ECE 329 – Fall 2022

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

Lecture 8

Lecture 8 – Outline

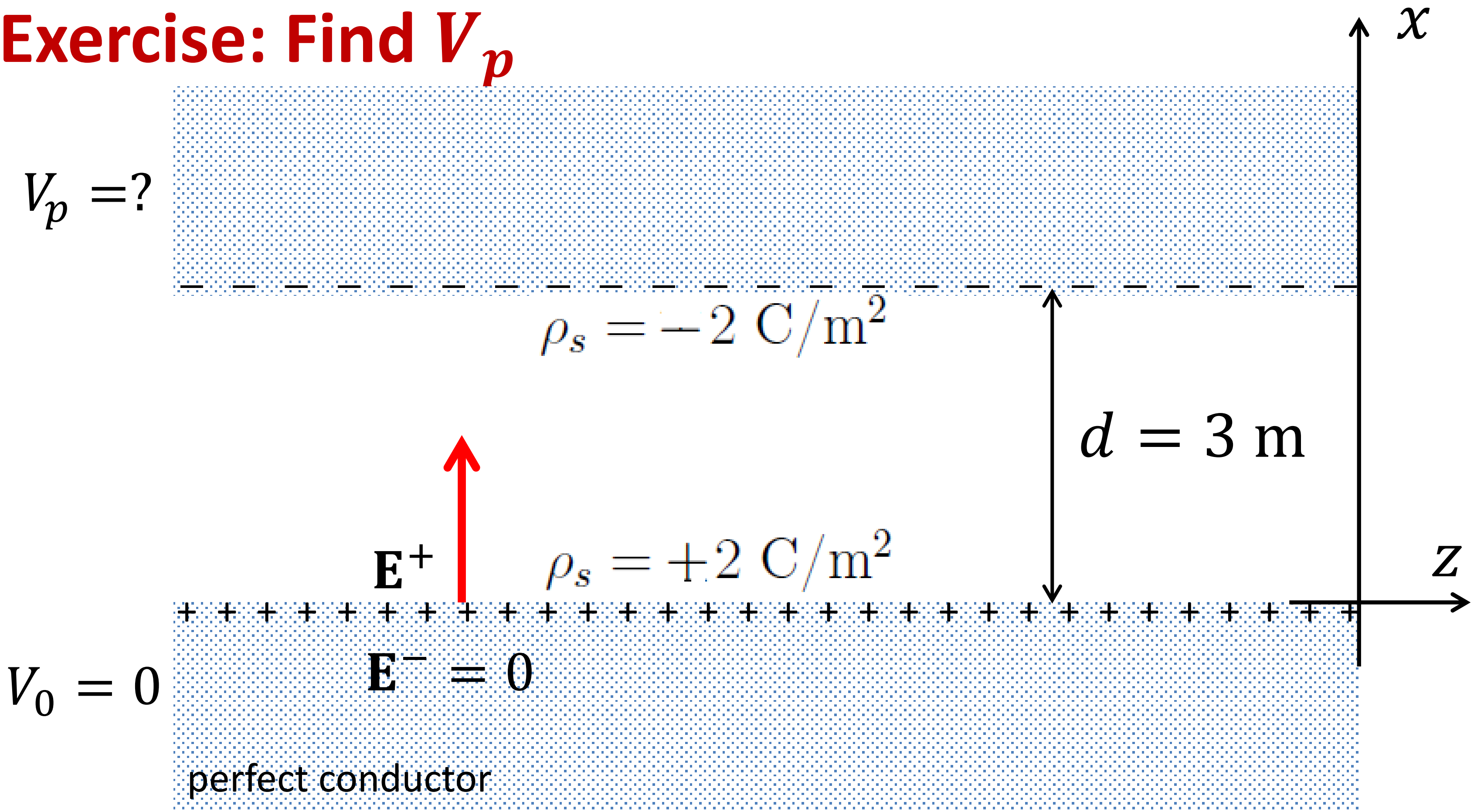
- **More on Poisson Equation**
- **Realistic Materials**
  - **Conductors**
  - **Dielectrics**
  - **Polarizability of materials**

Reading assignment

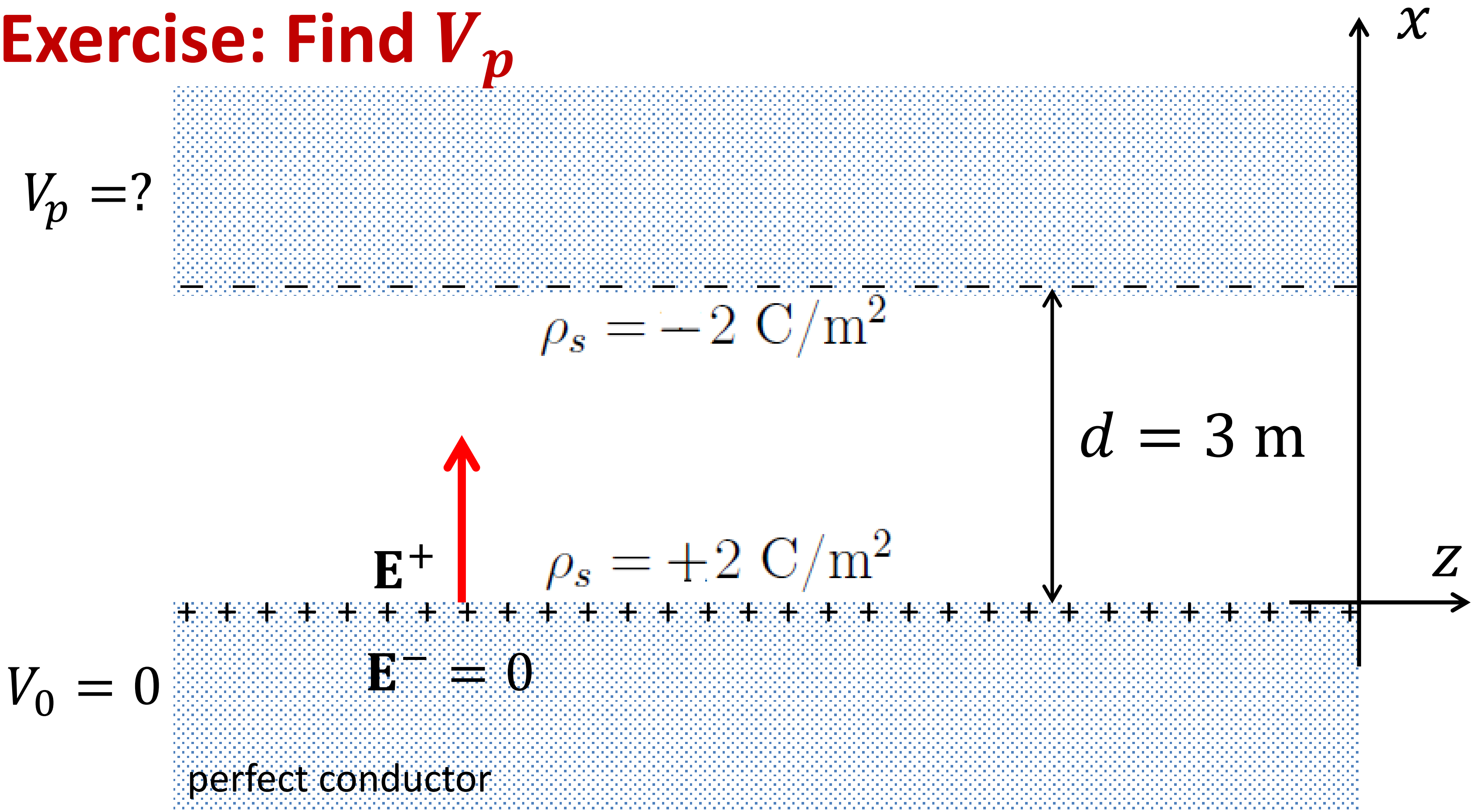
Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:

8) Conductors, dielectrics, and polarization

Exercise: Find V_p



Exercise: Find V_p



$\boxed{\mathbf{D}^- = 0}$ $\boxed{\mathbf{D}^+ = \hat{x}\epsilon_o E_x}$ $\hat{x}\cdot(\mathbf{D}^+ - \mathbf{D}^-) = \epsilon_o E_x = 2\,\frac{\mathrm{C}}{\mathrm{m}^2}$

$\Rightarrow E_x = \frac{2}{\epsilon_o}\ \frac{\mathrm{V}}{\mathrm{m}}$ $V_o - V_p = E_x d = \frac{2}{\epsilon_o}3 = \frac{6}{\epsilon_o}\ \mathrm{V}$ $V_p = -\frac{6}{\epsilon_o}\ \mathrm{V}$

Poisson's Equation is used in place of Laplace's equation when charge is present inside the calculation domain.

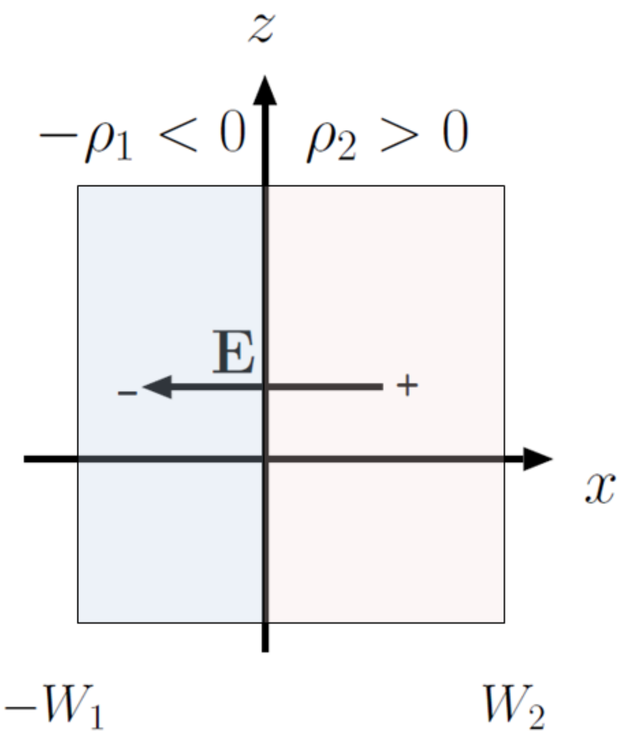
$$\nabla^2 V = -\frac{\rho}{\epsilon_o}$$

Analytical solution is fairly easy for elementary structures and simple charge distribution. For more general situations, numerical solution of Poisson's equation is a very popular approach.

In Lecture 3 we looked at a structures involving two layers of opposite volumetric charge, which is a common elementary model for basic semiconductor diodes. Now we can calculate the potential distribution in that structure.

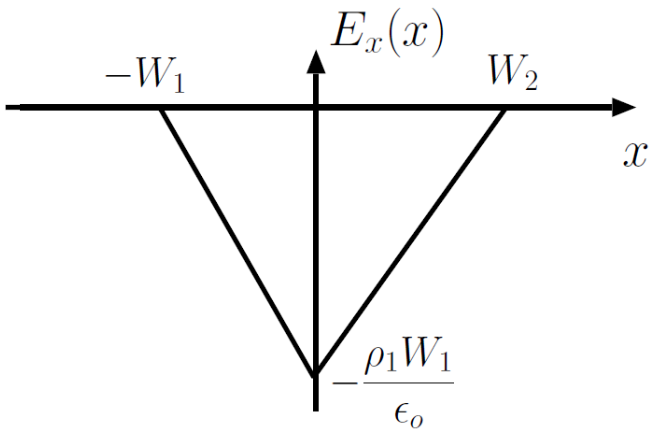
From Lecture 4

Example – Two Slabs with opposite volumetric charge



The system is charge neutral when

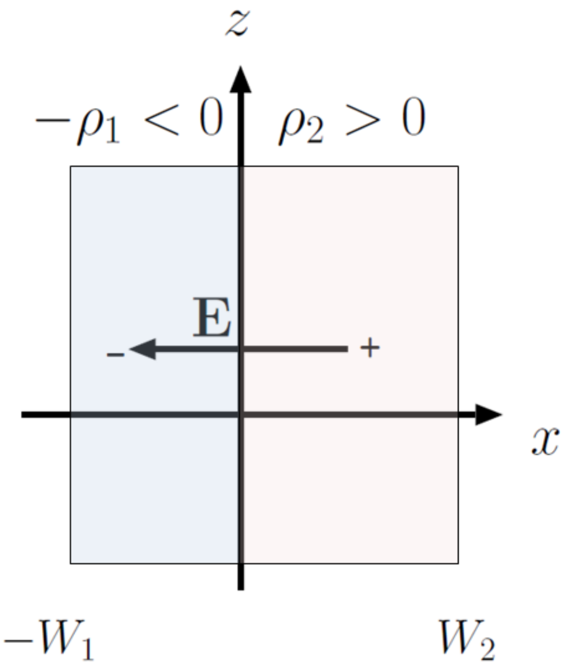
$$W_1\rho_1 = W_2\rho_2$$



We found earlier

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \begin{cases} -\hat{x}\frac{\rho_1(x+W_1)}{\epsilon_o}, & \text{for } -W_1 < x < 0 \\ \hat{x}\frac{\rho_2(x-W_2)}{\epsilon_o}, & \text{for } 0 < x < W_2 \\ 0, & \text{otherwise.} \end{cases}$$

Example – Two Slabs with opposite volumetric charge



$$\frac{d^2V}{dx^2} = -\frac{\rho(x)}{\epsilon_o}$$

Integration over x yields

$$\frac{dV}{dx} = -\mathbf{E}$$

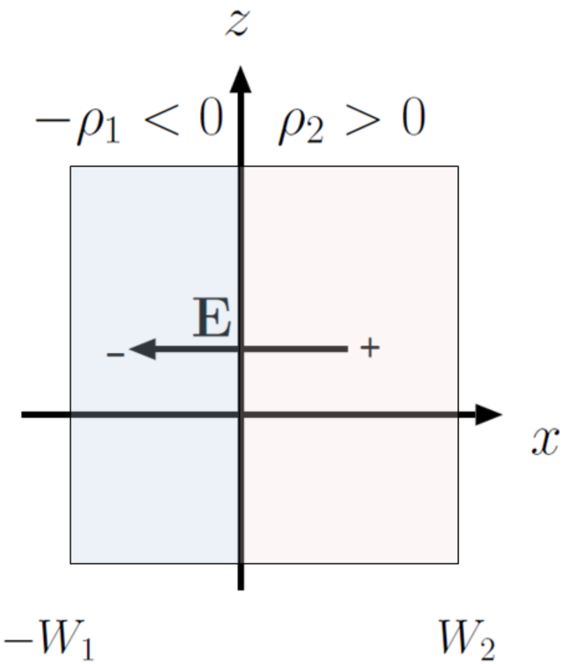
$-W_1 < x < 0$

$$\int_{-W_1}^{x} -\frac{\rho(x)}{\epsilon_0} dx = \frac{\rho_1}{\epsilon_0}[x]_{-W1}^{x} = \frac{\rho_1}{\epsilon_0}[x - (-W_1)]$$

$0 < x < W_2$

$$\int_{W_2}^{x} -\frac{\rho(x)}{\epsilon_0} dx = -\frac{\rho_2}{\epsilon_0}[x]_{W_2}^{x} = -\frac{\rho_2}{\epsilon_0}[x - W_2]$$

Example – Two Slabs with opposite volumetric charge



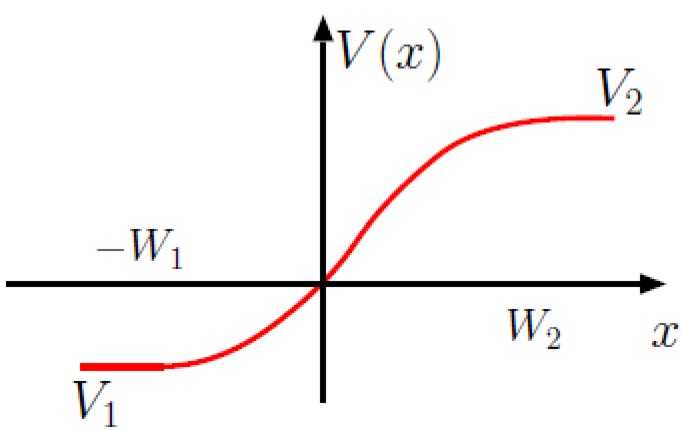
$$\frac{d^2V}{dx^2} = -\frac{\rho(x)}{\epsilon_o}$$

Integration over x yields

$$\frac{dV}{dx} = \begin{cases} \frac{\rho_1(x+W_1)}{\epsilon_o}, & \text{for } -W_1 < x < 0 \\ -\frac{\rho_2(x-W_2)}{\epsilon_o}, & \text{for } 0 < x < W_2 \end{cases}$$

and a second integration

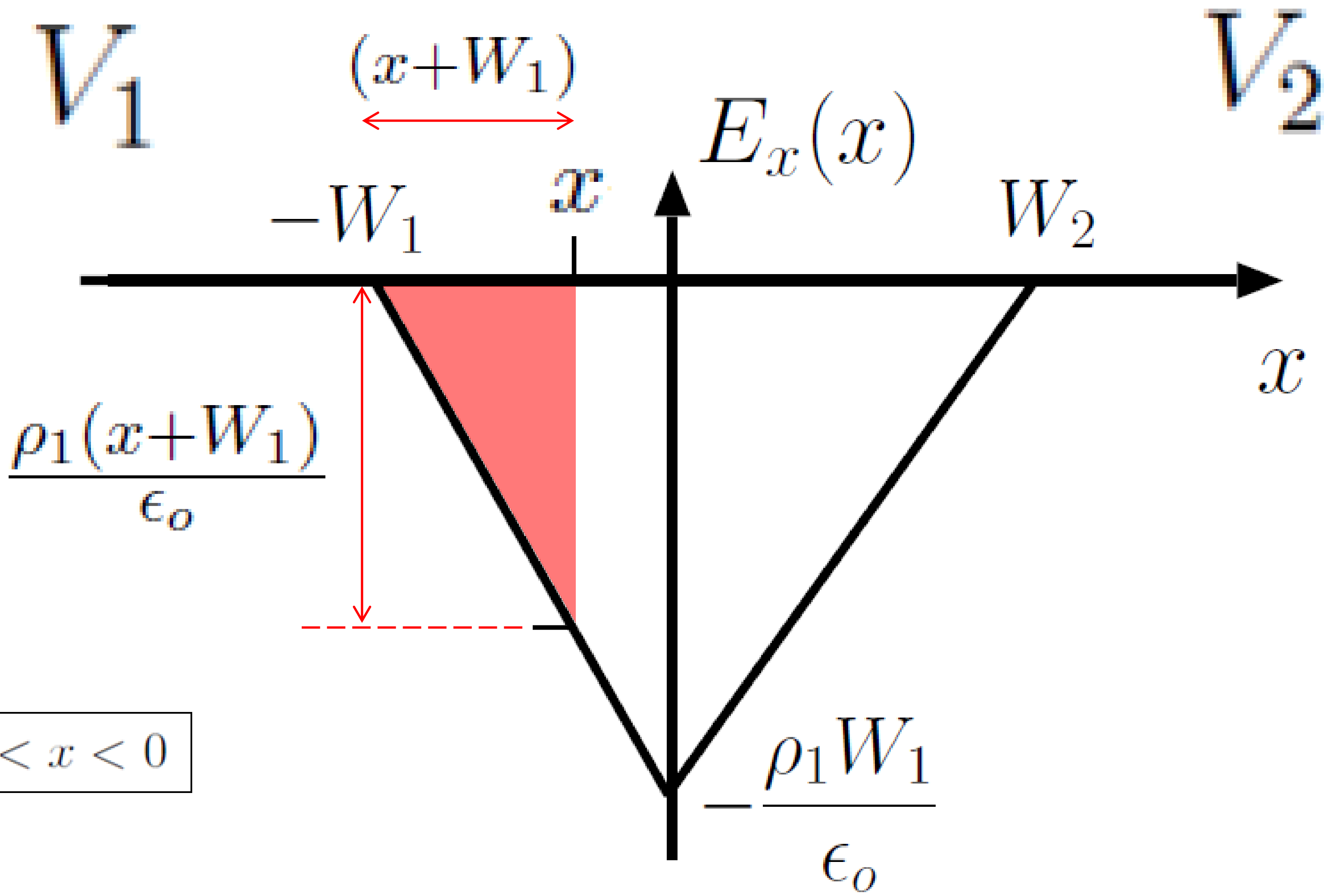
$$V(x) = \begin{cases} \frac{\rho_1(x+W_1)^2}{2\epsilon_o} + V_1, & \text{for } -W_1 < x < 0 \\ -\frac{\rho_2(x-W_2)^2}{2\epsilon_o} + V_2, & \text{for } 0 < x < W_2 \end{cases}$$



$\boxed{x = 0}$ solutions have to match

$$V_{21} = V_2 - V_1 = \frac{\rho_2 W_2^2 + \rho_1 W_1^2}{2\epsilon_o}$$

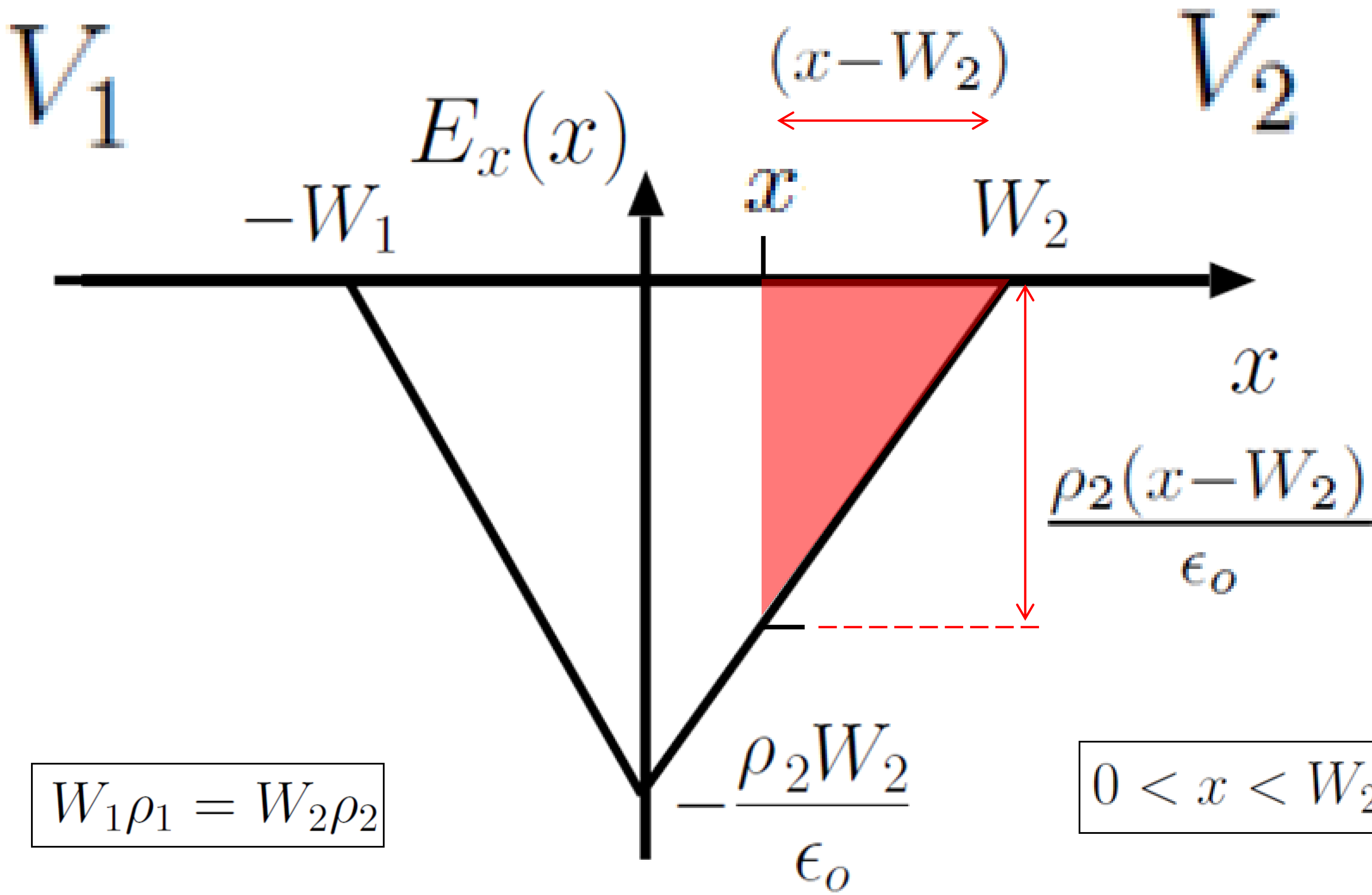
Geometric representation of the integral (left side)



$$\boxed{-W_1 < x < 0}$$

$$\boxed{V_1 + \frac{\rho_1(x+W_1)^2}{2\epsilon_o}}$$

Geometric representation of the integral (right side)



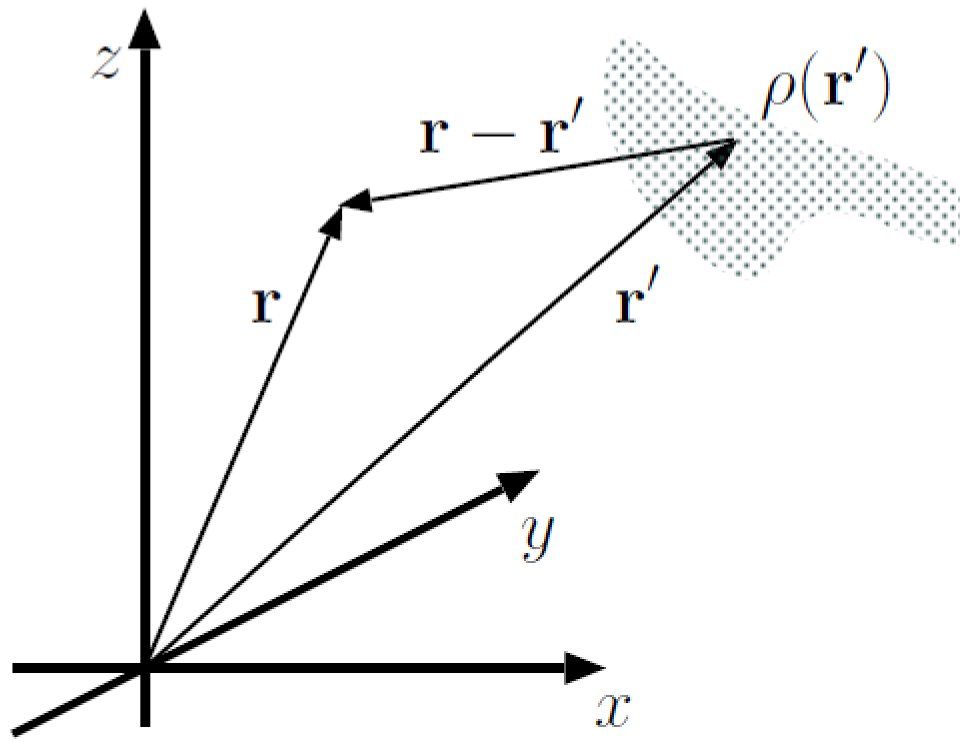
$$\boxed{W_1 \rho_1 = W_2 \rho_2}$$

$$\boxed{0 < x < W_2}$$

$$\boxed{V_2 - \frac{\rho_2 (x - W_2)^2}{2\epsilon_o}}$$

For arbitrary charge distributions, an analytical framework could be set up where the potential at any point in space is obtained through integration over elements of charge which make up the distribution

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$



Each element of charge is representable with a delta function in space which contributes to the total potential through a function expressing physical response (usually called "Green's function").

$$\frac{1}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|}$$

potential due to a unitary charge

In summary:

Charge representation in terms of delta (impulse) functions

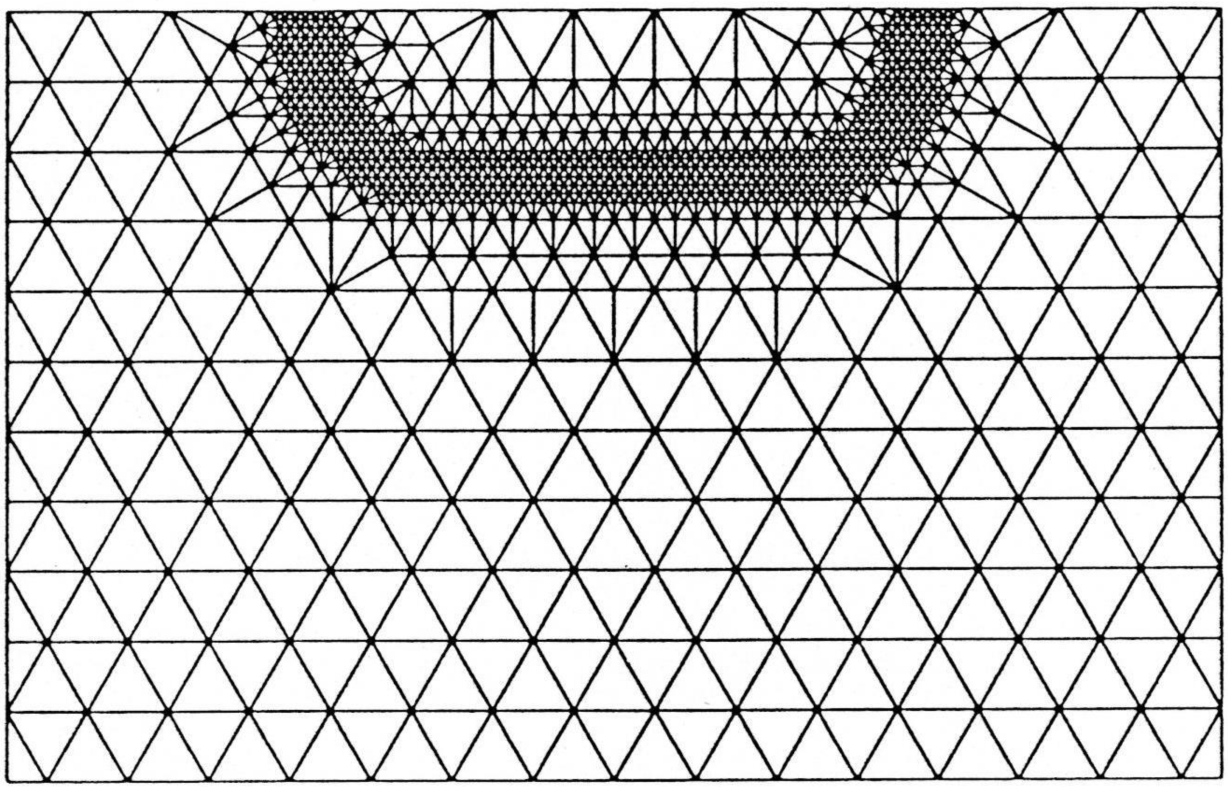
$$\rho(\mathbf{r}) = \int \rho(\mathbf{r}') \underbrace{\delta(\mathbf{r}-\mathbf{r}')}_{\text{delta (impulse) function}} d^3\mathbf{r}'$$

is mapped into the distribution of potential by knowing the appropriate Greens function that solves the physics of Poisson's equation (the response of the system to the impulse)

$$V(\mathbf{r}) = \int \rho(\mathbf{r}') \underbrace{\frac{1}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|}}_{\text{potential due to a unitary charge}} d^3\mathbf{r}'$$

For general cases, direct integration can be difficult. In alternative, numerical approaches have been developed for approximation of the differential operator (e.g., finite differences, finite elements).

The domain is subdivided into a "grid" and solution is sought only at the "grid nodes".



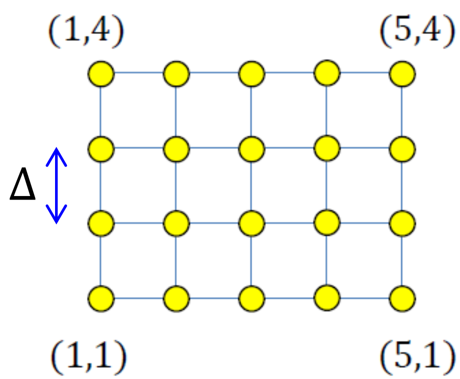
Example of highly non-uniform grid

More grid nodes are placed where more precision is needed

The analytical problem is transformed into a "linear algebra problem" involving solution of large systems of linear equations with matrices.

$$
\begin{bmatrix}
-4 & 1 & 0 & 1 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 \\
0 & 1 & -4 & 0 & 0 & 1 \\
1 & 0 & 0 & -4 & 1 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 \\
0 & 0 & 1 & 0 & 1 & -4
\end{bmatrix}
\begin{bmatrix}
V(2,2) \\ V(3,2) \\ V(4,2) \\ V(2,3) \\ V(3,3) \\ V(4,3)
\end{bmatrix}
=
\begin{bmatrix}
-\frac{\Delta^2 \rho(2,2)}{\epsilon} - V_{21} - V_{12} \\
-\frac{\Delta^2 \rho(3,2)}{\epsilon} - V_{31} \\
-\frac{\Delta^2 \rho(4,2)}{\epsilon} - V_{41} - V_{52} \\
-\frac{\Delta^2 \rho(2,3)}{\epsilon} - V_{13} - V_{24} \\
-\frac{\Delta^2 \rho(3,3)}{\epsilon} - V_{34} \\
-\frac{\Delta^2 \rho(4,3)}{\epsilon} - V_{44} - V_{53}
\end{bmatrix}
$$

boundary nodes



Linear system matrix problem arising from this uniform discretization of a rectangular domain with finite differences (note: corner points are not used in the solution)

Numerical solution of Poisson's equation is used extensively in the simulation of semiconductor devices and is an essential component of CAD software for electronics in industry.

MATERIALS

Materials consist of assemblies of atoms. Based on the type and ordering of the atoms, materials exhibit various physical properties to which measured parameters are associated.

"Constitutive parameters" which are relevant for electromagnetics are:

- **electrical permittivity ϵ**
- **magnetic permeability μ**
- **conductivity σ**

A material is "homogeneous" if its constitutive parameters do not vary from point to point.

A material is "isotropic" if its constitutive parameters do not depend on direction.

Unless specified otherwise, we will assume that materials are homogeneous and isotropic.

Just keep in mind that in electromagnetics and in electronics there are also important materials which are anisotropic, for instance some "crystals".

Conductors and Dielectrics

The ***conductivity*** of a material is a measure of how easily electrons or other charges (e.g., ions in a liquid) can travel through the material under the influence of an externally applied electric field.

- Materials are classified as ***conductors*** (e.g., metals) or ***dielectrics*** (insulators) according to the magnitudes of their conductivities.

Conductivity

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- Materials are classified as ***conductors*** (e.g., metals) or ***dielectrics*** (insulators) according to the magnitudes of their conductivities.
- Special materials called ***semiconductors*** (typically crystals) have relatively low conductivity in their pure state but this can be increased dramatically with the introduction of certain impurities called ***dopants***. Learn more about it in **ECE 340**.

Conductors

A solid conductor normally has a large number of loosely bonded electrons in the outermost energy levels of its atoms.

At rest, electrons move randomly producing on the average zero current. However, with the application of an external electric field, these "**conduction electrons**" can easily move from atom to atom. The collective movement generates a "**conduction current density**"

$$\mathbf{J} = \sigma \mathbf{E} \quad (\mathrm{A/m^2}) \quad \textbf{(Ohm's law)}.$$

where the units of σ are Siemens/meter (S/m).

Limit Case

A ***perfect electric conductor*** (PEC) has $\sigma \to \infty$.

As a consequence of Ohm's law, $\mathbf{E} \to \mathbf{0}$ regardless of J.

Circuit analogue: Short Circuit

When the electric field is zero everywhere, the electric potential must be constant everywhere. Therefore, a **PEC** is an **equipotential medium**.

Also, we must have $\boxed{\rho = \nabla \cdot \mathbf{D} = \nabla \cdot \epsilon_o \mathbf{E} = 0}$

This means that net charge cannot be present in the interior of a **PEC**. Only the surface can have non-zero charge, where $\boxed{\hat{n} \cdot \mathbf{D} = \rho_s \quad \text{and} \quad \hat{n} \times \mathbf{E} = 0}$

Conductivity σ of metals increases with decreasing temperature. This is because, while the number of available carriers does not vary significantly with temperature, the atomic vibrations which slow down conduction by causing electron to scatter, are quenched in a cooler material.

Some materials even become ***superconductors***, approaching the behavior of a perfect conductor, when reaching extremely low temperatures. The physics behind this phenomenon is rather sophisticated.

Dielectrics

In a solid dielectric electrons are tightly bound to the atoms. Application of an electric field does not produce significant current.

In general, the conductivity $\boldsymbol{\sigma}$ of *dielectrics* and *semiconductors* increases with temperature, because electrons gain thermal energy and may reach excited energy states where they are free to move.

Limit Case

A ***perfect dielectric (insulator)*** has $\sigma \to 0$.

As a consequence, $\boldsymbol{J} \to \mathbf{0}$ regardless of $\mathbf{E}$.

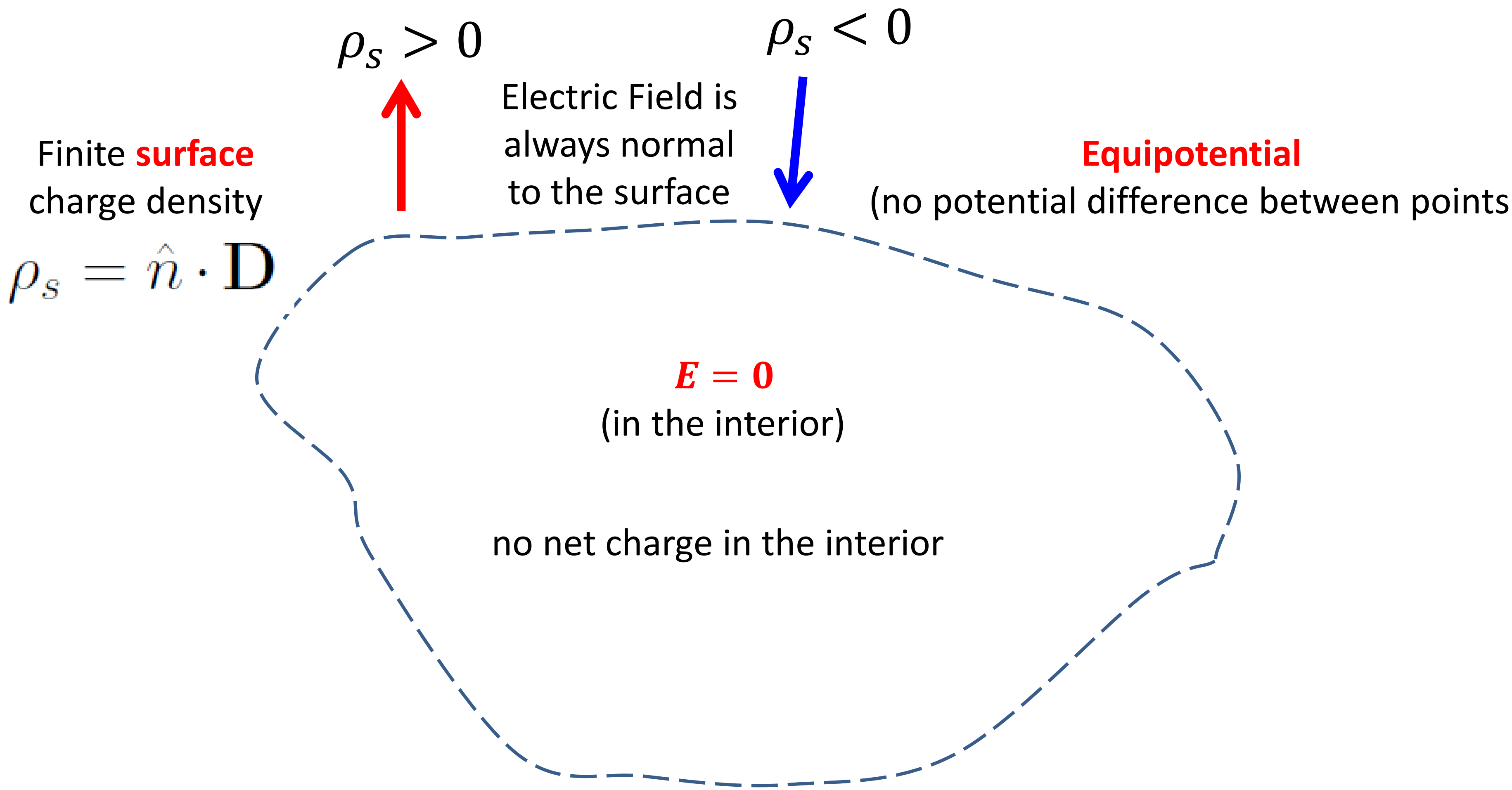
Circuit analogue: Open Circuit

Conductivity of some common materials at room temperature

Material	Conductivity, σ (S/m)
Conductors	
Silver	6.2×10^7
Copper	5.8×10^7
Gold	4.1×10^7
Aluminum	3.5×10^7
Iron	10^7
Mercury	10^6
Carbon	3×10^4
Semiconductors	
Pure germanium	2.2
Pure silicon	4.4×10^{-4}
Insulators	
Glass	10^{-12}
Paraffin	10^{-15}
Mica	10^{-15}
Fused quartz	10^{-17}

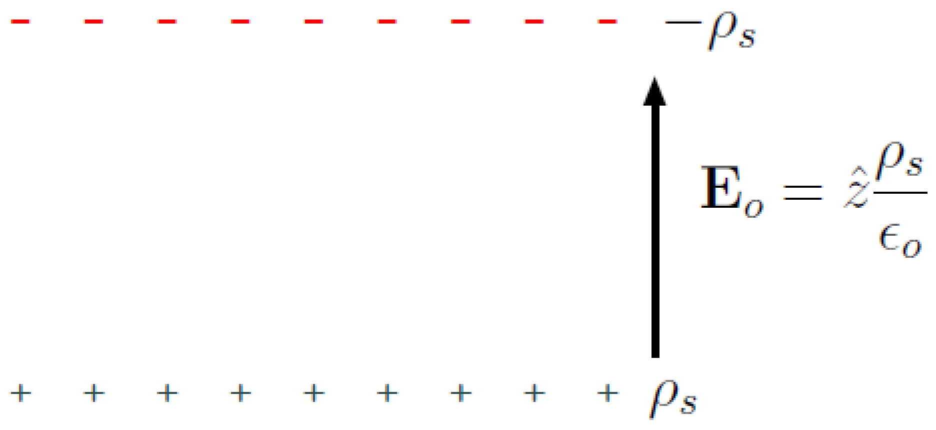
In Summary:

Perfect conductor model for metals in electrostatics

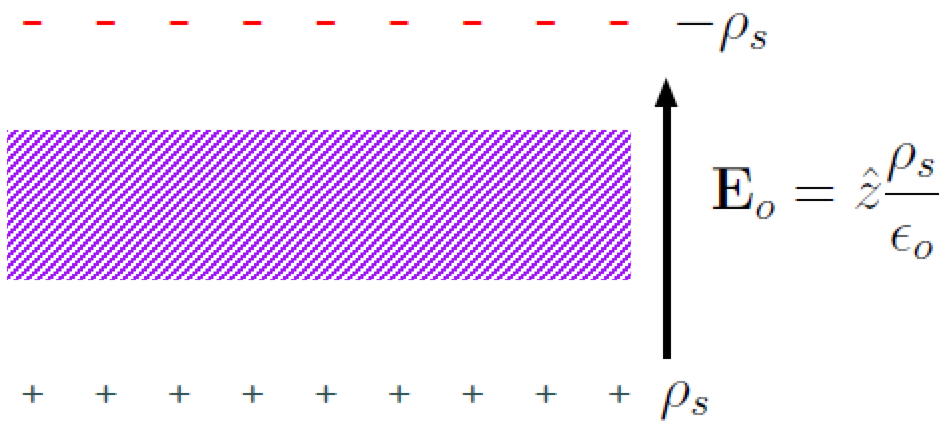


Example – Conducting slab inserted in a uniform field region

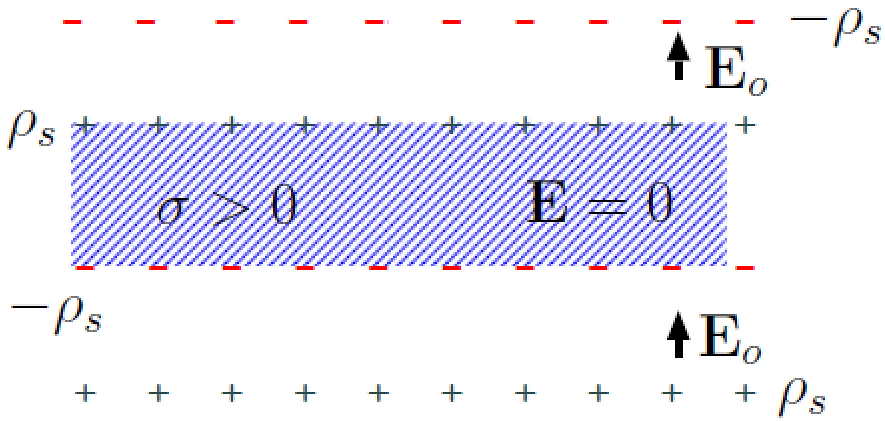
Consider this configuration with a uniform electric field established between charge sheets



What happens if a conducting slab is inserted somewhere in the middle?



Example – Conducting slab inserted in a uniform field region



Electrons move inside the metal to form surface layers of charge which provide boundary conditions for the original field to still exist outside the metal slab. At the same time the electric field inside the slab collapses to zero.

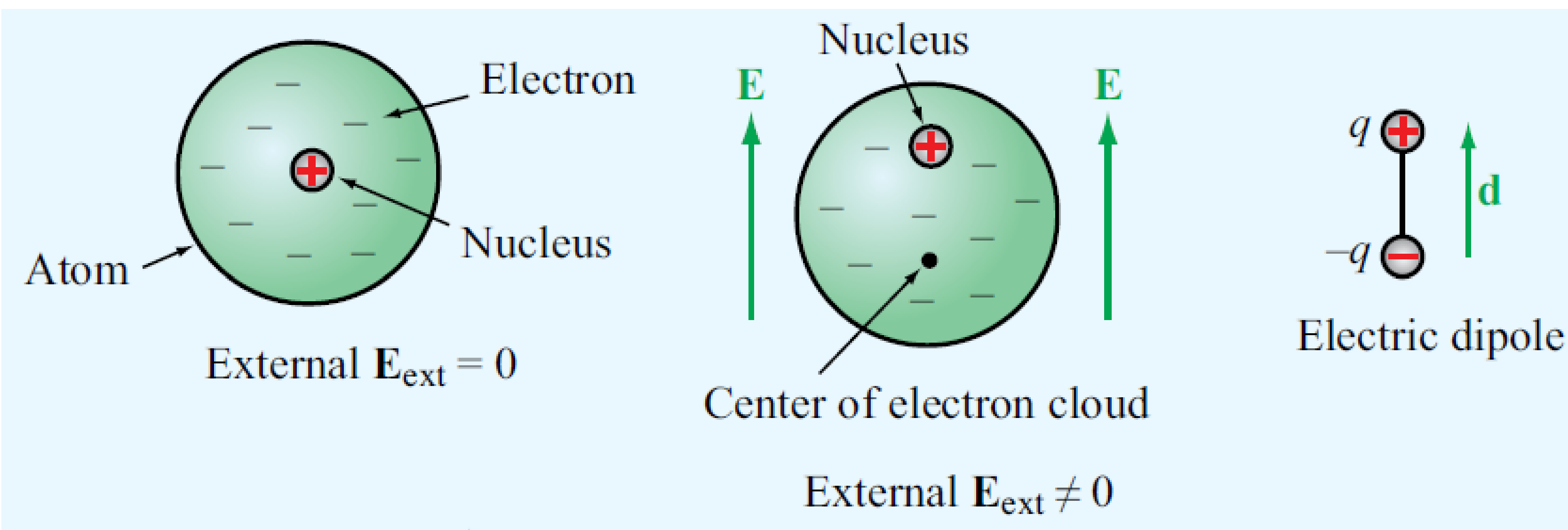
There is a transient to establish this new equilibrium configuration which is on the order of $\approx 10^{-18}$ s for metals (much less in practice than the time necessary to insert the slab). $\Rightarrow$ **Static conditions can be assumed.**

Polarization of Dielectrics

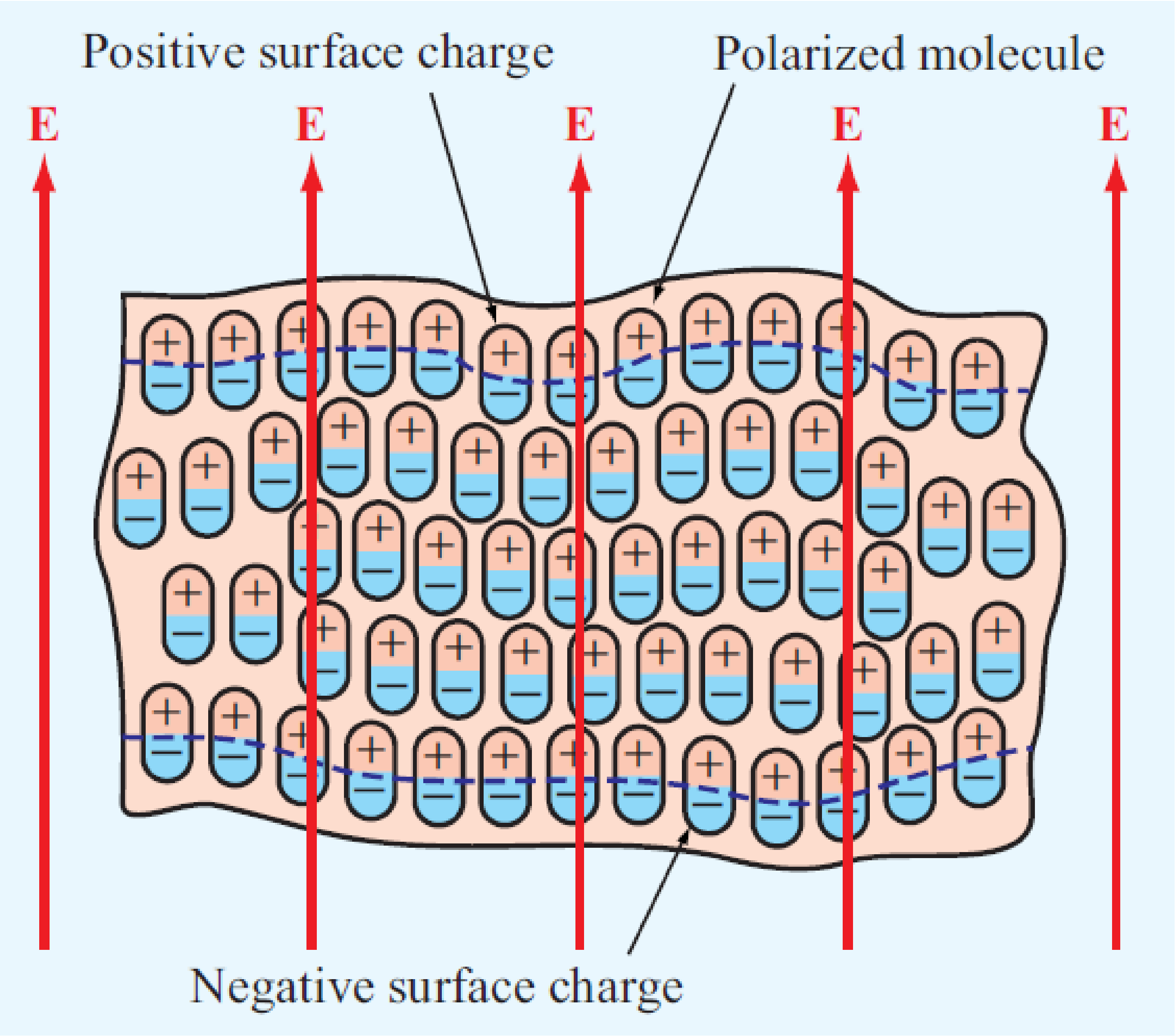
(non-polar materials with no permanent dipoles)

In a dielectric, an external electric field $\mathbf{E}$ cannot cause large movement of charges since they are not able to move freely. The field $\mathbf{E}$ will ***polarize*** the atoms or molecules in the material by moving the center of the electron cloud away from the nucleus.

The polarized atom or molecule may be represented by an electric dipole consisting of charges $+q$ in the nucleus and $-q$ at the center of the electron cloud.

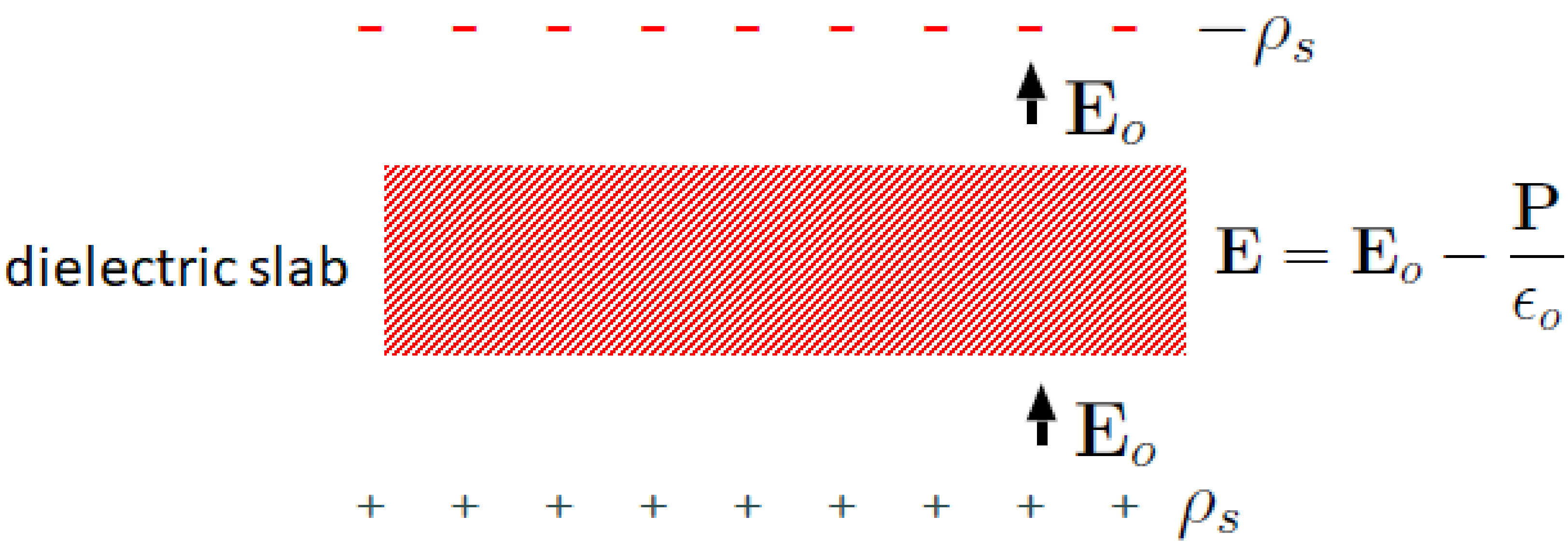


Dielectric medium polarized by an external field E



A “Perfect Dielectric” has no mobile charge carriers ($\sigma = 0$)

The electric field is not zero in the interior as for conductors but it is weaker than it would be if there were no polarization effects.



Polarization Field

Free space $$\mathbf{D} = \epsilon_0 \mathbf{E}$$

Material medium $$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

Electric Polarization Field $$\mathbf{P}$$

For linear, isotropic and homogeneous medium $$\mathbf{P} = \epsilon_0 \chi_{\mathrm{e}} \mathbf{E}$$

Electric Susceptibility

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \overbrace{\epsilon_0 \chi_e \mathbf{E}}^{\mathbf{P}}$$

$$= \epsilon_0 (1 + \chi_e) \mathbf{E}$$

$$= \epsilon \mathbf{E}$$

Permittivity of the Material $\epsilon = \epsilon_0 \underbrace{(1 + \chi_e)}_{\epsilon_r}$

Relative Permittivity ϵ_r

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Relative Permittivity ϵ_r

In static conditions the relative permittivity in natural media is > 1 since the susceptibility is always positive.

At high frequencies in certain metals, in certain ferromagnetic materials, in a state of matter called "plasma", and in synthetic structures called "metamaterials" susceptibility may become < 1 (even negative!). However, in general these are not "static" effects. They affect how EM waves propagate in the medium.

Relative Permittivity ϵ_r of Common Materials

$$\epsilon = \epsilon_r \epsilon_0 \text{ and } \epsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$$

Material	Relative Permittivity, ϵ_r	Material	Relative Permittivity, ϵ_r
Vacuum	1	Dry soil	2.5–3.5
Air (at sea level)	1.0006	Plexiglass	3.4
Styrofoam	1.03	Glass	4.5–10
Teflon	2.1	Quartz	3.8–5
Petroleum oil	2.1	Bakelite	5
Wood (dry)	1.5–4	Porcelain	5.7
Paraffin	2.2	Formica	6
Polyethylene	2.25	Mica	5.4–6
Polystyrene	2.6	Ammonia	22
Paper	2–4	Seawater	72–80
Rubber	2.2–4.1	Distilled water	81

These are low-frequency values at room temperature (20° C).

For most metals, $\epsilon_r \simeq 1$