ECE 329 – Fall 2022

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

From Lecture 12

Lecture 12 – Outline

- **More from Lecture 11**
- **Magnetostatics**
  - **Magnetic flux density**
  - **Magnetic force**
  - **Ampere's Law**
  - **Magnetic interaction with current flowing in parallel wires: relativistic explanation**

Reading assignment

Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:

12) Magnetic force and fields and Ampere's Law

Constant Applied Electric Field (DC)

A steady state solution of the equation of motion is reached

$$m\frac{d\mathbf{v}}{dt} = 0 = q\mathbf{E} - m\frac{\mathbf{v}}{\tau}$$

$$\boxed{\mathbf{v} = \frac{q\tau}{m}\mathbf{E}}$$ **steady-state velocity**

$$\mu = \left|\frac{q\tau}{m}\right|$$ **is called mobility**

Current Density

Assume N charge carriers per unit volume (carrier density) moving at the steady-state average velocity (drift velocity) in the direction of the electric field.

This corresponds to an average flux of charge density (current density)

$$\mathbf{J} = Nq\mathbf{v} = \frac{Nq^2\tau}{m}\mathbf{E} = \frac{Nq^2}{m\,\nu}\mathbf{E} \quad \left[\frac{\mathrm{C/s}}{\mathrm{m}^2}\right]$$

Current density can also be defined equivalently as number of charges crossing a unit area per second.

$$\mathbf{J} = Nq\mathbf{v} = \frac{Nq^2\tau}{m}\mathbf{E} = \frac{Nq^2}{m\,\nu}\mathbf{E} \qquad \left[\frac{\mathrm{C/s}}{\mathrm{m}^2}\right]$$

$$\frac{Nq^2\tau}{m} = \frac{Nq^2}{m\,\nu} = \sigma$$

Conductivity

$$\Rightarrow \quad \boxed{\mathbf{J} = \sigma\mathbf{E}}$$

This is the expression we used earlier in Lecture 8

Time-varying Electric Field (AC)

This regime can be studied in the frequency domain using phasor techniques, for instance:

$$\mathbf{E}(t) = \mathrm{Re}\{\tilde{\mathbf{E}} e^{j\omega t}\} \quad \Longleftrightarrow \quad \tilde{\mathbf{E}}$$

$$\mathbf{J}(t) = \mathrm{Re}\{\tilde{\mathbf{J}} e^{j\omega t}\} \quad \Longleftrightarrow \quad \tilde{\mathbf{J}}$$

Phasor transformed force balance equation

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} - m\frac{\mathbf{v}}{\tau} \quad \Rightarrow \quad mj\omega\tilde{\mathbf{v}} = q\tilde{\mathbf{E}} - m\frac{\tilde{\mathbf{v}}}{\tau}$$

$$\tilde{\mathbf{J}} = \sigma\tilde{\mathbf{E}}$$

$$\tilde{\mathbf{J}} = \sigma \tilde{\mathbf{E}}$$

The conductivity should be a function of frequency

$$\sigma = \sum_s \sigma_s \quad \text{and} \quad \sigma_s = \frac{N_s q_s^2}{m_s(\nu_s + j\omega)}$$

However, in many cases the DC conductivity can still be a good approximation of AC conductivity if the frequency of operation is much smaller than the collision frequencies.

Polarization effects in dielectrics

As we have seen, all dielectric materials, including perfect dielectrics, can be polarized and possess:

susceptibility

$$\chi_e \neq 0$$

relative permittivity

$$\epsilon_r = 1 + \chi_e > 1$$

In a perfect dielectric there are no free carriers and $\sigma = 0$. In a realistic dielectric there are some free carriers with $\sigma \neq 0$ (but very small) which can support some DC current.

In contrast to metals, where charges are free to move throughout the material, in dielectrics most charges are attached to specific atoms and molecules.

These polarization charges are known as bound charges.

Bound charges are able, however, to be displaced within an atom or a molecule. Such cumulative microscopic displacements account for the characteristic behavior of dielectric materials.

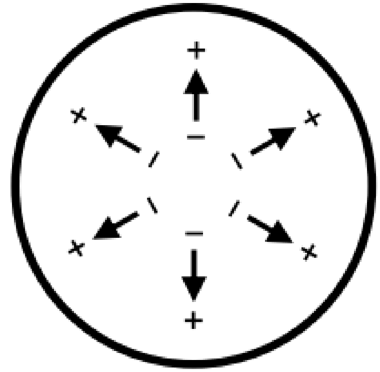
Since the polarized bound charge is not free to move from atom to atom, it cannot contribute to DC current. However, it can "oscillate" and contribute to AC current.

Some authors distinguish charge density as "unpaired" (free to move, contributing to DC current) and "paired" (bound charge with dipole configuration, due to polarization)

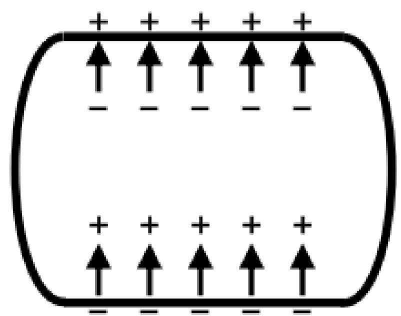
$$\rho = \rho_u + \rho_p \qquad \textbf{same as} \qquad \rho = \rho_{free} + \rho_p$$

There are two contributions to bound charge ρ_p:

- **Volume charge density**



- **Surface charge density**



The volume density of bound charge can be expressed as

$$\rho_v(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r})$$

Note: here the vector $\mathbf{r}$ represents (x, y, z)

The surface density of bound charge can be expressed as

$$\rho_s(\mathbf{r}) = \mathbf{P}(\mathbf{r}) \cdot \mathbf{n}$$

This is actually a result of the formula above with an abrupt change of $\mathbf{P}$ at the surface

To summarize, for a dielectric we can write

$$\rho_{total} = \rho_{free} + \rho_P = \rho - \nabla \cdot \mathbf{P}$$

The divergence of the electric field is

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\left[\rho - \nabla \cdot \mathbf{P}\right]}$$

and for the electric displacement

$$\boxed{\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}} \qquad \boxed{\nabla \cdot \mathbf{D} = \rho}$$

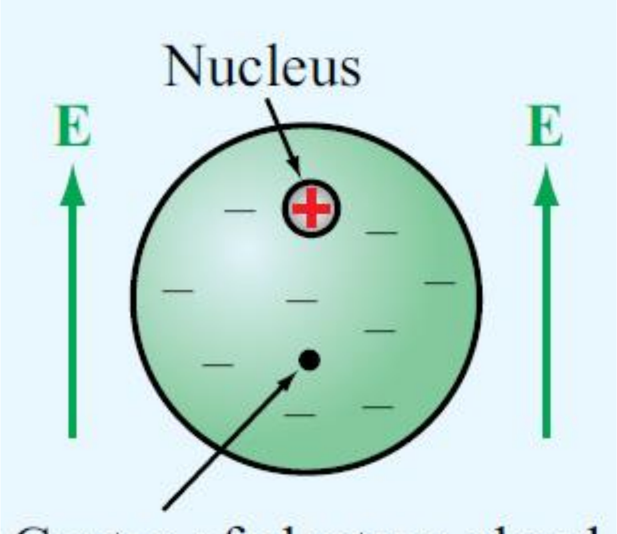
Susceptibility model

In the Lorentz-Drude model, each polarized atom (or molecule) is represented by a vector dipole moment p

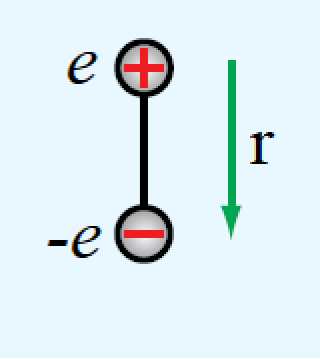
$$\mathbf{p} = -e\mathbf{r}$$

displacement of electron cloud from nucleus

electronic charge



Center of electron cloud



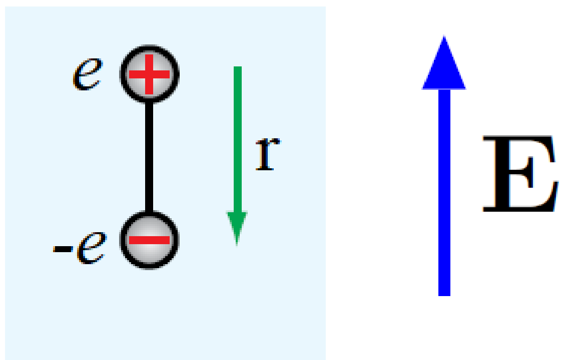
Equivalent dipole

We find two equivalent definitions in the literature (it can be confusing)

$$\mathbf{p} = -e\mathbf{r}$$

vector $\boldsymbol{r}$ from positive to negative

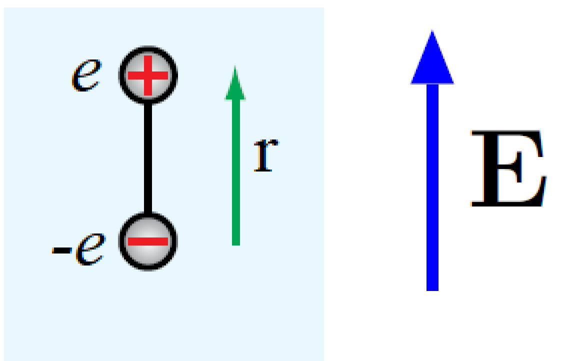
(This is what we have in our class notes)



$$\mathbf{p} = e\mathbf{r}$$

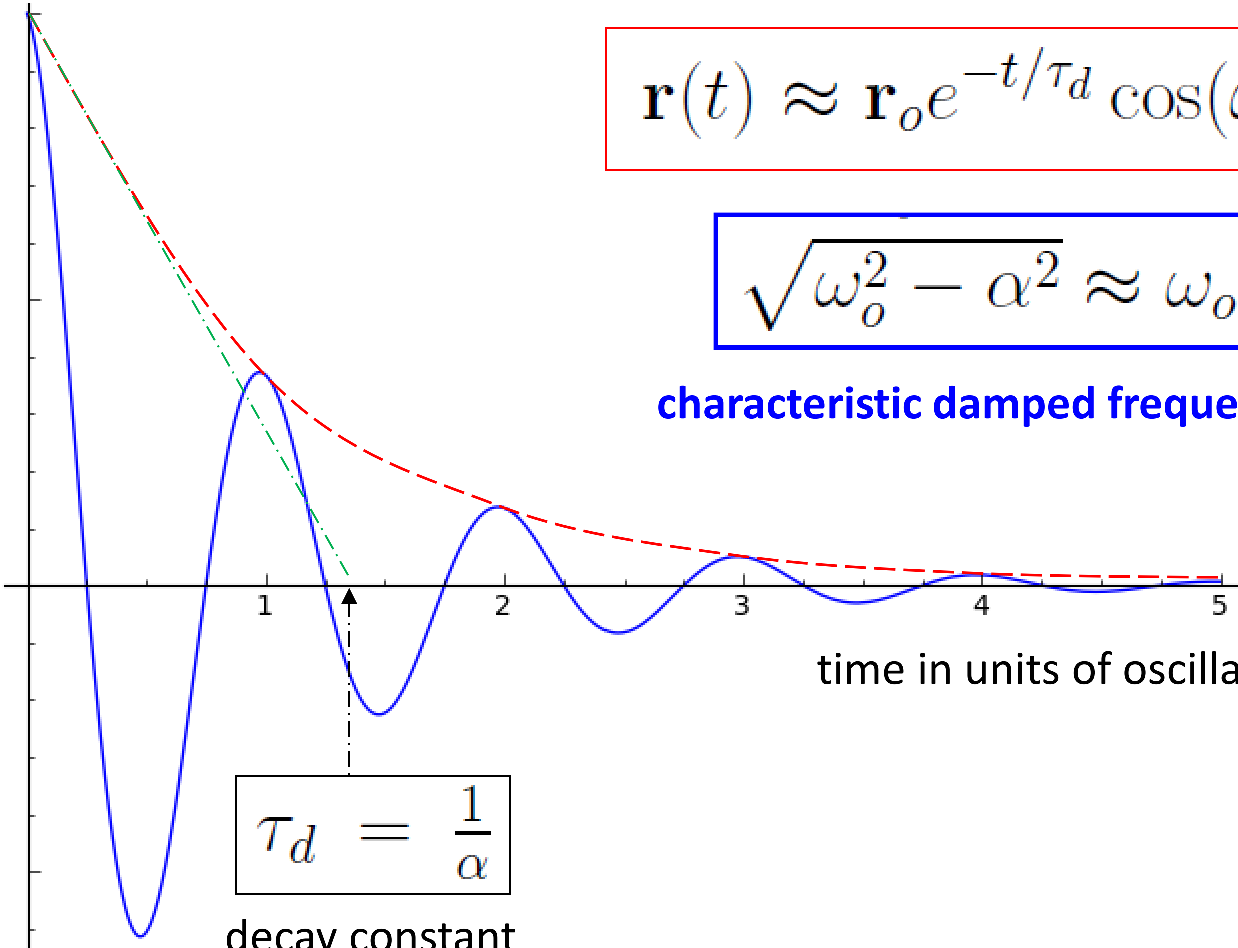
vector $\boldsymbol{r}$ from negative to positive

(Model in Rao's book)



The polarizing force due to the electric field is released at t = 0 ⇒ the dipole charges relax to equilibrium causing the dipole field to decay following the observed damped co-sinusoidal oscillation

$\mathbf{E}_p \propto \mathbf{p} \propto \mathbf{r}$ dipole field proportional to charge displacement



$$\boxed{\mathbf{r}(t) \approx \mathbf{r}_o e^{-t/\tau_d} \cos(\omega_o t)}$$

$$\boxed{\sqrt{\omega_o^2 - \alpha^2} \approx \omega_o}$$

characteristic damped frequency

time in units of oscillation periods

$$\boxed{\tau_d = \frac{1}{\alpha}}$$

decay constant

The behavior can be described by the Lorentz oscillator equation for the displacement vector

$$\boxed{t = 0}$$

$$m\frac{d^2\mathbf{r}}{dt^2} = -e\cancel{\mathbf{E}} - m\omega_o^2\mathbf{r} - m2\alpha\frac{d\mathbf{r}}{dt}$$

friction dissipative force

"spring" binding force

external force

acceleration

mass

The same Lorentz oscillator equation may be used do describe the steady-state condition after a field has been applied

steady-state steady-state

$$\cancel{m\frac{d^2\mathbf{r}}{dt^2}} = \underbrace{-e\mathbf{E}}_{\neq \mathbf{0}} - m\omega_o^2\mathbf{r} - \cancel{m2\alpha\frac{d\mathbf{r}}{dt}}$$

$$-e\mathbf{E} = m\omega_o^2\mathbf{r} \quad \Longrightarrow \quad \boxed{\mathbf{r} = -\frac{e}{m\omega_o^2}\mathbf{E}}$$

from which we can determine the atomic dipole moment

$$\mathbf{p} = -e\mathbf{r} = \frac{e^2}{m\omega_o^2}\mathbf{E}$$

Consider a medium consisting of N_d polarized particles per unit volume (dipole density)

$$\boxed{\mathbf{p} = -e\mathbf{r} = \frac{e^2}{m\omega_o^2}\mathbf{E}}$$

polarization field in the dielectric material

$$\boxed{\mathbf{P} = N_d\mathbf{p} = \frac{N_d e^2}{m\omega_o^2}\mathbf{E}}$$

From prior definition, we get an expression for the DC susceptibility

$$\mathbf{P} = \epsilon_o\chi_e\mathbf{E}$$

$$\boxed{\chi_e \equiv \frac{N_d\, e^2}{m\epsilon_o\, \omega_o^2}}$$

AC Conditions: Polarization Current

Let's consider now a time-varying Electric Field in a dielectric medium. If the frequency of the field is $\omega \ll \omega_o$ the DC susceptibility is still a good approximation of the AC value

$$\boxed{\chi_e \equiv \frac{N_d e^2}{m\epsilon_o\, \omega_o^2}}$$

and we can continue to use also

$$\boxed{\mathbf{r} = -\frac{e}{m\omega_o^2}\mathbf{E}} \qquad \boxed{\mathbf{P} = \epsilon_o \chi_e \mathbf{E}}$$

AC Conditions: Polarization Current

For a field $\mathbf{E}(t)$

$$\boxed{\mathbf{r}(t) = -\frac{e}{m\omega_o^2}\mathbf{E}(t)}$$

Electrons are displaced back and forth, with an effective spatial velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\frac{e}{m\omega_o^2}\frac{d\mathbf{E}}{dt}$$

This movement of bound charges in the dielectric medium generates an effective AC current density. With N_d bound electrons per unit volume

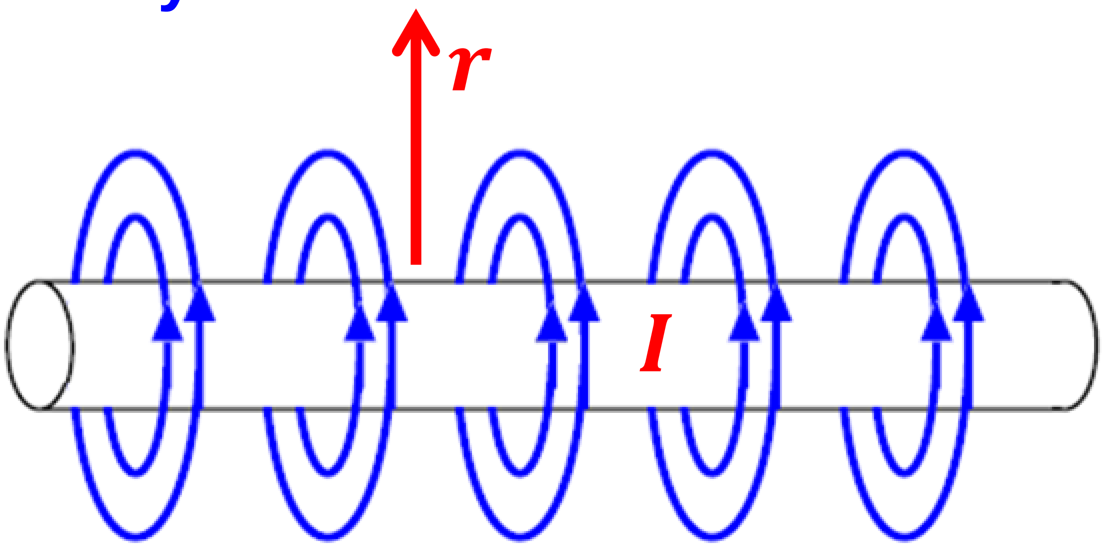
$$\mathbf{J}_p = -eN_d\mathbf{v} = \frac{N_d e^2}{m\omega_o^2}\frac{d\mathbf{E}}{dt} = \frac{d\mathbf{P}}{dt}$$

polarization current density (AC conditions)

Magnetic Flux density

Consider an infinite, straight current filament I with cylindrical coordinates wrapping around the wire.

At a distance r we measure a Magnetic Flux (or Magnetic Induction) which encircles the wire with azimuthal symmetry



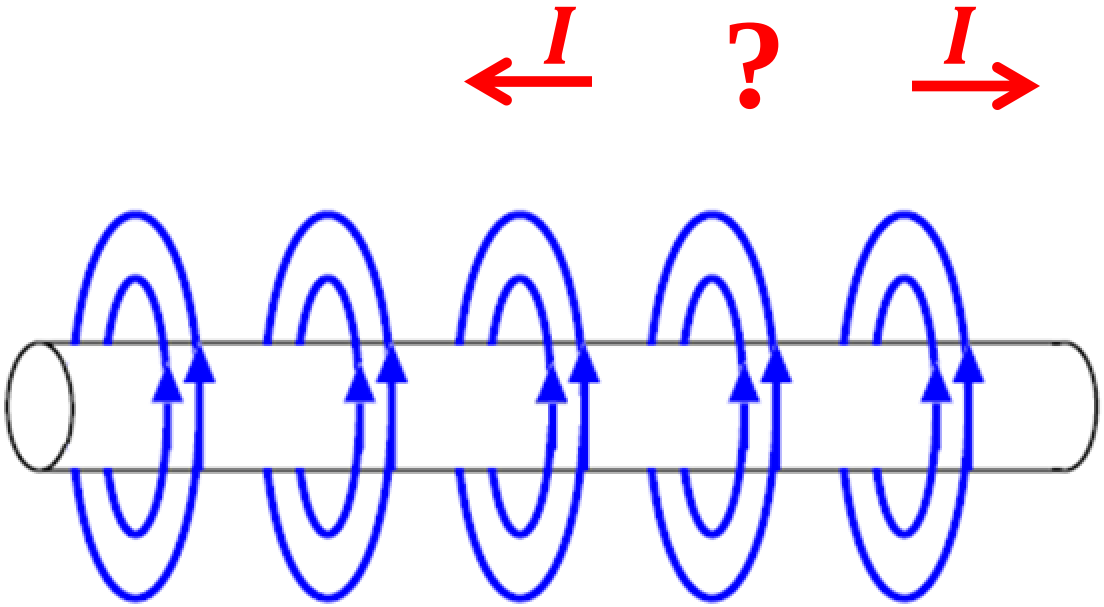
$$\mathbf{B} \equiv \frac{\mu_o I}{2\pi r}\hat{\phi}$$

A charge in proximity will experience the Lorentz force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

Question

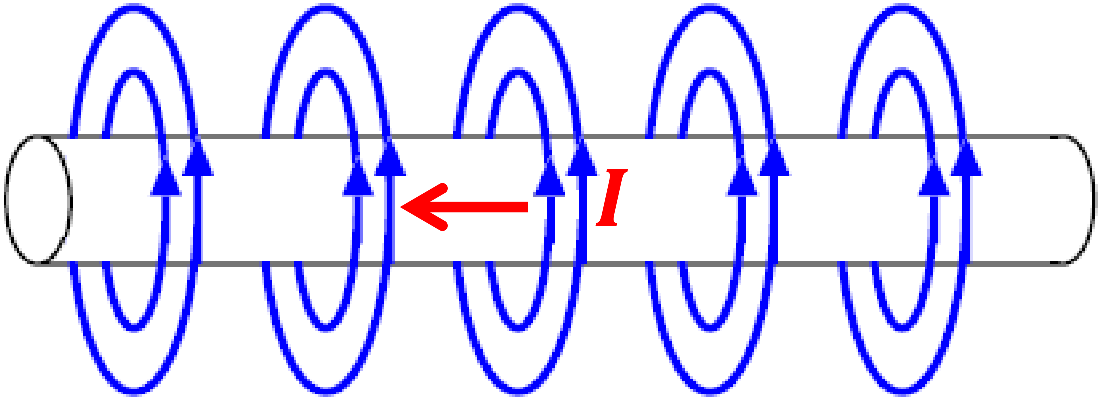
Which direction does the current point to?

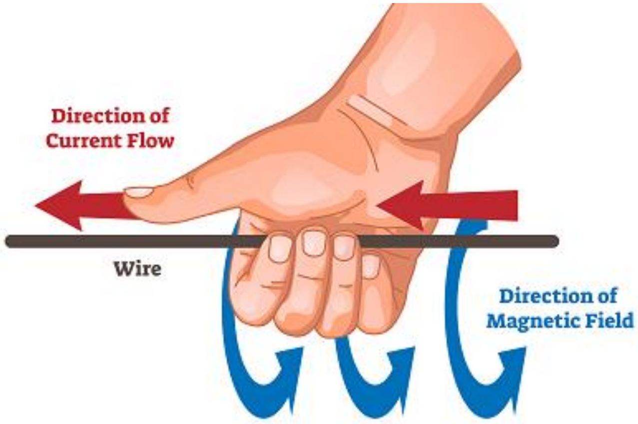


Question

Which direction does the current point to?

use the right-hand rule

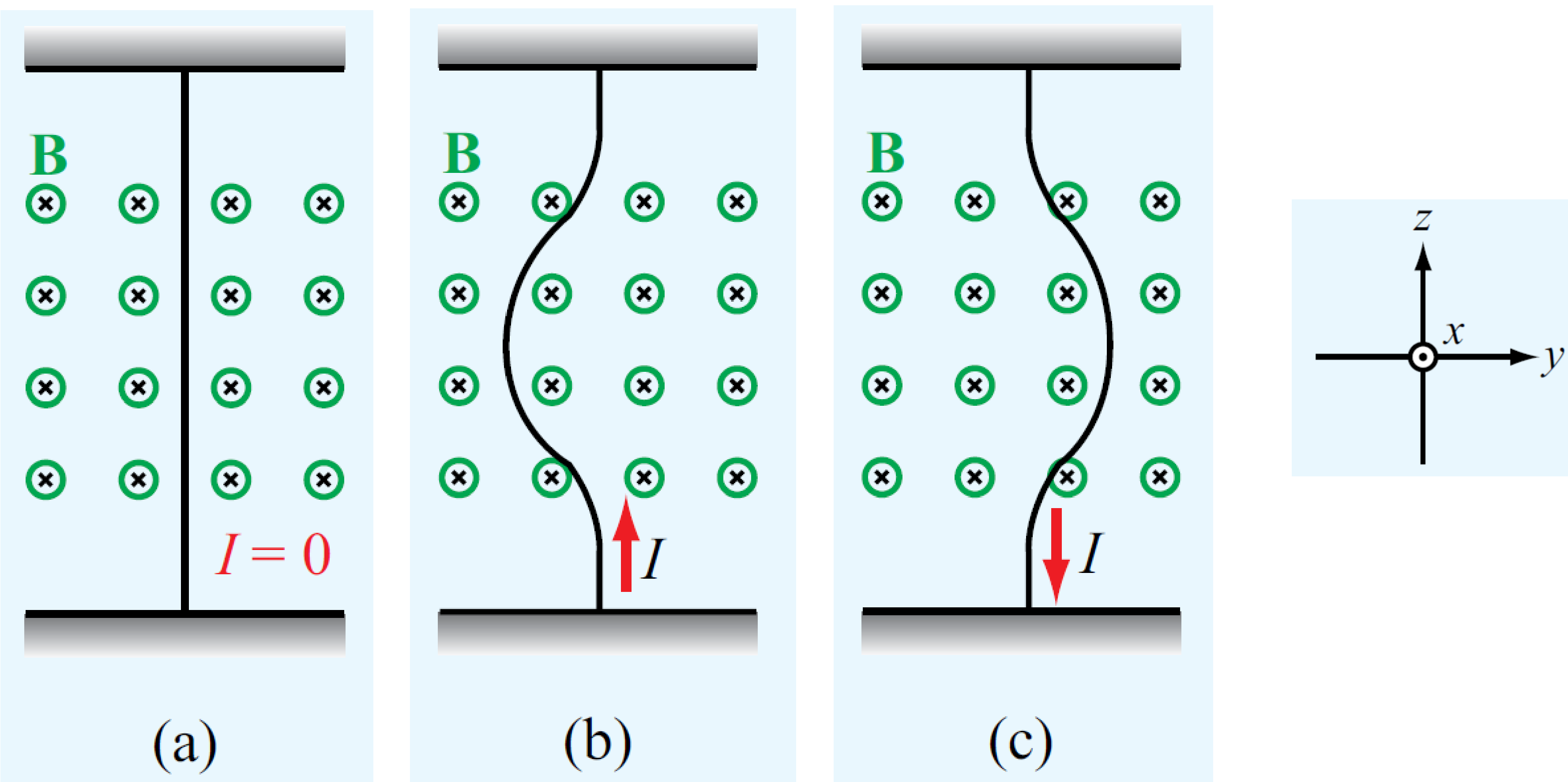




https://www.pasco.com/products/guides/right-hand-rule

Example: Magnetic Force

A flexible current filament immersed in a magnetic field is deformed by the magnetic force



Calculation of Magnetic Force on a wire

If the wire contains a free-electron charge density

$$\rho_e = -N_e e$$

the moving charge in an element dl of wire of area A is

$$dQ = \rho_e A\, dl = -N_e e A\, dl$$

The magnetic force in the presence of a field B

$$d\mathbf{F}_m = dQ\, \underbrace{\mathbf{v}_e}_{\text{drift velocity}} \times \mathbf{B} = -N_e e A\, \underbrace{dl\, \mathbf{v}_e} \times \mathbf{B}$$

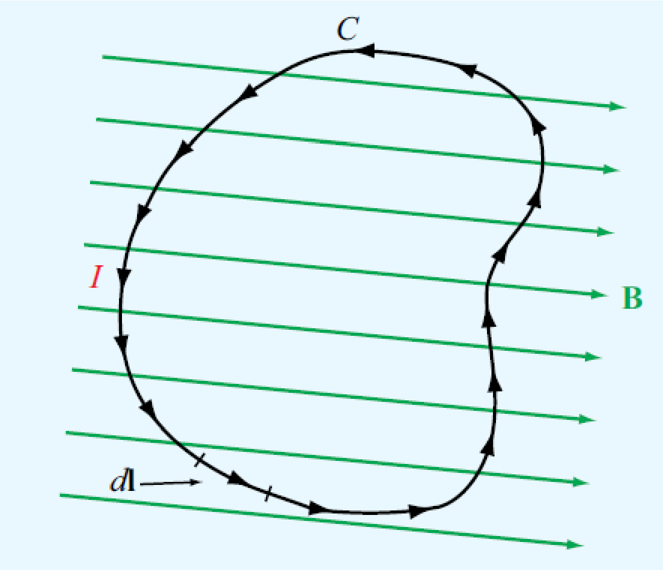
$\boxed{I = N_e e A v_e}$ $\hat{\mathbf{v}}$ and $d\boldsymbol{l}$ are aligned $\boxed{dl\, v_e \hat{\mathbf{v}} = d\boldsymbol{l}\, v_e}$

$$d\mathbf{F}_{\mathrm{m}} = I\, d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}$$

Integrate to find the total force over a closed loop

$$\mathbf{F}_{\mathrm{m}} = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}$$

If the field is uniform, it can be taken outside the integral

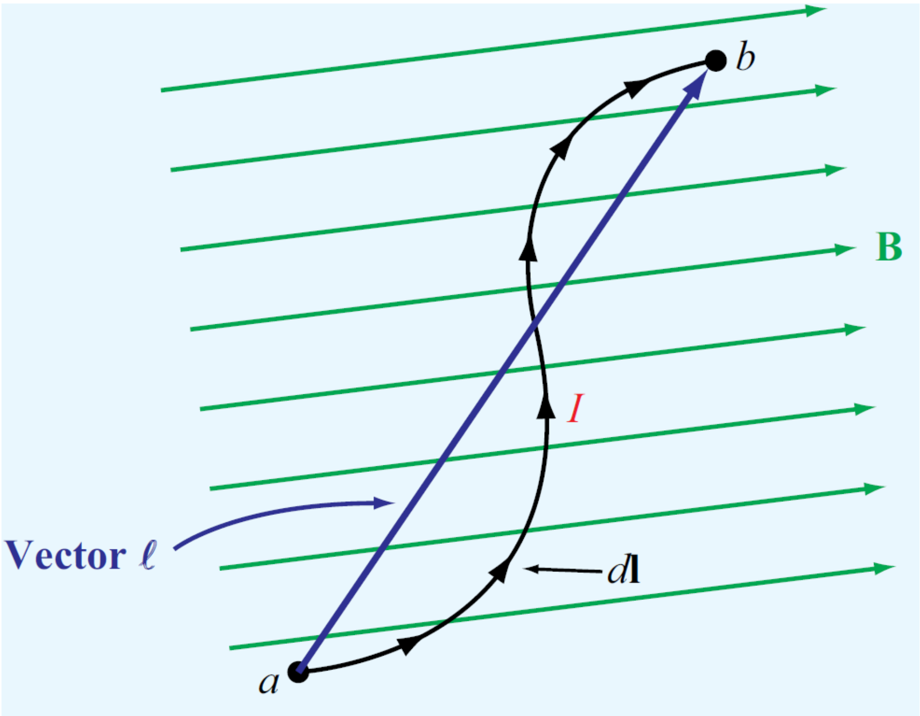


$$\mathbf{F}_{\mathrm{m}} = I \left(\oint_C d\mathbf{l} \right) \times \mathbf{B} = 0$$

To evaluate the total force of a segment of wire immersed in a uniform magnetic field

$$\mathbf{F}_{\mathrm{m}} = I\left(\int_{\ell} d\mathbf{l}\right) \times \mathbf{B} = I\boldsymbol{\ell} \times \mathbf{B}$$

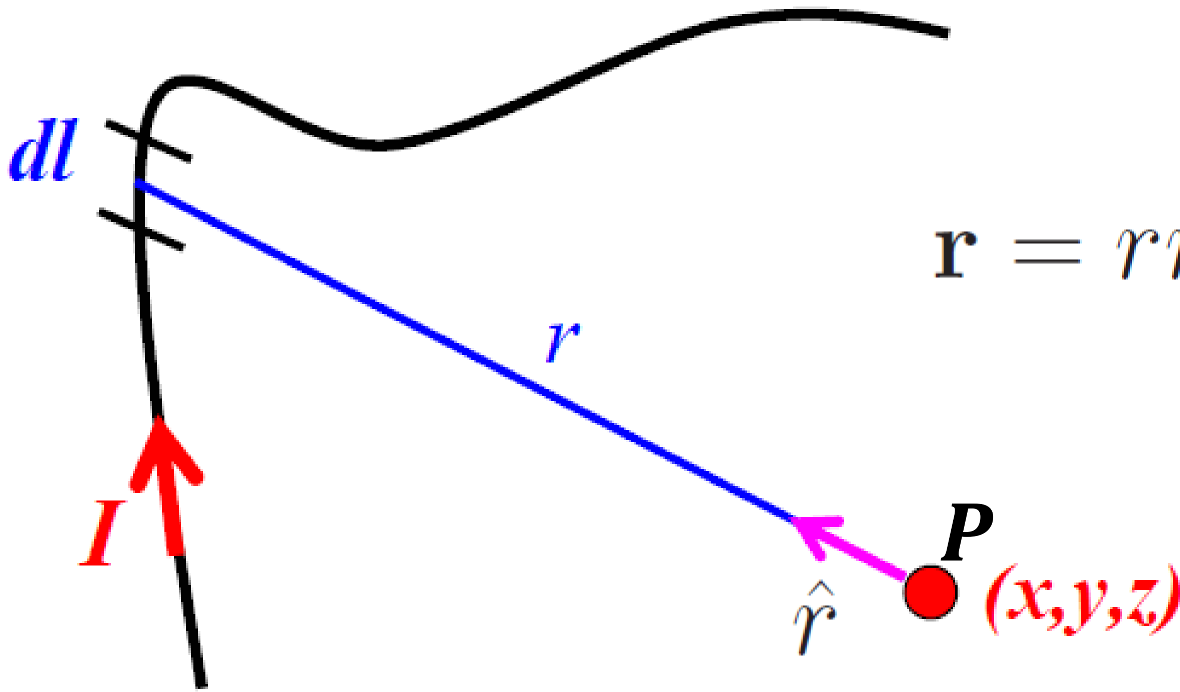
$\boldsymbol{\ell}$ **is the vector from a to b.**



Biot-Savart Law

Magnetic induction at a point *P* is obtained from superposition of the effects of infinitesimal elements of a current filament

$$d\mathbf{B} \equiv \frac{\mu_o I d\mathbf{l} \times \hat{r}}{4\pi r^2}$$



$$\mathbf{r} = r\hat{r}$$

Ampere's Law

Maxwell's equations for static magnetic fields

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

differential form of Ampere's Law

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

Take the flux of the curl through an open surface

$$\int_S \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = I_C$$

Stoke's Theorem

$$\boxed{\oint_C \mathbf{H} \cdot d\mathbf{l} = I_C}$$

integral form of Ampere's Law

current that crosses the area encircled by the closed circulation path around the surface

Volumetric current density is measured in $\mathrm{A/m^2}$ but in the limit we can have current flowing on a surface of infinitesimal thickness or on a filament of infinitesimal cross-section

Volumetric representation of

$$\mathbf{J}(x, y, z) = \mathbf{J}_s(y, z)\delta(x - x_o)$$

surface current density measured in $\mathrm{A/m}$

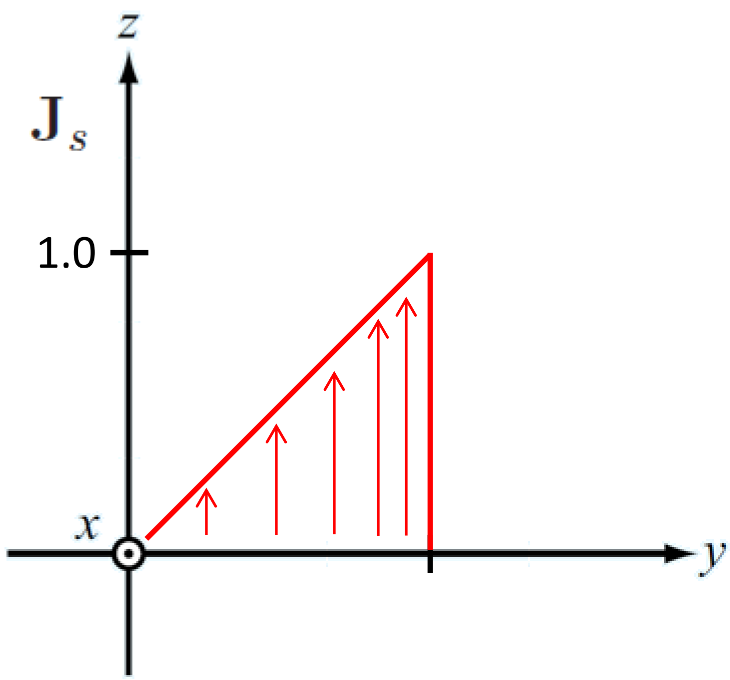
$$\mathbf{J}(x, y, z) = \hat{z}I(z)\delta(x - x_o)\delta(y - y_o)$$

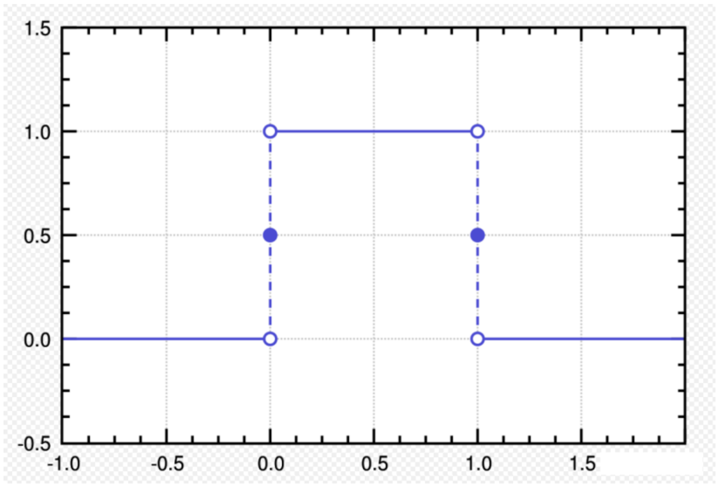
line current density along z-axis measured in A

Example – surface current density flowing on plane $x = 0$

$$\mathbf{J}_s = \hat{z} y \,\mathrm{rect}(y - 0.5) \text{ A/m}$$

Find the total current in Ampere units



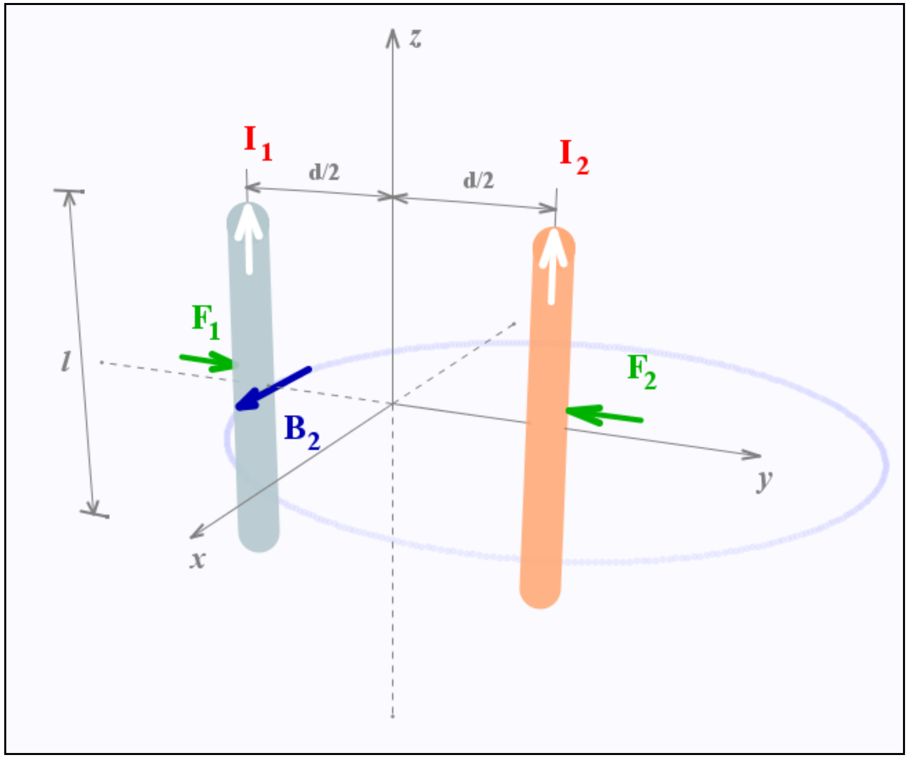


$\mathrm{rect}(y - 0.5)$

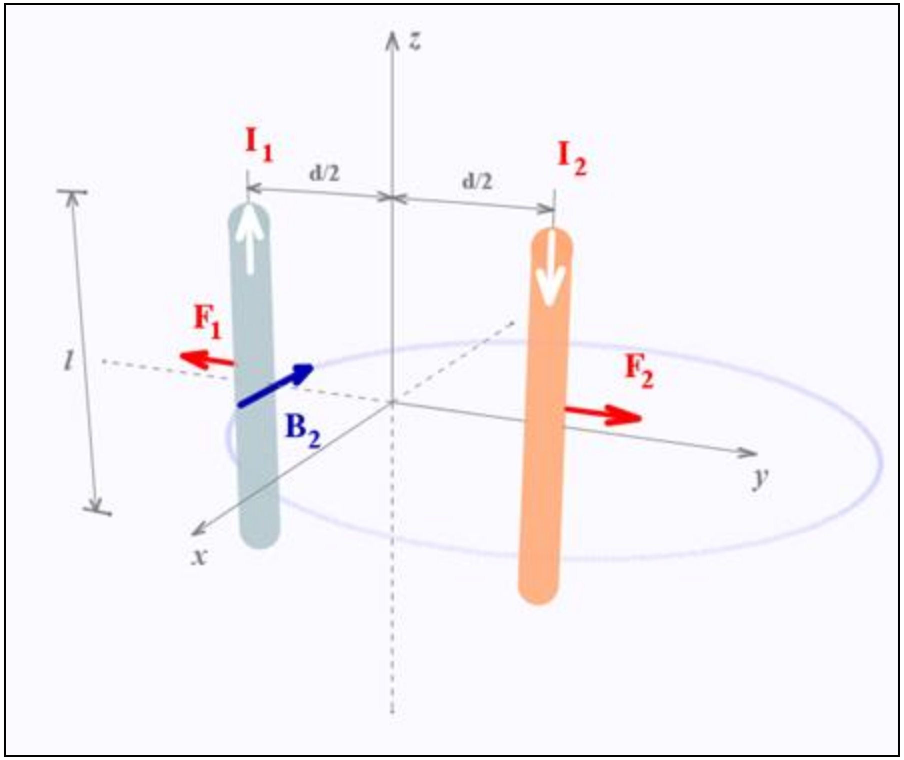
$$I = \int_{y=-\infty}^{\infty} \mathbf{J}_s \cdot \hat{z} dy = \int_{y=0}^{1} y\, dy = \frac{y^2}{2}\Big|_0^1 = \frac{1}{2} \text{ A}.$$

Example: Two parallel electrical wires carrying a current, interact with each other.

- If the wires have opposite currents, they repel each other.
- With currents in the same direction, they attract each other.



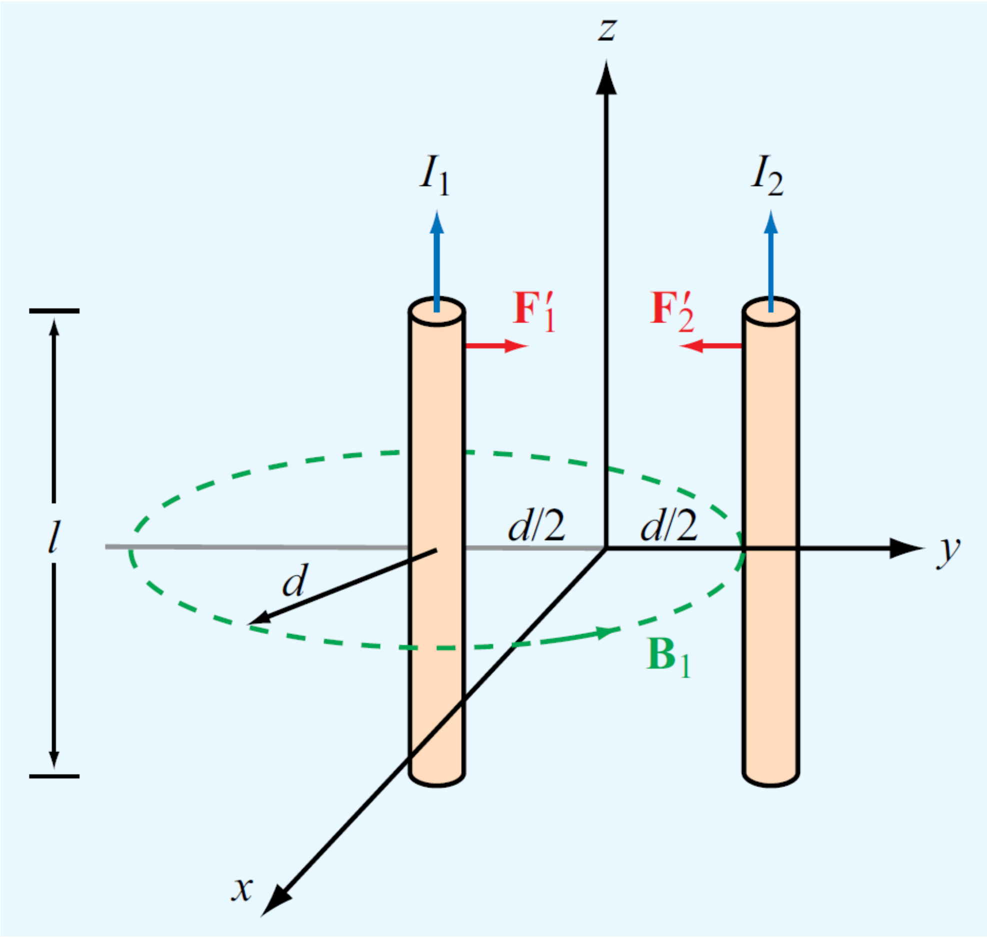
Wires attract each other with equal force



Wires repel each other with equal force

Wire 1 carries current I_1 and generates field $\mathbf{B}_1$ at distance d

$$\mathbf{B}_1 = -\hat{\mathbf{x}}\,\frac{\mu_0 I_1}{2\pi d}$$



Force exerted on a length of wire l with current I_2

$$\mathbf{F}_2 = I_2 l\hat{\mathbf{z}} \times \mathbf{B}_1$$

$$= I_2 l\hat{\mathbf{z}} \times (-\hat{\mathbf{x}})\,\frac{\mu_0 I_1}{2\pi d}$$

$$= -\hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2 l}{2\pi d}$$

Force per unit length

$$\mathbf{F}'_2 = \frac{\mathbf{F}_2}{l} = -\hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2}{2\pi d}$$

Similarly

$$\mathbf{F}'_1 = \hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2}{2\pi d}$$