ECE 329 – Fall 2021

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

From Lecture 10
Capacitance of diode junction

We have discussed earlier the case of two slabs with opposite volumetric charge
The potential across the junction was given by

\[ V_{21} = V_2 - V_1 = \frac{\rho_2 W_2^2 + \rho_1 W_1^2}{2\epsilon_o} \]

The charge neutrality assumption \( W_1 \rho_1 = W_2 \rho_2 \) gives

\[ V = \frac{\rho_2 W_2 (W_1 + W_2)}{2\epsilon_o} = \frac{\rho_1 W_1 (W_1 + W_2)}{2\epsilon_o} \]

and from simple manipulations:

\[ W_1 = \frac{2\epsilon_o V}{(W_1 + W_2)\rho_1} \quad \text{and} \quad W_2 = \frac{2\epsilon_o V}{(W_1 + W_2)\rho_2} \]

\[ \Rightarrow W_1 + W_2 = \sqrt{\frac{2\epsilon_o V}{\rho_1} + \frac{2\epsilon_o V}{\rho_2}} \]
Consider now a finite cross-section of area $A$ for the structure. We have total positive charge for $x > 0$

$$Q = \rho_2 W_2 A$$

From the previous result

$$W_1 + W_2 = \sqrt{2\varepsilon_0 V \frac{\rho_1 + \rho_2}{\rho_1 \rho_2}}$$

$$V = \frac{\rho_2 W_2 (W_1 + W_2)}{2\varepsilon_0} = \frac{Q \sqrt{2\varepsilon_0 V \frac{\rho_1 + \rho_2}{\rho_1 \rho_2}}}{2\varepsilon_0 A}$$

$$Q = A \sqrt{\frac{2\varepsilon_0 \rho_1 \rho_2}{\rho_1 + \rho_2}} \sqrt{V}$$

non-linear charge-voltage relation
We define the small-signal capacitance differentiating

\[ Q = A \sqrt{\frac{2\epsilon_0 \rho_1 \rho_2}{\rho_1 + \rho_2}} \sqrt{V} \]

and obtaining

\[ C = \frac{dQ}{dV} = A \sqrt{\frac{\epsilon_0}{2V} \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)}} \]

which depends on the potential as \( C \propto V^{-1/2} \).

In the case of parallel plate capacitor, the expression is linear and the value of the capacitance is constant

\[ C = \frac{Q}{V} \]
NOTE: In ECE 340 you will learn how the potential across this “space charge region” varies when the applied battery voltage is changed.

This will require some elementary knowledge of semiconductor theory and it will justify the expression for the $I-V$ curve of a diode, which you have seen in ECE 110.
ECE 329 – Fall 2021

Prof. Ravaioli – Office: 2062 ECEB
Section E – 1:00pm

Lecture 11
Lecture 10 – Outline

• Conductivity as random motion of charges
• Drude models for conductivity and susceptibility

Reading assignment
Prof. Kudeki’s ECE 329 Lecture Notes on Fields and Waves:
11) Lorentz-Drude models for conductivity and susceptibility and polarization current
Random motion of charges

Simple microscopic model by Drude, extended by Lorentz (established before the development of quantum mechanics).

Mobile charges are treated with Newton’s dynamic laws as classical particles which are scattered by atoms.

In the absence of electric field, as a result of many consecutive random collisions, the time-average velocity is zero.
Equation of motion in electric field

\[ m \frac{dv}{dt} = qE - m \frac{v}{\tau} \]

- mass \times\text{ acceleration} = \text{electric force} – \text{friction due to collisions}

\[ \nu = \frac{1}{\tau} \]

- \( \nu \) = collisions per unit time (frequency of collisions)

These parameters represent the average behavior of a large ensemble of particles at any given time.
In the absence of electric field

\[ m \frac{dv}{dt} = qE - \frac{m v}{\tau} \]

\[ \frac{dv}{dt} = - \frac{v}{\tau} \]

solution is an exponential decay

\[ v(t) = v(0)e^{-t/\tau} \]

turn field off

\[
\begin{align*}
E &= 0 \\
t &= 0
\end{align*}
\]
Exponential decay

\[ v(t) = v(0)e^{-t/\tau} \]
Constant Applied Electric Field (DC)

A steady state solution of the equation of motion is reached

\[ m \frac{d\mathbf{v}}{dt} = 0 = q\mathbf{E} - m \frac{\mathbf{v}}{\tau} \]

\[ \mathbf{v} = \frac{q \tau}{m} \mathbf{E} \]

\( \mu = \left| \frac{q \tau}{m} \right| \) is called mobility
## Typical electron mobility values

<table>
<thead>
<tr>
<th>Material</th>
<th>Mobility (cm² V⁻¹ s⁻¹)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Metals</strong> (e.g., Al, Au, Cu, Ag)</td>
<td>10 – 50</td>
</tr>
<tr>
<td>Pure crystalline silicon</td>
<td>1,400</td>
</tr>
<tr>
<td>Polycrystalline silicon</td>
<td>~ 100</td>
</tr>
<tr>
<td>Amorphous silicon</td>
<td>~ 1</td>
</tr>
<tr>
<td>Carbon nanotube</td>
<td>~ 70,000 (room T)</td>
</tr>
<tr>
<td>Graphene</td>
<td>~ 200,000 (low T)</td>
</tr>
<tr>
<td>AlGaAs/GaAs heterojunction</td>
<td>~ 30,000,000</td>
</tr>
</tbody>
</table>
Current Density

Assume $N$ charge carriers per unit volume (carrier density) moving at the steady-state average velocity (drift velocity) in the direction of the electric field.

This corresponds to an average flux of charge density (current density)

$$
\mathbf{J} = N q v = \frac{N q^2 \tau}{m} \mathbf{E} = \frac{N q^2}{m \nu} \mathbf{E} \quad \left[ \frac{\text{C/s}}{\text{m}^2} \right]
$$

Current density can also be defined equivalently as number of charges crossing a unit area per second.
\[ J = Nq \mathbf{v} = \frac{Nq^2 \tau}{m} \mathbf{E} = \frac{Nq^2}{m \nu} \mathbf{E} \left[ \frac{\text{C/s}}{\text{m}^2} \right] \]

\[ \frac{Nq^2 \tau}{m} = \frac{Nq^2}{m \nu} = \sigma \]

Conductivity

\[ J = \sigma \mathbf{E} \]

This is the expression we used earlier in Lecture 8
Conductivity of some common materials at room temperature

<table>
<thead>
<tr>
<th>Material</th>
<th>Conductivity, $\sigma$ (S/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Conductors</strong></td>
<td></td>
</tr>
<tr>
<td>Silver</td>
<td>$6.2 \times 10^7$</td>
</tr>
<tr>
<td>Copper</td>
<td>$5.8 \times 10^7$</td>
</tr>
<tr>
<td>Gold</td>
<td>$4.1 \times 10^7$</td>
</tr>
<tr>
<td>Aluminum</td>
<td>$3.5 \times 10^7$</td>
</tr>
<tr>
<td>Iron</td>
<td>$10^7$</td>
</tr>
<tr>
<td>Mercury</td>
<td>$10^6$</td>
</tr>
<tr>
<td>Carbon</td>
<td>$3 \times 10^4$</td>
</tr>
<tr>
<td><strong>Semiconductors</strong></td>
<td></td>
</tr>
<tr>
<td>Pure germanium</td>
<td>2.2</td>
</tr>
<tr>
<td>Pure silicon</td>
<td>$4.4 \times 10^{-4}$</td>
</tr>
<tr>
<td><strong>Insulators</strong></td>
<td></td>
</tr>
<tr>
<td>Glass</td>
<td>$10^{-12}$</td>
</tr>
<tr>
<td>Paraffin</td>
<td>$10^{-15}$</td>
</tr>
<tr>
<td>Mica</td>
<td>$10^{-15}$</td>
</tr>
<tr>
<td>Fused quartz</td>
<td>$10^{-17}$</td>
</tr>
</tbody>
</table>

**Question**

If metals have such a low mobility, why do they have very high conductivity?
If there are different conducting species participating in the current density, the different conductivities are simply summed up

\[ \sigma = \sum_s \sigma_s \]

where

\[ \sigma_s = \frac{N_s q_s^2}{m_s \nu_s} \]
Time-varying Electric Field (AC)

This regime can be studied in the frequency domain using phasor techniques, for instance:

\[ E(t) = \text{Re}\{\tilde{E}e^{j\omega t}\} \quad \text{↔} \quad \tilde{E} \]

\[ J(t) = \text{Re}\{\tilde{J}e^{j\omega t}\} \quad \text{↔} \quad \tilde{J} \]

Phasor transformed force balance equation

\[ m \frac{dv}{dt} = qE - m \frac{v}{\tau} \quad \Rightarrow \quad mj\omega \tilde{v} = q\tilde{E} - m \frac{\tilde{v}}{\tau} \]

\[ \tilde{J} = \sigma \tilde{E} \]
\[ \tilde{J} = \sigma \tilde{E} \]

The conductivity should be a function of frequency

\[ \sigma = \sum_s \sigma_s \quad \text{and} \quad \sigma_s = \frac{N_s q_s^2}{m_s (\nu_s + j\omega)} \]

However, in many cases the DC conductivity can still be a good approximation of AC conductivity if the frequency of operation is much smaller than the collision frequencies.
Polarization effects in dielectrics

As we have seen, all dielectric materials, including perfect dielectrics, can be polarized and possess:

\[ \chi_e \neq 0 \]

\[ \varepsilon_r = 1 + \chi_e > 1 \]

In a perfect dielectric there are no free carriers and \( \sigma = 0 \). In a realistic dielectric there are some free carriers with \( \sigma \neq 0 \) (but very small) which can support some DC current.
In contrast to metals, where charges are free to move throughout the material, in dielectrics most charges are attached to specific atoms and molecules.

These polarization charges are known as bound charges.

Bound charges are able, however, to be displaced within an atom or a molecule. Such cumulative microscopic displacements account for the characteristic behavior of dielectric materials.

Since the polarized bound charge is not free to move from atom to atom, it cannot contribute to DC current. However, it can “oscillate” and contribute to AC current.

Some authors distinguish charge density as “unpaired” (free to move, contributing to DC current) and “paired” (bound charge with dipole configuration, due to polarization)

\[ \rho = \rho_u + \rho_p \quad \text{same as} \quad \rho = \rho_{free} + \rho_p \]
There are two contributions to bound charge $\rho_p$:

- **Volume charge density**
  
- **Surface charge density**

The volume density of bound charge can be expressed as

$$\rho_v (\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r})$$

Note: here the vector $\mathbf{r}$ represents $(x, y, z)$

The surface density of bound charge can be expressed as

$$\rho_s (\mathbf{r}) = \mathbf{P}(\mathbf{r}) \cdot \mathbf{n}$$

This is actually a result of the formula above with an abrupt change of $\mathbf{P}$ at the surface.
To summarize, for a dielectric we can write

$$\rho_{\text{total}} = \rho_{\text{free}} + \rho_P = \rho - \nabla \cdot P$$

The divergence of the electric field is

$$\nabla \cdot E = \frac{1}{\varepsilon_0} \left[ \rho - \nabla \cdot P \right]$$

and for the electric displacement

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\nabla \cdot \mathbf{D} = \rho$$
Susceptibility model

In the Lorentz-Drude model, each polarized atom (or molecule) is represented by a vector dipole moment $p$

\[ p = -er \]

displacement of electron cloud from nucleus

electronic charge

Equivalent dipole
We find two equivalent definitions in the literature (it can be confusing)

\[ p = -er \]

(This is what we have in our class notes)

\[ p = er \]

(e.g., in Prof. Rao’s book)

vector \( r \) from positive to negative

vector \( r \) from negative to positive
Release the polarizing force due to the electric field is released at $t = 0 \Rightarrow$ the dipole charges relax to equilibrium causing the dipole field to decay following the observed damped co-sinusoidal oscillation.

$$E_p \propto p \propto r$$

Dipole field proportional to charge displacement

$$r(t) \approx r_0 e^{-t/\tau_d} \cos(\omega_0 t)$$

Characteristic damped frequency

$$\sqrt{\omega_0^2 - \alpha^2} \approx \omega_0$$

$$\tau_d = \frac{1}{\alpha}$$

Decay constant
The behavior can be described by the **Lorentz oscillator equation** for the displacement vector

\[ m \frac{d^2 \mathbf{r}}{dt^2} = -e \mathbf{E} - m \omega_0^2 \mathbf{r} - m 2 \alpha \frac{d\mathbf{r}}{dt} \]

- \( t = 0 \)
- **friction dissipative force**
- **“spring” binding force**
- **external force**
- **acceleration**
- **mass**
The same **Lorentz oscillator equation** may be used to describe the steady-state condition after a field has been applied.

\[
\frac{m}{dt^2} \frac{d^2 \mathbf{r}}{dt^2} = -e \mathbf{E} - m \omega_0^2 \mathbf{r} - m2\alpha \frac{d\mathbf{r}}{dt} \neq 0
\]

\[
-e \mathbf{E} = m \omega_0^2 \mathbf{r}
\]

\[
\mathbf{r} = -\frac{e}{m \omega_0^2} \mathbf{E}
\]

from which we can determine the atomic dipole moment

\[
\mathbf{p} = -e \mathbf{r} = \frac{e^2}{m \omega_0^2} \mathbf{E}
\]
Consider a medium consisting of $N_d$ polarized particles per unit volume (dipole density)

$$p = -er = \frac{e^2}{m\omega_0^2}E$$

polarization field in the dielectric material

$$P = N_dp = \frac{N_de^2}{m\omega_0^2}E$$

From prior definition, we get an expression for the DC susceptibility

$$P = \varepsilon_0\chi_eE$$

$$\chi_e \equiv \frac{N_de^2}{m\varepsilon_0\omega_0^2}$$
AC Conditions: Polarization Current

Let’s consider now a time-varying Electric Field in a dielectric medium. If the frequency of the field is \( \omega \ll \omega_0 \) the DC susceptibility is still a good approximation of the AC value

\[
\chi_e \equiv \frac{N_d e^2}{m\varepsilon_0 \omega_0^2}
\]

and we can continue to use also

\[
r = -\frac{e}{m\omega_0^2}E
\]

\[
P = \varepsilon_0 \chi_e E
\]
AC Conditions: Polarization Current

For a field $\mathbf{E}(t)$

Electrons are displaced back and forth, with an effective spatial velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\frac{e}{m\omega^2} \frac{d\mathbf{E}}{dt}$$

This movement of bound charges in the dielectric medium generates an effective AC current density. With $N_d$ bound electrons per unit volume

$$\mathbf{J}_p = -eN_d \mathbf{v} = \frac{N_de^2}{m\omega^2} \frac{d\mathbf{E}}{dt} = \frac{d\mathbf{P}}{dt}$$

polarization current density (AC conditions)