Lecture 10 – Outline

• Parallel Plate Capacitor
• Coaxial Cable
• Diode Junction Capacitance
• Conductance in the imperfect dielectric of a capacitor

Reading assignment
Prof. Kudeki’s ECE 329 Lecture Notes on Fields and Waves: 10) Capacitance and Conductance
Excess charge on a conductor

1. When excess electric charge is present in a conductor surrounded by an insulator, it is distributed on the surface to maintain a zero electric field everywhere in the interior.

2. The field due to the charges is normal to the conductor surface while the tangent field is zero (otherwise there would be current flowing).

\[ E_n = \hat{n} \cdot E = \frac{\rho_s}{\epsilon} \]

3. The electric potential is the same at every point in the conductor (equipotential condition).
Any two conducting bodies, separated by a dielectric medium, form a capacitor regardless of their shapes and sizes.

Field lines originate from the positive charges and terminate on the negative ones.
The capacitance of a two-conductor configuration is defined as

\[ C = \frac{Q}{V} \quad \text{(C/V or F)} \]

The capacitance \( C \) is independent of the electric field magnitude. It depends only on geometry (size, shape, positions) and on permittivity of the insulator.
The charge is equal to the integral of the density over the surface

\[ E_n = \hat{n} \cdot E = \frac{\rho_s}{\varepsilon} \]

\[ Q = \int_S \rho_s \, ds = \int_S \varepsilon \hat{n} \cdot E \, ds = \int_S \varepsilon E \cdot ds \]

The voltage between two arbitrary points on the two conductors is

\[ V = V_{12} = -\int_{P_2}^{P_1} E \cdot dl \]

on surface with positive charge
From the previous results, we obtain that

\[
C = \frac{Q}{V}
\]

where \( l \) is the path of integration between two points on different conductors.
If the material between the conductors is not a perfect dielectric but has a small conductivity \( \sigma \), some particle current can flow through the material between the conductors, and the material exhibits a resistance \( R \)

\[
R = \frac{\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}
\]

(\( \Omega \))

For a material with uniform conductivity and permittivity

\[
RC = \frac{\varepsilon}{\sigma}
\]
Parallel-plate capacitor – pair of conducting plates of finite area $A$ separated by a distance $d$ in free space.

Charge density on plates:

$$\rho = \pm \frac{Q}{A}$$

May ignore fringing fields if:

$$d \ll \sqrt{A}$$
Fields approximate the ones in the configuration with infinite plates

Constant Displacement Field

\[ D = \hat{x} \frac{Q}{A} \]

satisfies normal boundary condition on left plate and Gauss’ law between plates

\[ \nabla \cdot D = 0 \]

Electrostatic Field

\[ E = \frac{D}{\varepsilon_0} = \hat{x} \frac{Q}{\varepsilon_0 A} \]
Capacitance of parallel plate capacitor

\[ \rho_s = \frac{Q}{A} \quad \text{E} = -\hat{x}E \]

\[ E = \frac{\rho_s}{\varepsilon} = \frac{Q}{\varepsilon A} \]

\[ V = -\int_{0}^{d} \mathbf{E} \cdot d\mathbf{l} = -\int_{0}^{d} (-\hat{x}E) \cdot \hat{x} \, dz = Ed \]

\[ C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\varepsilon A}{d} \]
A charged capacitor connected to an external circuit generates a current

\[ I = \frac{dQ}{dt} \]

\[ Q = CV \]

\[ I = C \frac{dV}{dt} \]

This model is generally valid in quasi-static conditions, when the side of the plates is much smaller than the minimum wavelength associated with \( V(t) \).
Power absorbed by the capacitor

\[ P = V I = V C \frac{dV}{dt} = \frac{d}{dt} \left( \frac{1}{2} C V^2 \right) \]

Stored Energy

\[ W = \frac{1}{2} CV^2 = \frac{1}{2} \frac{\epsilon A}{d} \left( E_x d \right)^2 \]

\[ = \frac{1}{2} \epsilon \left| E_x \right|^2 A d \]

stored electrostatic energy per unit volume  
volume between plates
For a perfect dielectric, charge is stored indefinitely in the plates. If the dielectric is imperfect, current flows in the dielectric, with density

\[ J_x = \sigma E_x \]

The current discharges the plates, and the stored energy is dissipated as heat in the dielectric.

The total current is

\[ I = A J_x = A \sigma E_x = A \sigma \frac{V}{d} = GV \]

\[ G \equiv \sigma \frac{A}{d} \]

\[ R \equiv \frac{1}{G} = \frac{d}{A \sigma} \]

conductance

resistance
Coaxial Capacitor

Cylindrical symmetry — For \( \ell \gg b \) we can neglect fringing and assume perfectly radial field

Consider a concentric Gaussian surface between inner and outer conductor with radius \( a < r < b \)

Gauss Law

\[
\epsilon \oint_S \mathbf{E} \cdot d\mathbf{S} = Q_V \quad \rightarrow \quad \epsilon E_r 2\pi r \ell = Q
\]
Then we have

\[ E_r = \frac{Q}{2\pi \varepsilon lr} \]

Using

\[ V = \int_{r=a}^{b} E_r \, dr = \int_{r=a}^{b} \frac{Q}{2\pi \varepsilon lr} \, dr = \frac{Q}{2\pi \varepsilon l} \int_{r=a}^{b} \frac{dr}{r} = \frac{Q}{2\pi \varepsilon l} \ln \frac{b}{a} \]

we obtain the capacitance for the coaxial structure

\[ C = \frac{2\pi}{\ln \frac{b}{a}} \varepsilon l \]
The capacitance per unit length is simply

\[ C = \frac{2\pi}{\ln \frac{b}{a}} \epsilon \]

Since for a capacitor with imperfect dielectric

\[ RC = \frac{\epsilon}{\sigma} \quad \text{or} \quad G = \frac{\sigma}{\epsilon} C \]

we can express the conductance per unit length as

\[ G = \frac{2\pi}{\ln \frac{b}{a}} \sigma \]

Suggested reading about realistic electrolytic capacitors:
https://eepower.com/technical-articles/electrolytic-capacitor-leakage-current/
Capacitance of diode junction

We have discussed earlier the case of two slabs with opposite volumetric charge.
The potential across the junction was given by

\[ V_{21} = V_2 - V_1 = \frac{\rho_2 W_2^2 + \rho_1 W_1^2}{2\varepsilon_0} \]

The charge neutrality assumption \( W_1\rho_1 = W_2\rho_2 \) gives

\[ V = \frac{\rho_2 W_2(W_1 + W_2)}{2\varepsilon_0} = \frac{\rho_1 W_1(W_1 + W_2)}{2\varepsilon_0} \]

and from simple manipulations:

\[ W_1 = \frac{2\varepsilon_0 V}{(W_1 + W_2)\rho_1} \quad \quad W_2 = \frac{2\varepsilon_0 V}{(W_1 + W_2)\rho_2} \]

\[ \Rightarrow \quad W_1 + W_2 = \sqrt{2\varepsilon_0 V \frac{\rho_1 + \rho_2}{\rho_1\rho_2}} \]
Consider now a finite cross-section of area $A$ for the structure. We have total positive charge for $x > 0$

$$Q = \rho_2 W_2 A$$

From the previous result

$$W_1 + W_2 = \sqrt{2\varepsilon_0 V \frac{\rho_1 + \rho_2}{\rho_1 \rho_2}}$$

$$V = \frac{\rho_2 W_2 (W_1 + W_2)}{2\varepsilon_0} = \frac{Q \sqrt{2\varepsilon_0 V \frac{\rho_1 + \rho_2}{\rho_1 \rho_2}}}{2\varepsilon_0 A}$$

$$Q = A \sqrt{\frac{2\varepsilon_0 \rho_1 \rho_2}{\rho_1 + \rho_2} \sqrt{V}}$$

non-linear charge-voltage relation
We define the small-signal capacitance differentiating and obtaining

\[ Q = A \sqrt{\frac{2\epsilon_o \rho_1 \rho_2}{\rho_1 + \rho_2}} \sqrt{V} \]

which depends on the potential as

\[ C \propto V^{-1/2}. \]

In the case of parallel plate capacitor, the expression is linear and the value of the capacitance is constant

\[ C = \frac{Q}{V} \]
NOTE: In ECE 340 you will learn how the potential across this “space charge region” varies when the applied battery voltage is changed.

This will require some elementary knowledge of semiconductor theory and it will justify the expression for the $I-V$ curve of a diode, which you have seen in ECE 110.