify by inspection that for the given field all the terms of the cylindrical curl are zero.