ECE 329 – Fall 2022

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

Lecture 10

Lecture 10 – Outline

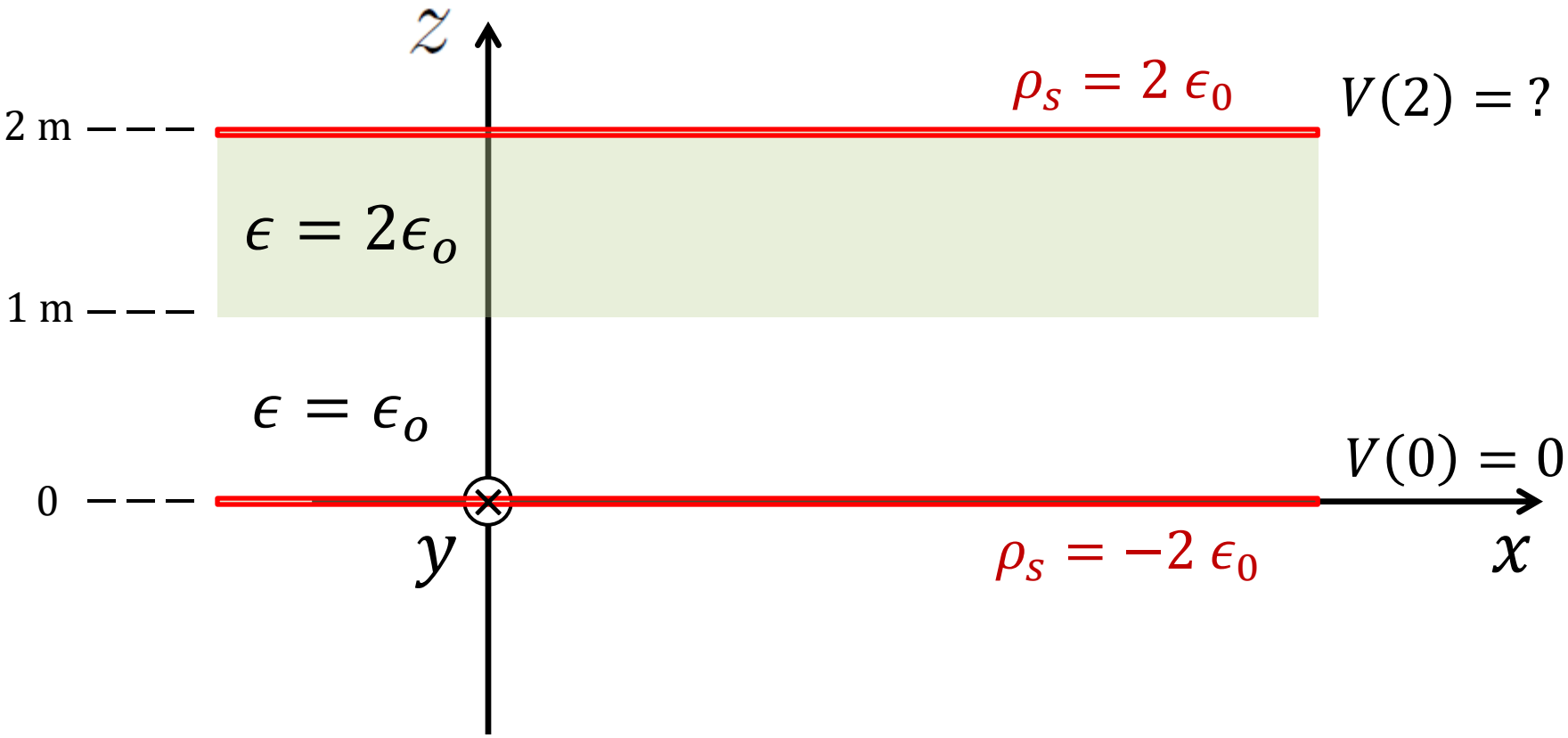
- **More Problems**
- **Parallel Plate Capacitor**
- **Coaxial Cable**
- **Diode Junction Capacitance**
- **Conductance in the imperfect dielectric of a capacitor**

Reading assignment

Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:

10) Capacitance and Conductance

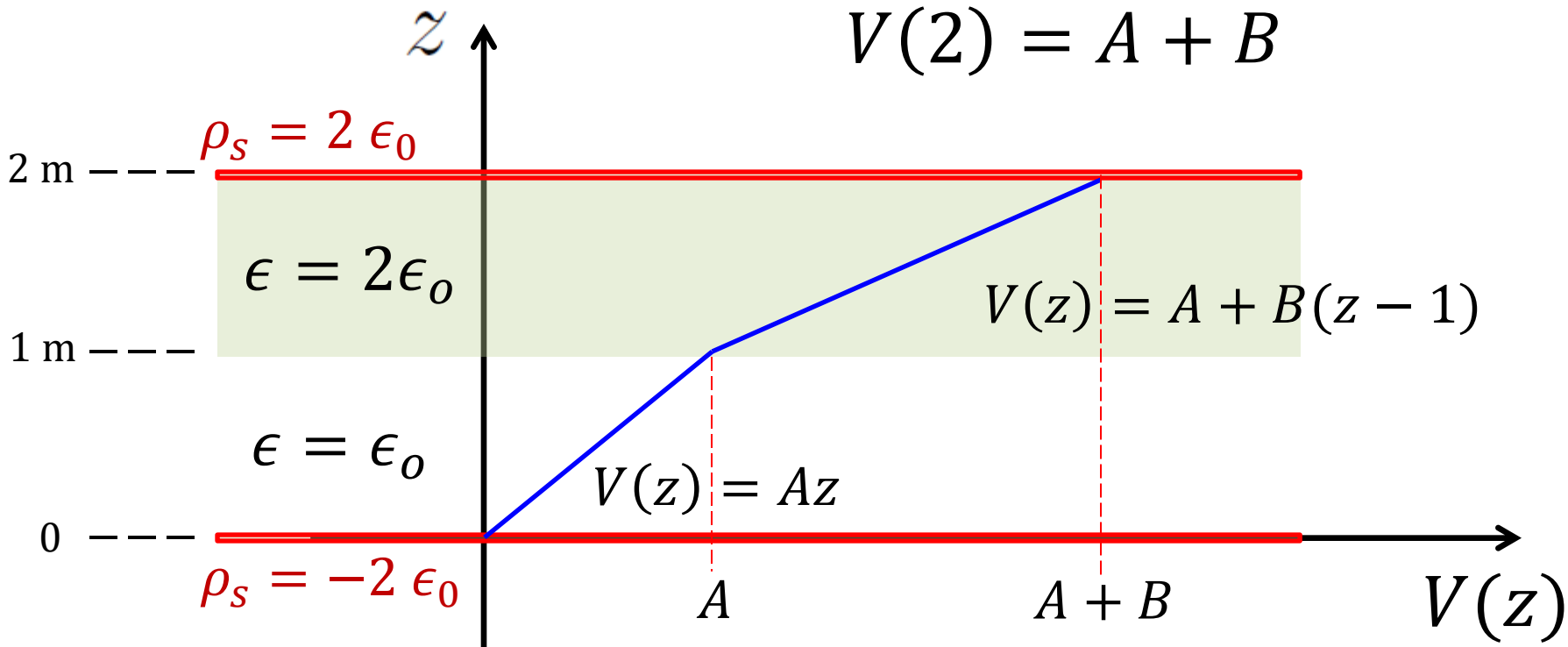
Example – Two infinite conducting plates separated by two layers of dielectric. Find $V(2)$.



Example – Two infinite conducting plates separated by two layers of dielectric. Find $V(2)$.

$$V(1) = A$$

$$V(2) = A + B$$



$$\boxed{0 < z < 1\ \mathrm{m}}$$

$$-\int_0^{1\mathrm{m}} E\,dz = V(0) - V(1) = -V(1) = -A$$

B.C. at z = 0 $\boxed{\hat{z}\cdot\mathbf{D}(0) = \rho_s = -2\epsilon_o}$ → $\boxed{A = 2}$

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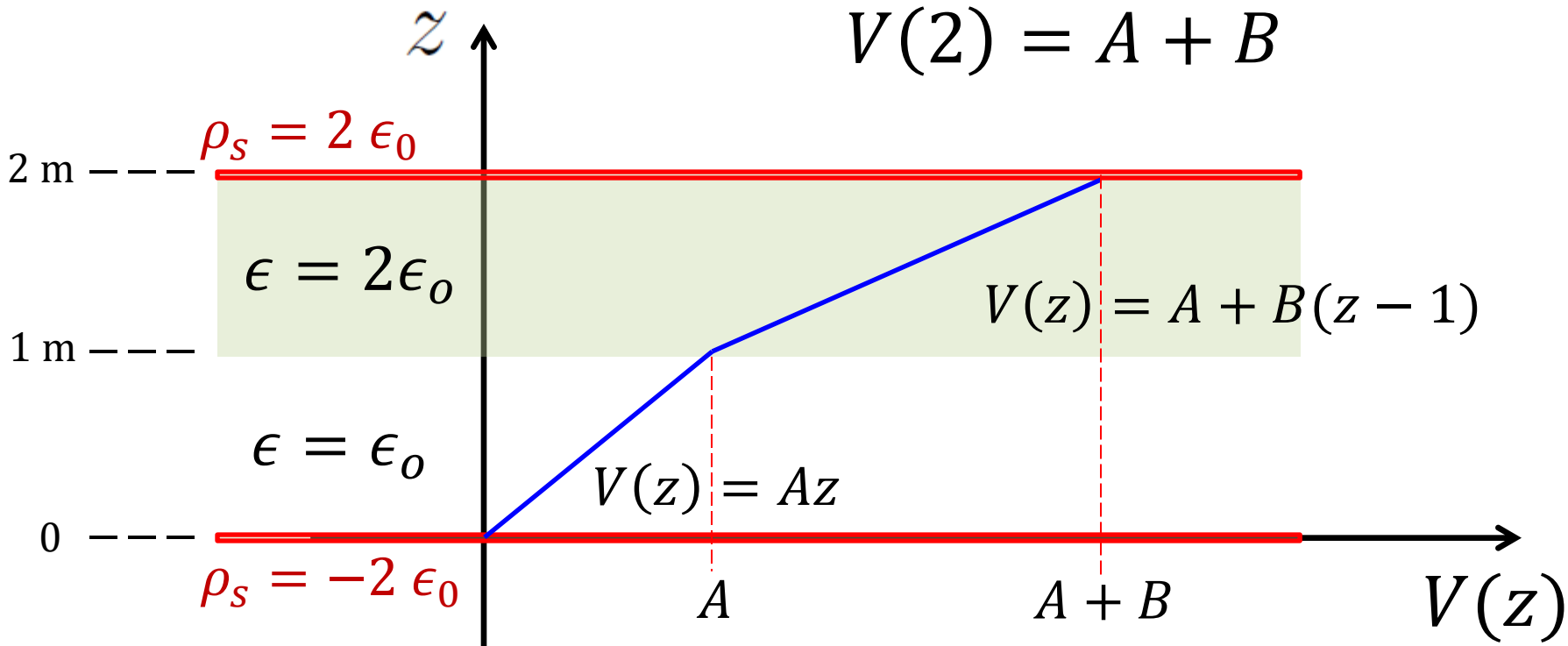

$$\epsilon_0\, E_n^- = 2\,\epsilon_0\, E_n^+$$

$$\epsilon_0\,(-2) = 2\,\epsilon_0\, E_n^+ \rightarrow \boxed{B = 1}$$

$$E_n^+ = -1$$

$$\boxed{z = 1\ \mathrm{m}} \qquad \mathbf{D}(1^+) \text{ ---------- } \mathbf{D}(1^-)$$

Example – Two infinite conducting plates separated by two layers of dielectric. Find $V(2)$.

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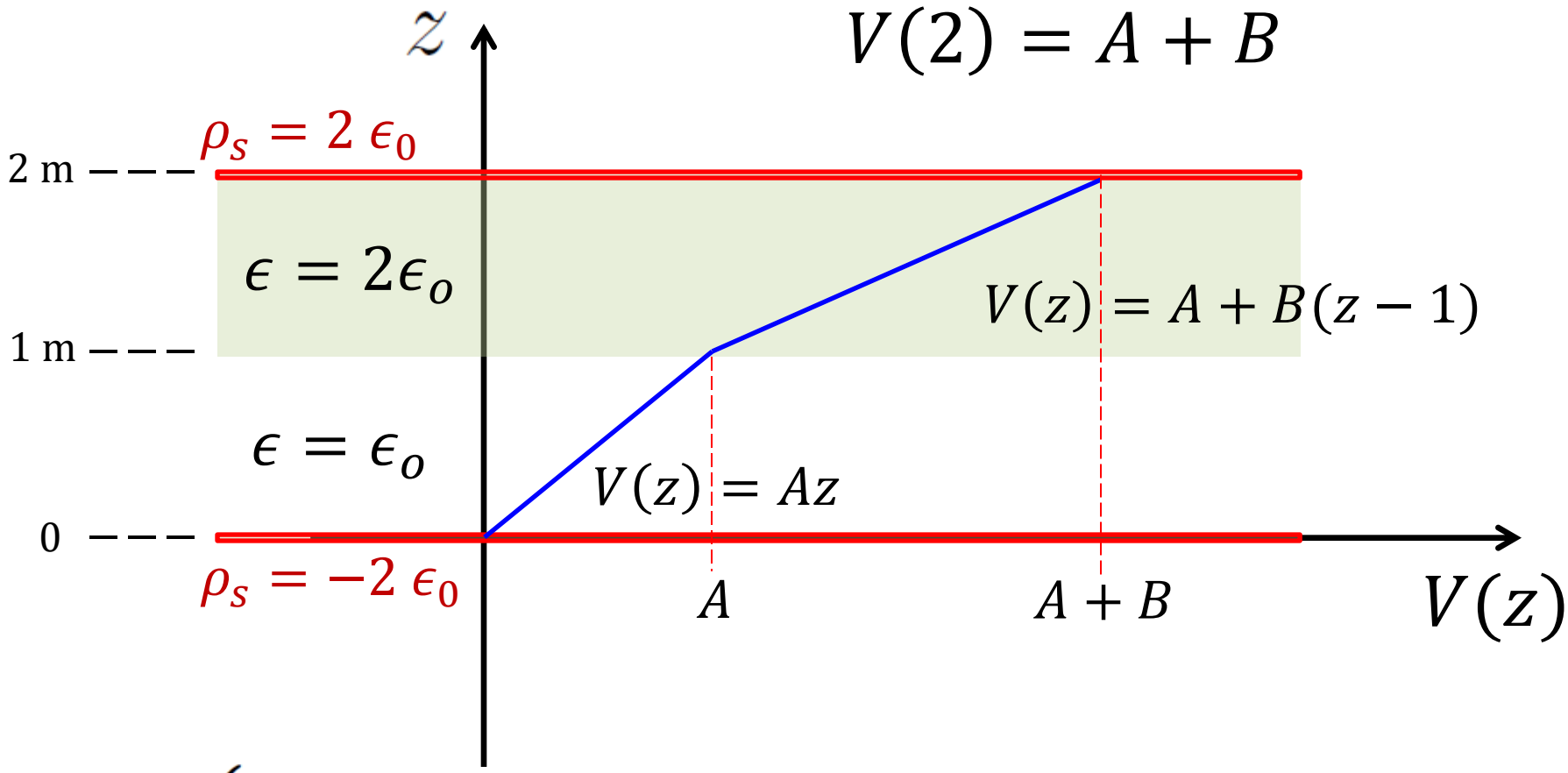
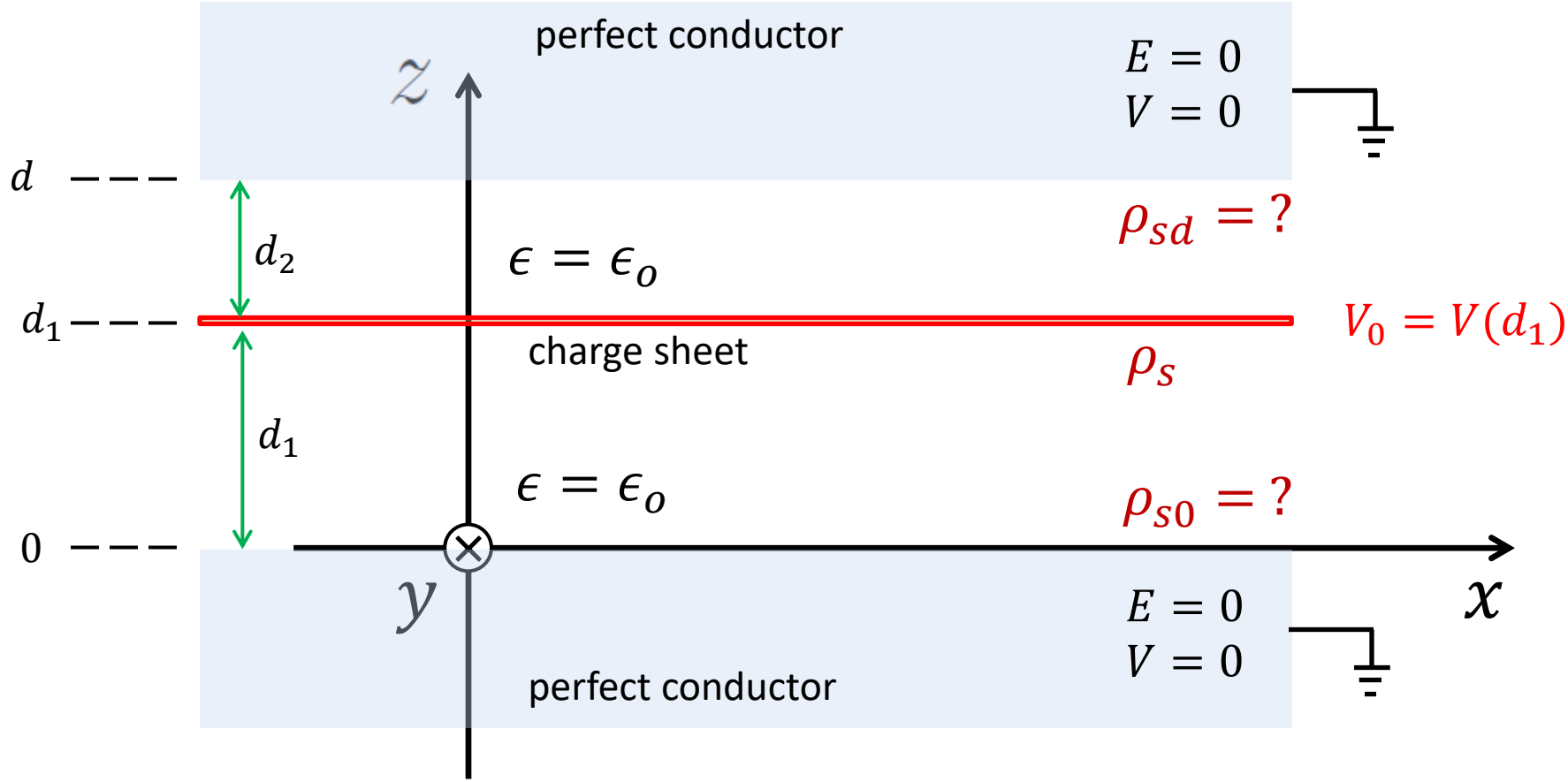


$$V(z) = \begin{cases} 2z\,\mathrm{V} & 0 < z < 1 \\ 2 + (z-1)\,\mathrm{V} & 1 < z < 2 \end{cases}$$

$$\boxed{V(2) = 3\ \mathrm{V}}$$

Example – Two parallel infinite conducting plates with an infinite charge sheet in the region between the plates.

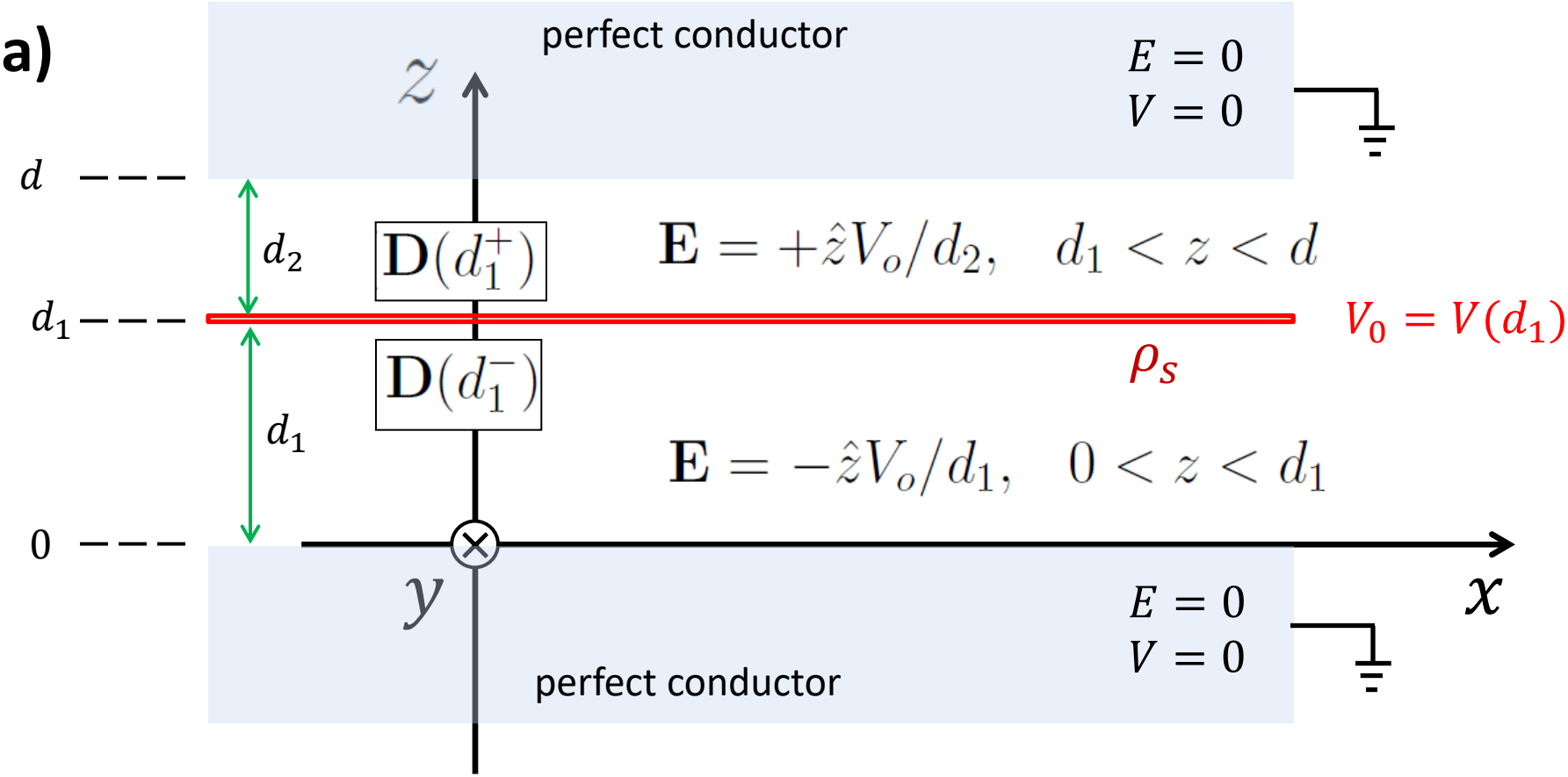


(a) Determine the electrostatic field between the plates

(b) The surface charge densities on the plates when the charge sheet is exactly in the middle ($d_1 = d/2$)

Example – Two parallel infinite conducting plates with an infinite charge sheet in the region between the plates.

(a)



Boundary conditions at charge sheet

$$\boxed{\hat{z} \cdot (\mathbf{D}(d_1^+) - \mathbf{D}(d_1^-)) = \rho_s}$$

Linear fields between charge sheet and conductors

$$\boxed{\mathbf{D} = \epsilon_o \mathbf{E} = \begin{cases} -\hat{z}\epsilon_o V_o/d_1, & 0 < z < d_1 \\ +\hat{z}\epsilon_o V_o/d_2, & d_1 < z < d \end{cases}}$$

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$$+\epsilon_o V_o/d_2 - \left(-\epsilon_o V_o/d_1\right) = \rho_s$$

D_n^+ D_n^-

$$V_o = \frac{\rho_s}{\epsilon_o}\left(\frac{1}{d_2} + \frac{1}{d_1}\right)^{-1} = \frac{\rho_s}{\epsilon_o}\frac{d_1 d_2}{d_1 + d_2} = \frac{\rho_s}{\epsilon_o}\frac{d_1 d_2}{d}$$

Example – Two parallel infinite conducting plates with an infinite charge sheet in the region between the plates.

Substituting

$$V_o = \frac{\rho_s}{\epsilon_o}\frac{d_1 d_2}{d} \longrightarrow \boxed{\mathbf{D} = \epsilon_o \mathbf{E} = \begin{cases} -\hat{z}\epsilon_o V_o/d_1, & 0 < z < d_1 \\ +\hat{z}\epsilon_o V_o/d_2, & d_1 < z < d \end{cases}}$$

$$\mathbf{E} = \begin{cases} -\hat{z}\frac{\rho_s}{\epsilon_o}\frac{d_2}{d}, & 0 < z < d_1 \\ +\hat{z}\frac{\rho_s}{\epsilon_o}\frac{d_1}{d}, & d_1 < z < d \end{cases}$$

Example – Two parallel infinite conducting plates with an infinite charge sheet in the region between the plates.

(b) The conducting plates are at the same zero potential and the surface charge is induced by the sheet of charge. The polarity of the charge on the plates has to be the same and it is opposite to the polarity of charge on the sheet in between.

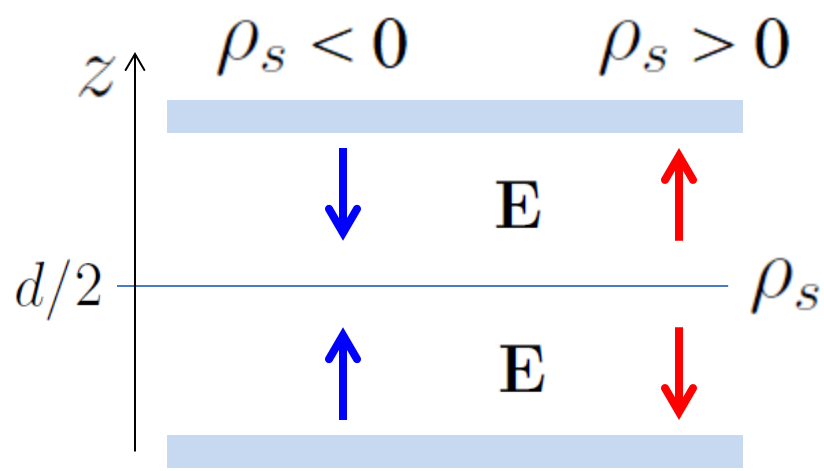
Remember that for this part of the problem $d_1 = d/2 \longrightarrow d_1 = d_2$

We evaluate the boundary conditions normal to the plates:

$$\hat{z} \cdot \mathbf{D} = \hat{z} \cdot \epsilon_o \mathbf{E}$$

$$\boxed{z = d} \qquad -\hat{z} \cdot \epsilon_o \mathbf{E}(d) = \rho_{sd}$$

$$\boxed{z = 0} \qquad \hat{z} \cdot \epsilon_o \mathbf{E}(0) = \rho_{s0}$$



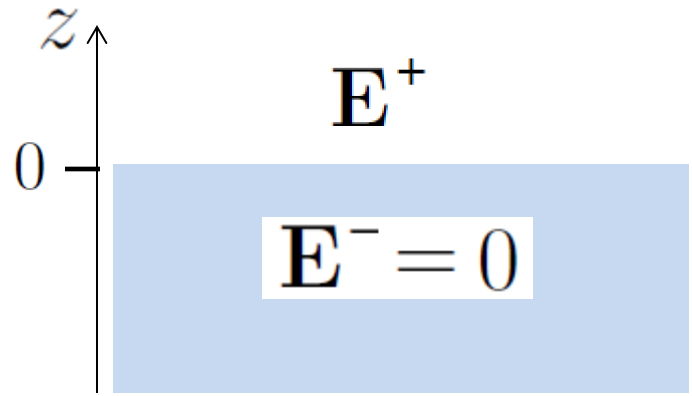
Example – Two parallel infinite conducting plates with an infinite charge sheet in the region between the plates.

$$\boxed{z = 0}$$

$$\hat{z} \cdot \epsilon_o \mathbf{E}(0) = \rho_{s0}$$

$$\epsilon_o \mathbf{E}^+ - \cancel{\epsilon_o \mathbf{E}^-} = \rho_{s0}$$

$$\rho_{s0} = \epsilon_o \mathbf{E}^+ = -\rho_s \frac{d_2}{d} = -\rho_s \frac{1}{2}$$

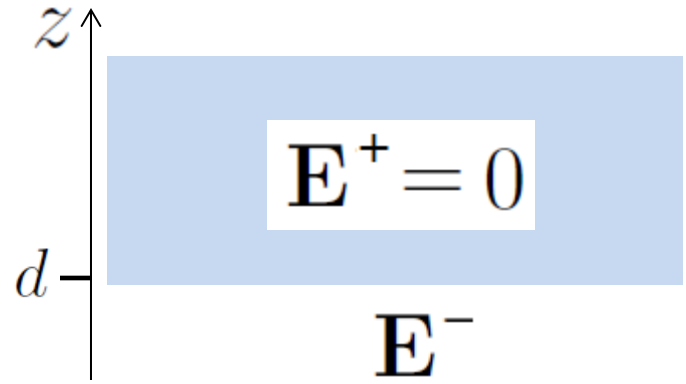


$$\boxed{z = d}$$

$$-\hat{z} \cdot \epsilon_o \mathbf{E}(d) = \rho_{sd}$$

$$\cancel{\epsilon_o \mathbf{E}^+} - \epsilon_o \mathbf{E}^- = \rho_{sd}$$

$$\rho_{sd} = -\epsilon_o \mathbf{E} = -\rho_s \frac{d_1}{d} = -\rho_s \frac{1}{2}$$



Summary – In a material medium, the displacement vector has been revised as

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_o \mathbf{E} + \mathbf{P}$$

$$\boxed{\epsilon = \epsilon_r \epsilon_o}$$

The fundamental equations are still valid

$$\boxed{\mathbf{E} = -\nabla V}$$

$$\boxed{\nabla \cdot \mathbf{D} = \rho}$$

In a homogeneous (uniform) medium, the permittivity is a constant and it is valid to write

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = \epsilon \nabla \cdot \mathbf{E}$$

and Poisson's equation then becomes

$$\nabla^2 V = -\frac{\rho}{\epsilon_r \epsilon_o} = -\frac{\rho}{\epsilon}$$

Laplace's equation is still

$$\nabla^2 V = 0$$

In inhomogeneous media, where the permittivity varies in space, Gauss' law is always valid and we can still formulate an equation for the potential

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = \nabla \cdot [\epsilon(-\nabla V)] = \rho$$

$$\boxed{\nabla \cdot [\epsilon \nabla V] = -\rho}$$

If a system consists of multiple homogeneous regions with different permittivity, Poisson (Laplace) equation can be solved in each separate region, matching the results at the boundaries with appropriate boundary conditions.

Excess charge on a conductor

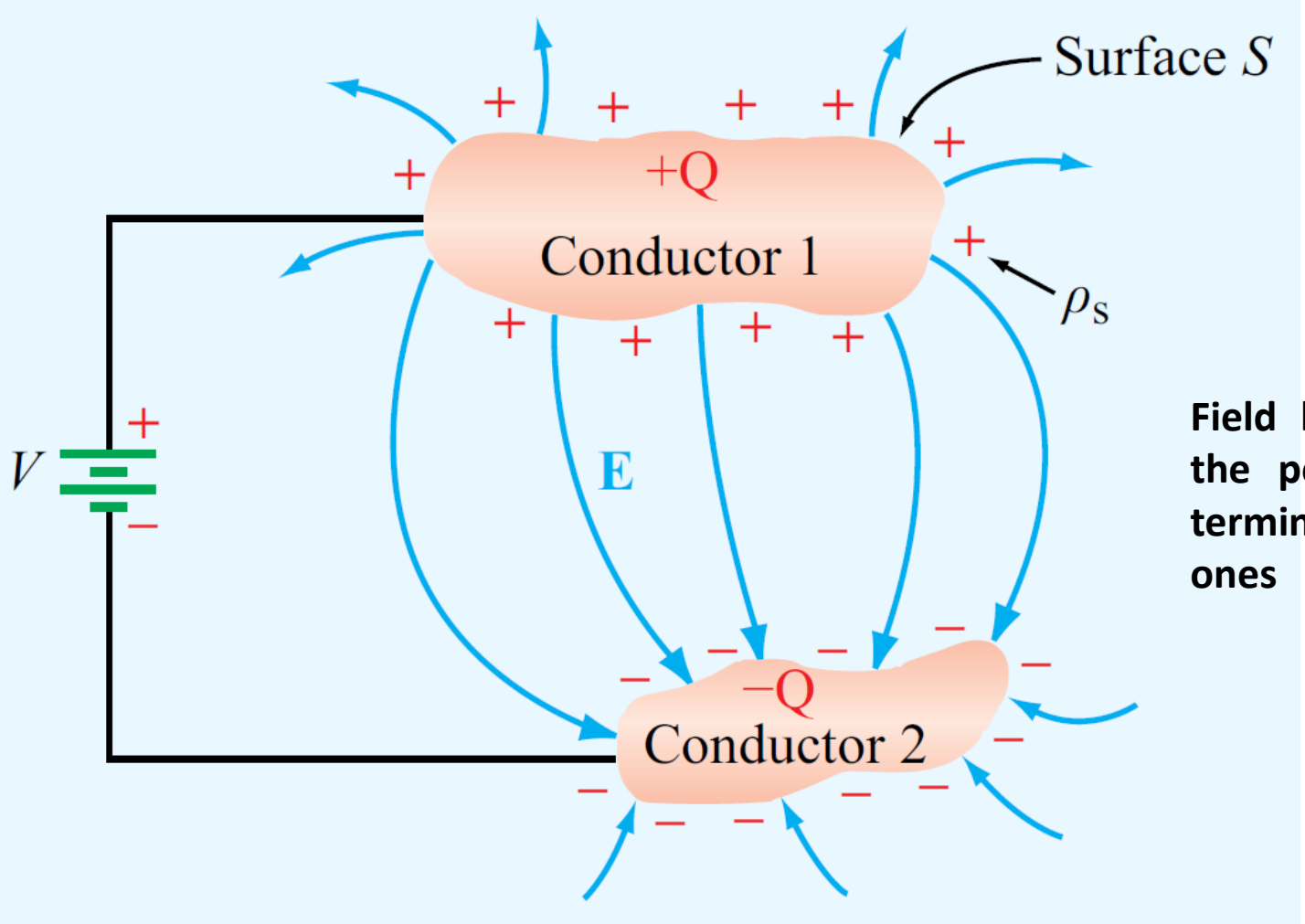
1. When excess electric charge is present in a conductor surrounded by an insulator, it is distributed on the surface to maintain a zero electric field everywhere in the interior.

2. The field due to the charges is normal to the conductor surface while the tangent field is zero (otherwise there would be current flowing).

$$E_{\mathrm{n}} = \hat{\mathbf{n}} \cdot \mathbf{E} = \frac{\rho_{\mathrm{S}}}{\epsilon}$$

3. The electric potential is the same at every point in the conductor (equipotential condition).

Any two conducting bodies, separated by a dielectric medium, form a ***capacitor*** regardless of their shapes and sizes.



Field lines originate from the positive charges and terminate on the negative ones

The capacitance of a two-conductor configuration is defined as

$$C = \frac{Q}{V} \qquad (\mathrm{C/V\ or\ F})$$

Coulomb/Volt

Farad

The capacitance C is independent of the electric field magnitude. It depends only on geometry (size, shape, positions) and on permittivity of the insulator (linear capacitor).

The charge is equal to the integral of the density over the surface

$$E_n = \hat{\mathbf{n}} \cdot \mathbf{E} = \frac{\rho_s}{\epsilon}$$

$$Q = \int_S \rho_s \, ds = \int_S \epsilon \hat{\mathbf{n}} \cdot \mathbf{E} \, ds = \int_S \epsilon \mathbf{E} \cdot d\mathbf{s}$$

The voltage between two arbitrary points on the two conductors is

P_1 ← on surface with positive charge

$$V = V_{12} = -\int_{P_2}^{P_1} \mathbf{E} \cdot d\mathbf{l}$$

From the previous results, we obtain that

$$C = \frac{Q}{V} \longrightarrow C = \frac{\int_S \epsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}} \qquad \text{(F)}$$

where l is the path of integration between two points on different conductors.

If the material between the conductors is not a perfect dielectric but has a small conductivity σ, some particle current can flow through the material between the conductors, and the material exhibits a resistance R

$$R = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}} \qquad (\Omega)$$

For a material with uniform conductivity and permittivity

$$RC = \frac{\epsilon}{\sigma}$$

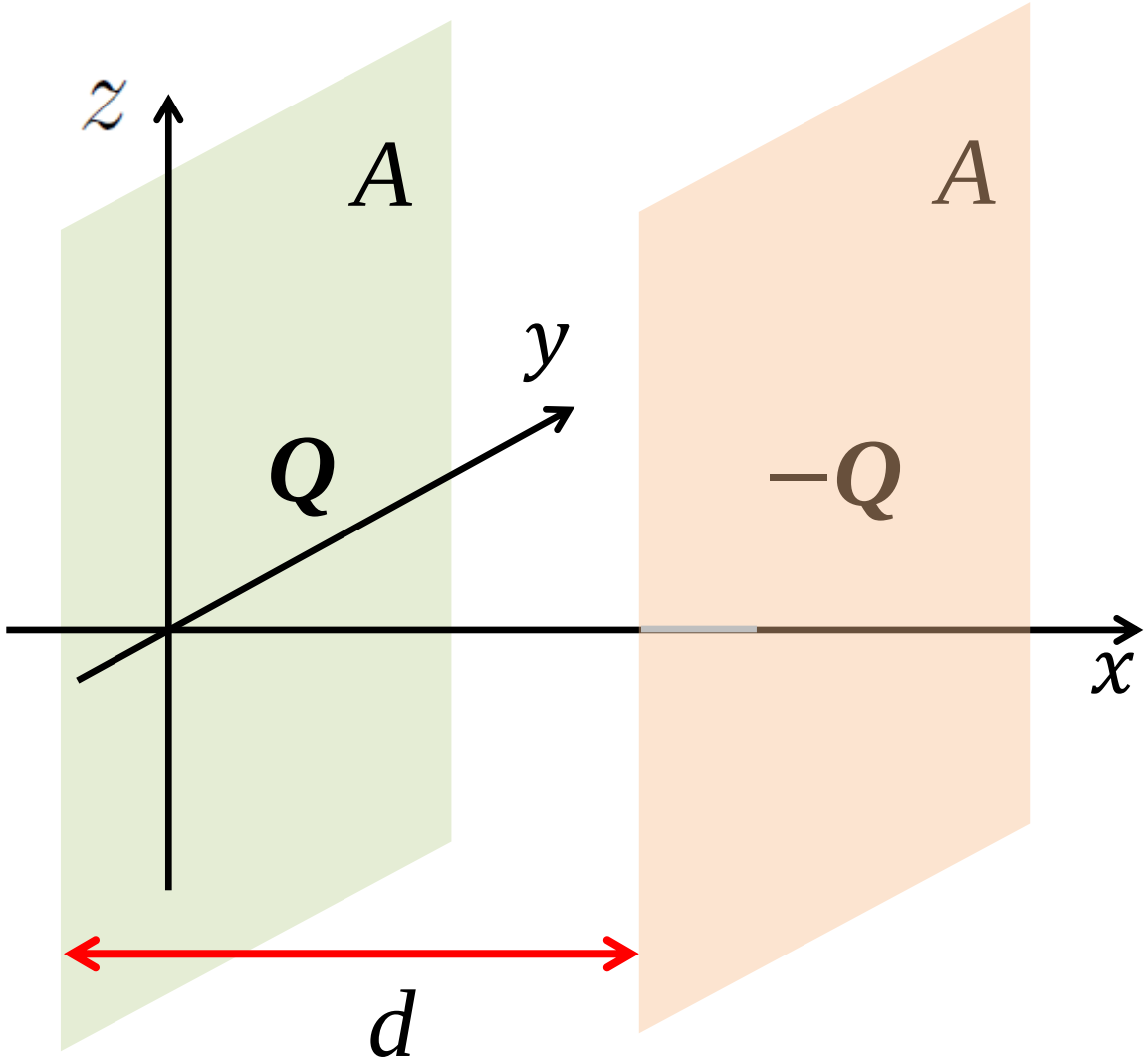
Parallel-plate capacitor – pair of conducting plates of finite area A separated by a distance d in free space

charge density on plates

$$\rho = \pm \frac{Q}{A}$$

may ignore fringing fields if

$$d \ll \sqrt{A}$$



Fields approximate the ones in the configuration with infinite plates

Constant Displacement Field

$$\mathbf{D} = \hat{x}\frac{Q}{A}$$

satisfies normal boundary condition on left plate and Gauss' law between plates

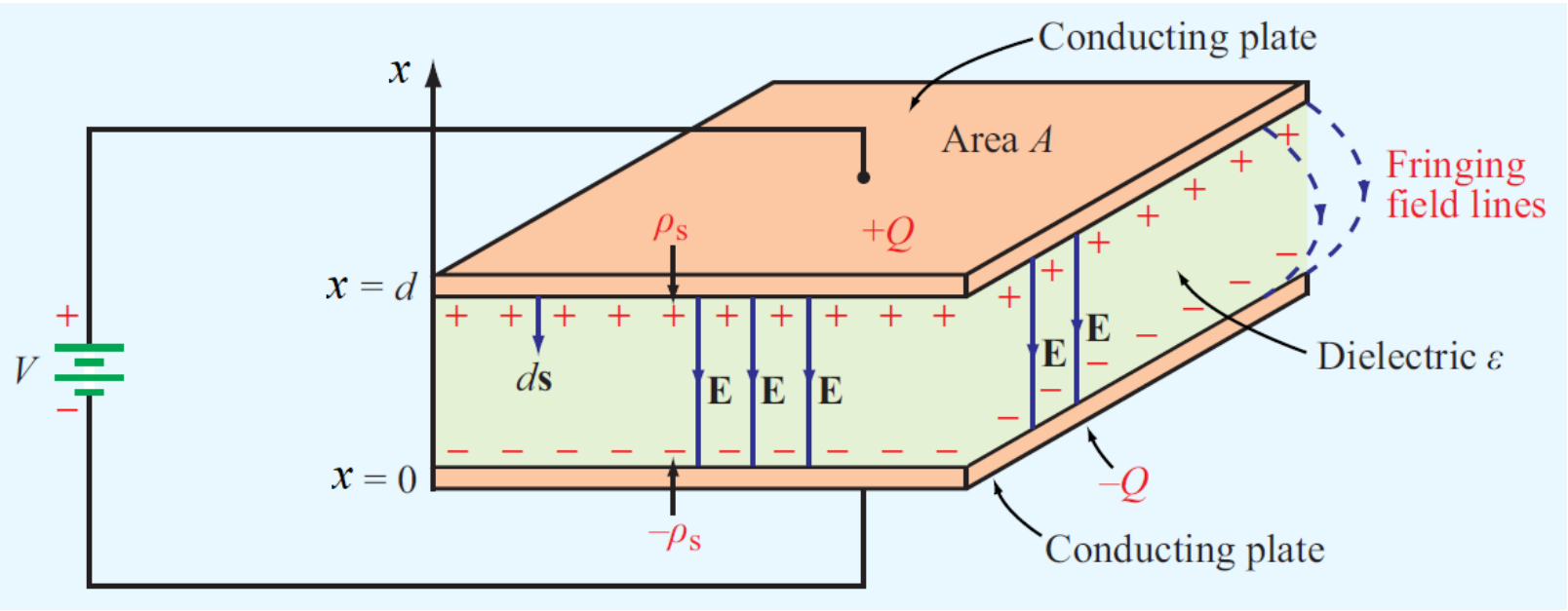
$$\nabla \cdot \mathbf{D} = 0$$

Electrostatic Field

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_o} = \hat{x}\frac{Q}{\epsilon_o A}$$

Capacitance of parallel plate capacitor

$$\rho_s = Q/A \qquad\qquad \mathbf{E} = -\hat{\mathbf{x}}E$$



$$E = \rho_s/\epsilon = Q/\epsilon A$$

$$V = -\int_0^d \mathbf{E}\cdot d\mathbf{l} = -\int_0^d (-\hat{\mathbf{x}}E)\cdot\hat{\mathbf{x}}\,dz = Ed$$

$$\boxed{C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\epsilon A}{d}}$$

A charged capacitor connected to an external circuit conducts a current

$$I = \frac{dQ}{dt}$$

$$\boxed{Q = CV}$$

$$I = C\frac{dV}{dt}$$

This model is generally valid in quasi-static conditions, when the side of the plates is much smaller than the minimum wavelength associated with $V(t)$.

Power absorbed by the capacitor

$$P = VI = VC\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{2}CV^2\right)$$

Stored Energy

$$W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\epsilon A}{d}\left(E_x d\right)^2$$

$$= \underbrace{\frac{1}{2}\epsilon\left|E_x\right|^2}_{\text{stored electrostatic energy per unit volume}} \underbrace{Ad}_{\text{volume between plates}}$$

For a perfect dielectric, charge is stored indefinitely in the plates. **If the dielectric is imperfect, current flows in the dielectric, with density**

$$J_x = \sigma E_x$$

The current discharges the plates, and the stored energy is dissipated as heat in the dielectric.

The total current is

$$I = A J_x = A\sigma E_x = A\sigma \frac{V}{d} = GV$$

$$\boxed{G \equiv \sigma \frac{A}{d}}$$

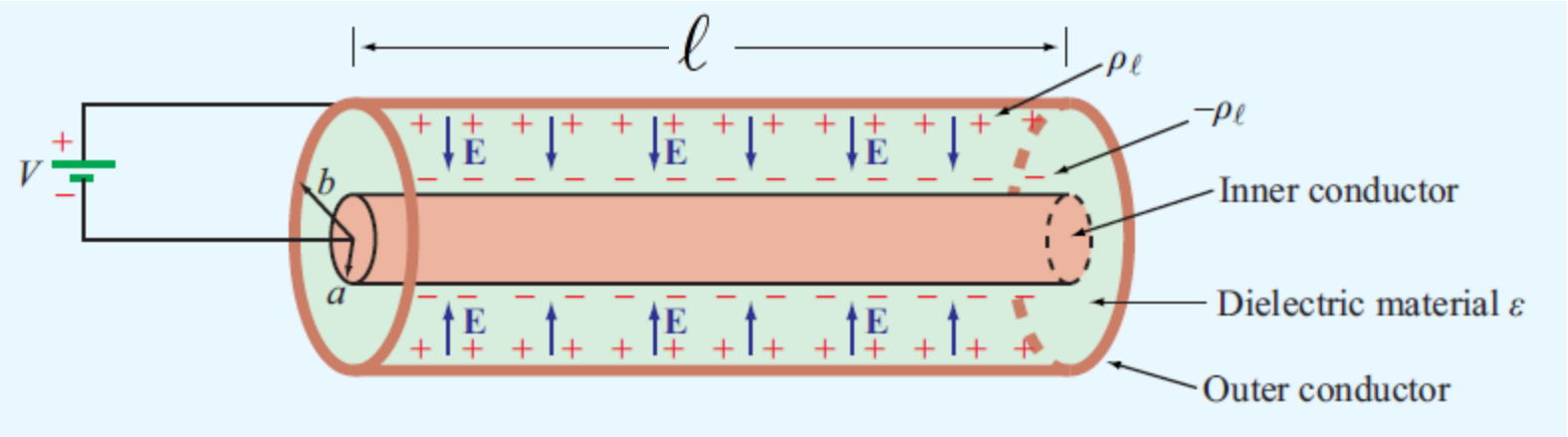
conductance

$$\boxed{R \equiv \frac{1}{G} = \frac{d}{A\sigma}}$$

resistance

Coaxial Capacitor

Cylindrical symmetry – For $\ell >> b$ we can neglect fringing and assume perfectly radial field



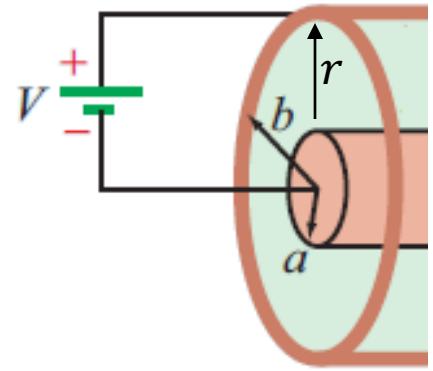
Consider a concentric Gaussian surface between inner and outer conductor with radius $a < r < b$

Gauss Law

$$\epsilon \oint_S \mathbf{E} \cdot d\mathbf{S} = Q_V \Longrightarrow \epsilon E_r 2\pi r \ell = Q$$

Then we have

$$E_r = \frac{Q}{2\pi\epsilon\ell r}$$



$$V = \int_{r=a}^{b} E_r dr = \int_{r=a}^{b} \frac{Q}{2\pi\epsilon\ell r} dr = \frac{Q}{2\pi\ell\epsilon} \int_{r=a}^{b} \frac{dr}{r} = \frac{Q}{2\pi\ell\epsilon} \ln\frac{b}{a}$$

Using

$$\boxed{Q = CV} \qquad \boxed{V = \frac{Q}{2\pi\ell\epsilon} \ln\frac{b}{a}}$$

we obtain the capacitance for the coaxial structure

$$C = \frac{2\pi}{\ln\frac{b}{a}} \ell\epsilon$$

The capacitance per unit length is simply

$$\boxed{\mathcal{C} = \frac{2\pi}{\ln \frac{b}{a}}\epsilon}$$

Since for a capacitor with imperfect dielectric

$$RC = \frac{\epsilon}{\sigma} \quad \text{or} \quad G = \frac{\sigma}{\epsilon}C$$

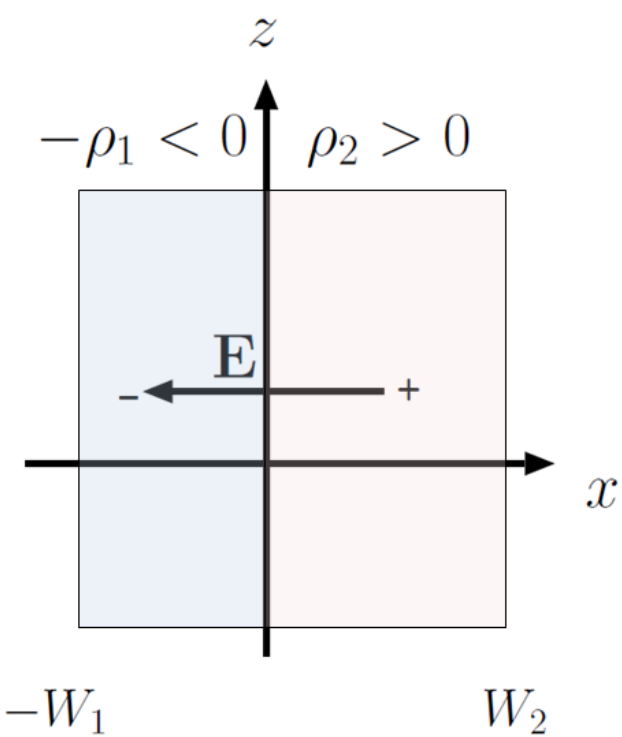
we can express the conductance per unit length as

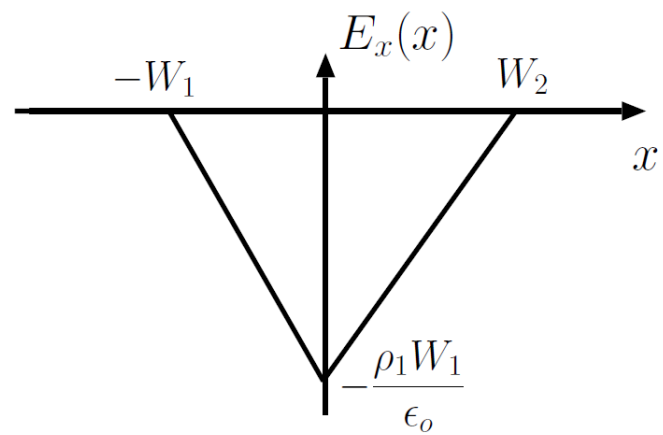
$$\boxed{\mathcal{G} = \frac{2\pi}{\ln \frac{b}{a}}\sigma}$$

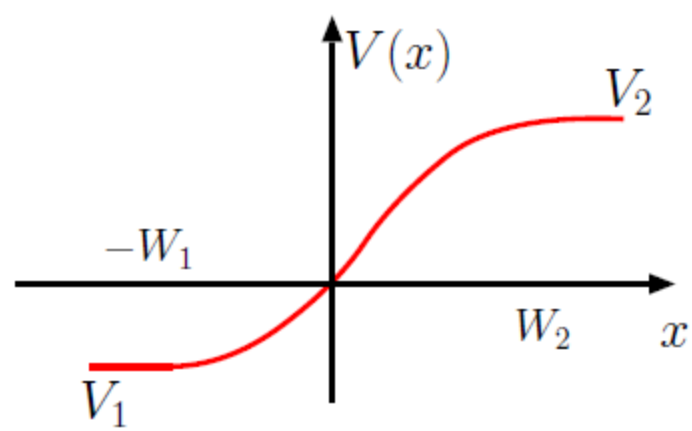
Suggested reading about realistic electrolytic capacitors:
https://eepower.com/technical-articles/electrolytic-capacitor-leakage-current/

Capacitance of diode junction

We have discussed earlier the case of two slabs with opposite volumetric charge







The potential across the junction was given by

$$V_{21} = V_2 - V_1 = \frac{\rho_2 W_2^2 + \rho_1 W_1^2}{2\epsilon_o}$$

The charge neutrality assumption $W_1\rho_1 = W_2\rho_2$ **gives**

$$V = \frac{\rho_2 W_2 (W_1 + W_2)}{2\epsilon_o} = \frac{\rho_1 W_1 (W_1 + W_2)}{2\epsilon_o}$$

and from simple manipulations:

$$W_1 = \frac{2\epsilon_o V}{(W_1 + W_2)\rho_1} \qquad W_2 = \frac{2\epsilon_o V}{(W_1 + W_2)\rho_2}$$

$$\Rightarrow \quad W_1 + W_2 = \sqrt{2\epsilon_o V \frac{\rho_1 + \rho_2}{\rho_1 \rho_2}}$$

Consider now a finite cross-section of area A for the structure. We have total positive charge for $x > 0$

$$\boxed{Q = \rho_2 W_2 A}$$

From the previous result $W_1 + W_2 = \sqrt{2\epsilon_o V \frac{\rho_1 + \rho_2}{\rho_1 \rho_2}}$

$$\boxed{V = \frac{\rho_2 W_2 (W_1 + W_2)}{2\epsilon_o} = \frac{Q\sqrt{2\epsilon_o V \frac{\rho_1 + \rho_2}{\rho_1 \rho_2}}}{2\epsilon_o A}}$$

$$\boxed{Q = A\sqrt{\frac{2\epsilon_o \rho_1 \rho_2}{\rho_1 + \rho_2}}\sqrt{V}}$$

non-linear charge-voltage relation

We define the small-signal capacitance differentiating

$$\boxed{Q = A\sqrt{\frac{2\epsilon_o \rho_1 \rho_2}{\rho_1 + \rho_2}}\sqrt{V}}$$

and obtaining

$$C = \frac{dQ}{dV} = A\sqrt{\frac{\epsilon_o}{2V}\frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)}}$$

which depends on the potential as $C \propto V^{-1/2}$.

In the case of parallel plate capacitor, the expression is linear and the value of the capacitance is constant

$$C = \frac{Q}{V}$$

NOTE: In ECE 340 you will learn how the potential across this "space charge region" varies when the applied battery voltage is changed.

This will require some elementary knowledge of semiconductor theory and it will justify the expression for the I-V curve of a diode, which you have seen in ECE 110.