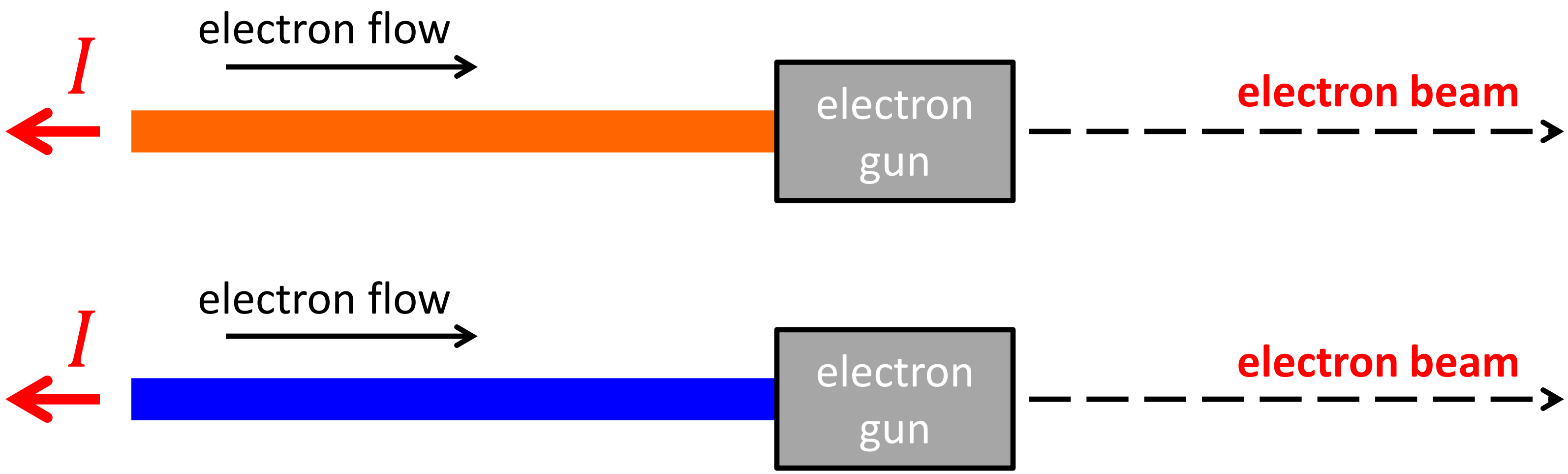
# ECE 329 – Fall 2021

**Prof. Ravaioli – Office: 2062 ECEB**

Section E – 1:00pm

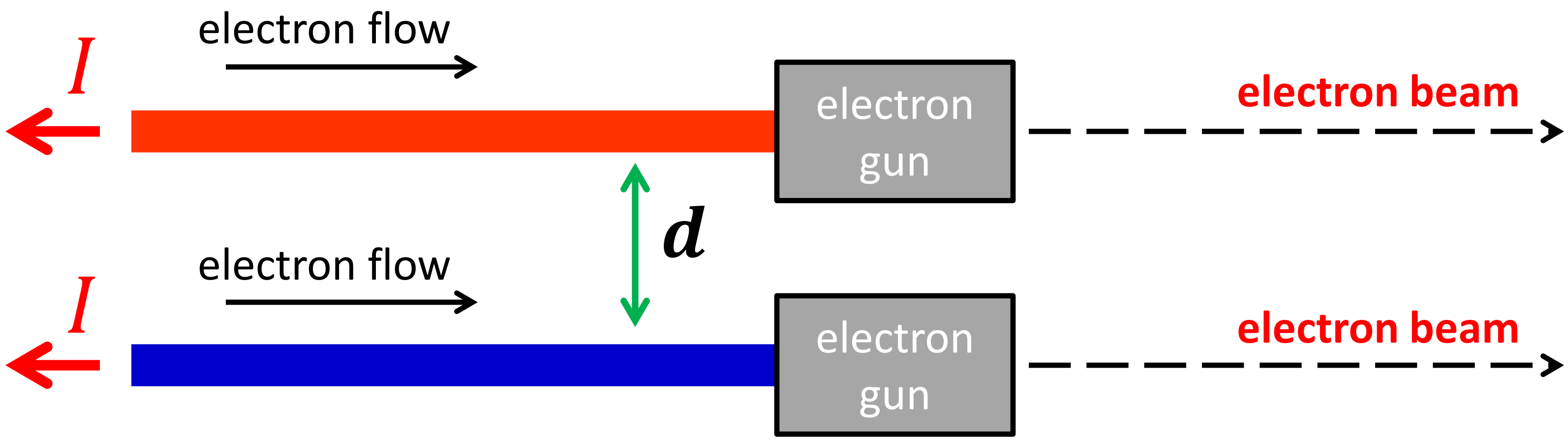
Lecture 12 - SUPPLEMENT

# Let's think of an experiment



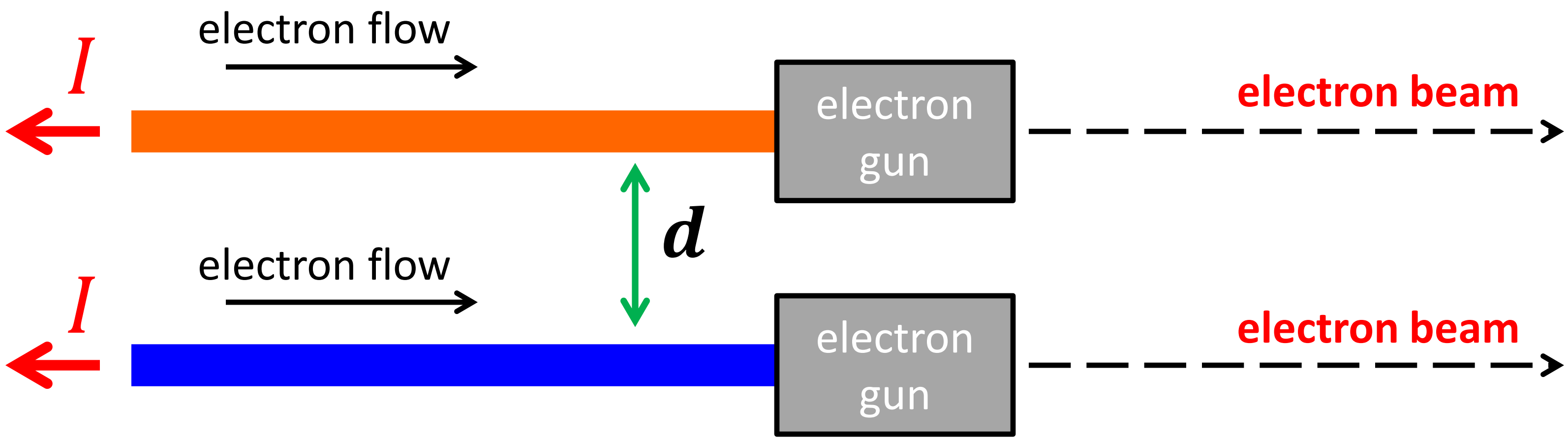

**The wires are electrically neutral (uncharged) because for every electron there is a proton.**

**Electron beams are not "neutral" since there are no protons.**




If there are $N$ electron per meter in the beam, and electron have constant velocity $v$, the beam carries a current $I = Nev$ and the same current flows in the wires. From theory, the wires attract each other by magnetic interaction with force

$$F_{attr} = \frac{\mu_0 (Nev)^2}{2\pi d} = \frac{\mu_0 (I)^2}{2\pi d} \quad \frac{\mathrm{N}}{\mathrm{m}}$$




**Also the electron beams should be magnetically attracted, but there are no protons, so there is also an electrical repulsive force**

$$F_{rep} = \frac{N^2 e^2}{2\pi\epsilon_0 d} \quad \frac{\mathrm{N}}{\mathrm{m}}$$

$$F_{attr} = \frac{N^2 e^2}{2\pi\epsilon_0 d}\epsilon_0\mu_0 v^2 \quad \frac{\mathrm{N}}{\mathrm{m}}$$

**same attractive force as before but conveniently rewritten**

# We have so far the forces

$$F_{rep} = \frac{N^2 e^2}{2\pi\epsilon_0 d} \quad \frac{\mathrm{N}}{\mathrm{m}}$$

$$F_{attr} = \frac{N^2 e^2}{2\pi\epsilon_0 d}\epsilon_0\mu_0 v^2 = \frac{N^2 e^2}{2\pi\epsilon_0 d}\frac{v^2}{c^2} \quad \frac{\mathrm{N}}{\mathrm{m}}$$

# Combining the two forces we obtain

$$F_{tot} = F_{rep} - F_{attr} = \frac{N^2 e^2}{2\pi\epsilon_0 d}\left(1 - \frac{v^2}{c^2}\right) \quad \frac{\mathrm{N}}{\mathrm{m}}$$

This is a term which appears in the Special Theory of Relativity

$$\frac{F_{attr}(\text{magnetic})}{F_{rep}(\text{electric})} = \frac{v^2}{c^2}$$

**These forces have been measured in the frame of reference of the laboratory.**

**But what if you imagined to "ride" an electron in one of the beams, and measure the forces in that frame of reference, moving at constant velocity $v$?**

**The electron configurations in the beams would appear stationary with each other, so there would be no perception of a current, no magnetism and only electric repulsive force…**

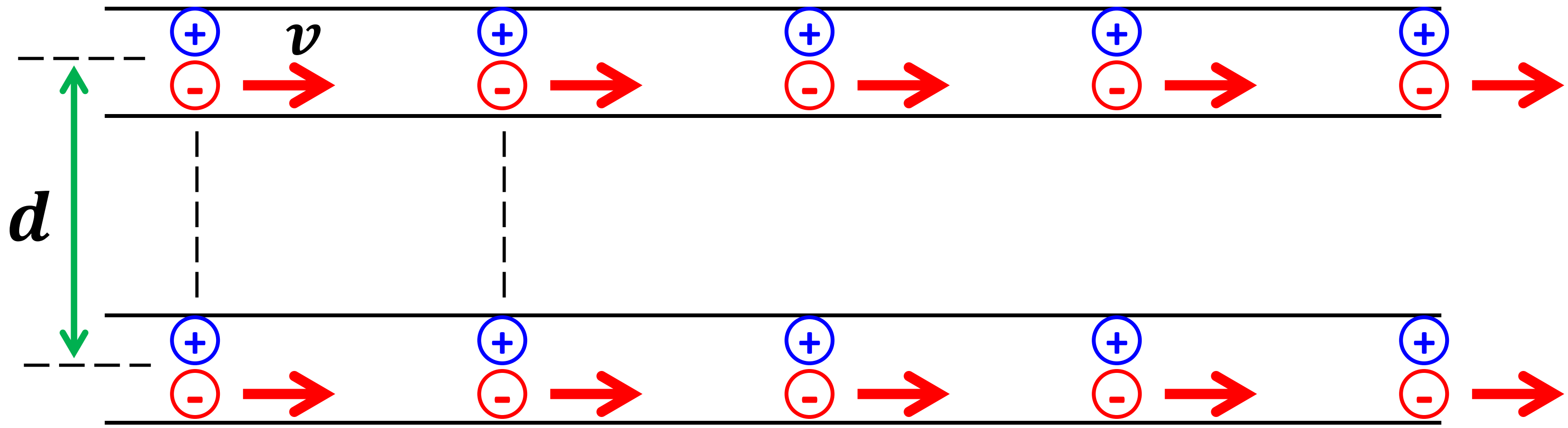
$$F_{tot} = F_{rep} = \frac{N^2 e^2}{2\pi\epsilon_0 d} \, \frac{\mathrm{N}}{\mathrm{m}}$$

**? ? ?**

There has to be a discrepancy and the fact that we got the term

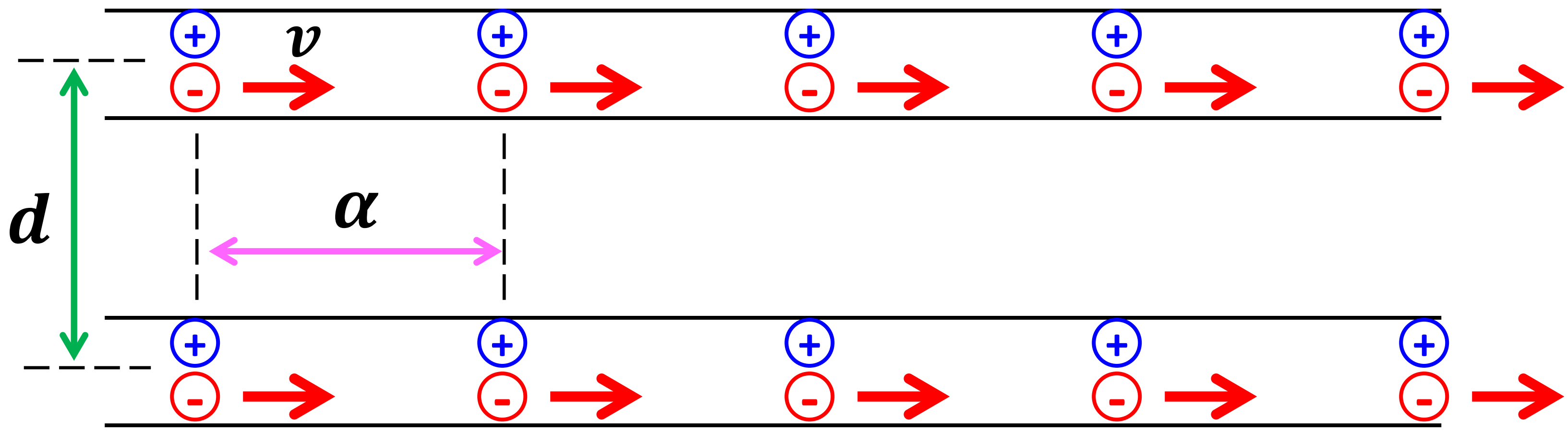
$$\left(1 - \frac{v^2}{c^2}\right)$$

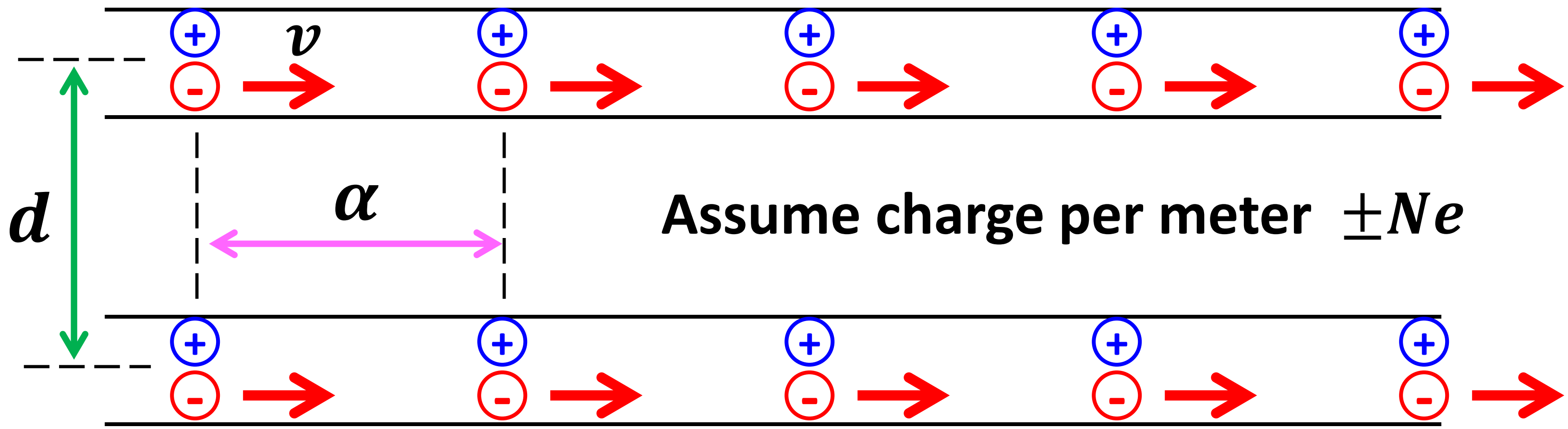
in the laboratory reference, suggests that we should consider relativity to understand the origins of magnetic forces. We will consider an idealized model of the parallel wires, with fixed protons and mobile electrons moving at constant velocity $v$.



**Magnetic force is observed experimentally but where does it come from?**

**Let's consider an idealized model of the parallel wires, with fixed protons and mobile electrons moving at constant velocity $v$. Fixed charges are evenly spaced and electrons move single file.**






Without relativity, one would say that there is perfect charge neutrality, so no net electric forces between charges. But electrons are moving …

Consider four forces (per meter) between:

(a) the two proton lines (+ +)
(b) lower proton line and upper electron line (+ –)
(c) upper proton line and lower electron line (– +)
(d) the two electron lines (– –)

**(+ +) is a repulsion**

$$F_{rep} = \frac{N^2 e^2}{2\pi\epsilon_0 d} \quad \frac{\mathrm{N}}{\mathrm{m}}$$

**(+ −) is an attraction**

$$F_{attr} = -\frac{N^2 e^2}{2\pi\epsilon_0 d} \quad \frac{\mathrm{N}}{\mathrm{m}}$$

**(− +) is an attraction**

$$F_{attr} = -\frac{N^2 e^2}{2\pi\epsilon_0 d} \quad \frac{\mathrm{N}}{\mathrm{m}}$$

**Even if electrons are moving, they are measured in the laboratory frame of reference ($S$) and must pass the protons simultaneously with the same charge density. The sum of the three forces then gives**

$$F_{1\_3} = -\frac{N^2 e^2}{2\pi\epsilon_0 d} \quad \frac{\mathrm{N}}{\mathrm{m}}$$

(– –) To estimate this force, we need to ride along with the electrons in their moving frame of reference ($S'$) in which they are stationary, applying the electric force equation.

The factor subject to relativity in the expression for the force is the number of electrons per meter, $N$. Note that the spacing between wires, $d$, is unaffected because it is normal to the propagation.

In the frame $S'$ we can write formally the repulsion force between electron lines as

$$F_4' = \frac{(N')^2 e^2}{2\pi\epsilon_0 d} \quad \frac{\text{N}}{\text{m}}$$

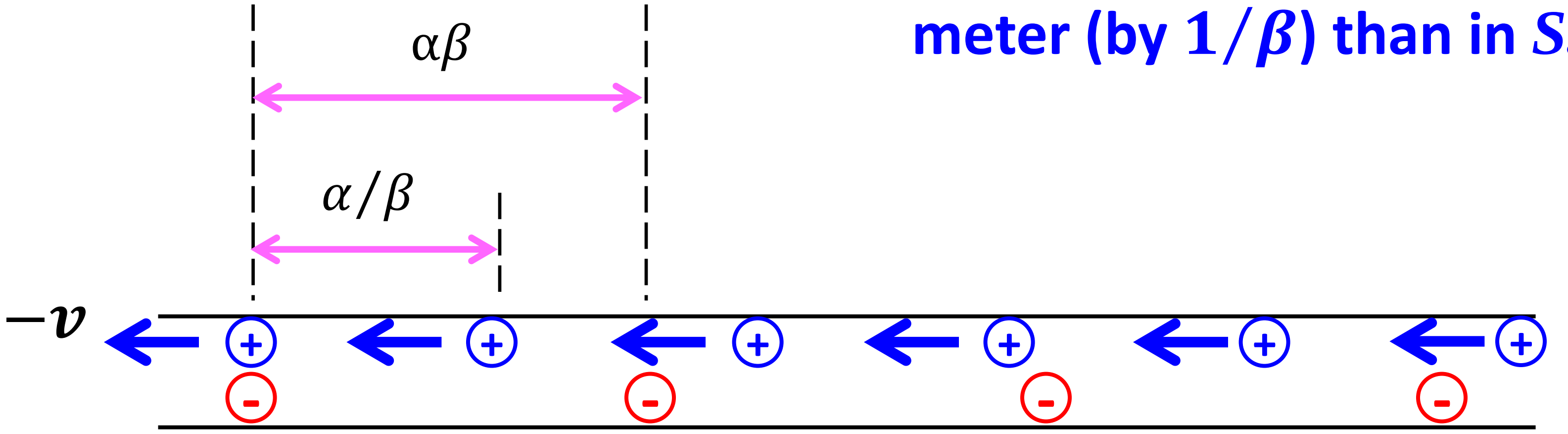
**In the frame of reference $S'$ moving with the electrons, it appears as if the protons residing in the frame $S$ are moving to the left with distances reduced by the factor**

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Lorentz factor

**In $S'$ the protons look closer than they did in $S$ and there are $\beta$ times more per meter.**

**For $S$ the electrons look further apart by a factor $\beta$ and so they are fewer per meter (by $1/\beta$) than in $S$.**



# Lorentz Contraction (e.g., https://www.feynmanlectures.caltech.edu/II_13.html)

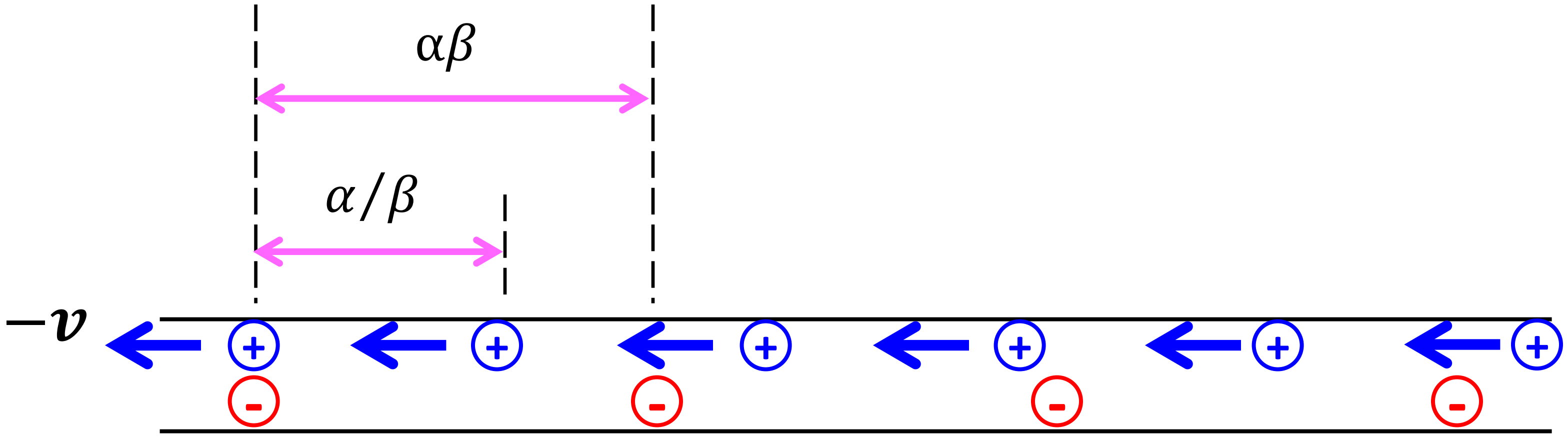
$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

**Lorentz factor**

$$L = L_0 \frac{1}{\beta} = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

**$L$ = Length observed by observer in motion relative to an object**

**$L_0$ = Length of the object in its rest frame**



**Because electrons are standing still in frame *S'* we can indeed use the standard equation for the repulsive force shown earlier, with primed parameters**

$$F_4' = \frac{(N')^2 e^2}{2\pi\epsilon_0 d} \quad \frac{\text{N}}{\text{m}}$$

**Going back to *S*, it is a principle of the theory of relativity that the laws of nature are the same in all inertial systems (moving at constant velocity).**

$$F_4 = F_4' = \frac{(N')^2 e^2}{2\pi\epsilon_0 d} = \frac{(N/\beta)^2 e^2}{2\pi\epsilon_0 d} = \frac{N^2 e^2}{2\pi\epsilon_0 d}\left(1 - \frac{v^2}{c^2}\right) \quad \frac{\text{N}}{\text{m}}$$

## We had before the sum of the first three forces

$$F_{1\_3} = -\frac{N^2 e^2}{2\pi\epsilon_0 d} \quad \frac{\mathrm{N}}{\mathrm{m}}$$

## and now the fourth force:

$$F_4 = \frac{N^2 e^2}{2\pi\epsilon_0 d}\left(1 - \frac{v^2}{c^2}\right) \quad \frac{\mathrm{N}}{\mathrm{m}}$$

$$F_{1\_3} + F_4 = -\frac{N^2 e^2}{2\pi\epsilon_0 d} + \frac{N^2 e^2}{2\pi\epsilon_0 d}\left(1 - \frac{v^2}{c^2}\right)$$
$$= -\frac{N^2 e^2}{2\pi\epsilon_0 d}\,\frac{v^2}{c^2} = -\frac{\mu_0 (Nev)^2}{2\pi d}$$
$$= -\frac{\mu_0 I^2}{2\pi d} \quad \frac{\mathrm{N}}{\mathrm{m}}$$

negative sign means attractive force by convention

**The result means that we have obtained the standard formula for the magnetic force of attraction between two parallel wires spaced $d$ meters apart and carrying a current $I$ in the same direction**

$$\boxed{F = -\frac{\mu_0 I^2}{2\pi d} \quad \frac{\text{N}}{\text{m}}}$$

**But we have arrived at it by considering a <u>purely electrical force</u>, due to imbalance of positive and negative charges when <u>relativity</u> is taken into account.**

**The same demonstration can be carried out for repulsion due to opposite currents, but it is somewhat more involved because two different velocities are considered.**

**Example**

**Consider $I = 1\text{A}$ in a metal wire with cross-section $1\text{mm}^2$. First, estimate the electron drift velocity.**

**Let's assume $10^{29}/\text{m}^3$ free carriers in our metal (copper for instance has $8.4918 \times 10^{28}$) . For a wire of 1m length, the volume is $10^{-6}$ cubic meters, which means $N = 10^{23}$.**

**Since $I = Nev = 1.0\ \text{A}$,**

$$v = \frac{I}{Ne} = \frac{1}{10^{23} \times 1.602 \times 10^{-19}}$$

**then $v \approx 6.25 \times 10^{-5}\ \text{m/s}$ or $0.0625\ \text{mm/s}$.**

**Now, consider the two parallel wires to be at a distance $d = 0.01\ \mathrm{m}$**

$$F_{1\_3} + F_4 = -\frac{N^2e^2}{2\pi\epsilon_0 d}\,\frac{v^2}{c^2}\quad \frac{\mathrm{N}}{\mathrm{m}}$$

$$\frac{v^2}{c^2} \approx \frac{(6.25\times10^{-5})^2}{(3\times10^{8})^2} \approx 4.3\times10^{-26}$$

$$-\frac{N^2e^2}{2\pi\epsilon_0 d} \approx \frac{(10^{23}\times1.602\times10^{-19})^2}{2\pi\times8.854\times10^{-12}\times0.01} \approx -4.6\times10^{20}\ \frac{\mathrm{N}}{\mathrm{m}}$$

$$F_{1\_3} + F_4 = -\frac{N^2e^2}{2\pi\epsilon_0 d}\,\frac{v^2}{c^2} \approx -1.98\times10^{-5}\ \frac{\mathrm{N}}{\mathrm{m}}$$

**With the standard formula**

$$F = -\frac{\mu_0 I^2}{2\pi d} = -\frac{4\pi\times10^{-7}}{2\pi\times0.01} = -2.0\times10^{-5} = \ \frac{\mathrm{N}}{\mathrm{m}}$$

**Results were pretty close. But let's use the exact values for the speed of light, permittivity, and electron charge**

$$v = \frac{1}{10^{23} \times 1.60217663 \times 10^{-19}} = 6.24150909 \text{ m/s}$$

$$\frac{v^2}{c^2} = \frac{(6.24150909 \times 10^{-5})^2}{(2.99792458 \times 10^{8})^2} = 4.33448804 \times 10^{-26}$$

$$-\frac{N^2 e^2}{2\pi\epsilon_0 d} = \frac{(10^{23} \times 1.60217663 \times 10^{-19})^2}{2\pi \times 8.85418782 \times 10^{-12} \times 0.01} = -4.614155078 \times 10^{20} \ \frac{\text{N}}{\text{m}}$$

## Results were pretty close. But let's use the exact values for the speed of light, permittivity, and electron charge

$$v = \frac{1}{10^{23} \times 1.60217663 \times 10^{-19}} = 6.24150909 \text{ m/s}$$

$$\frac{v^2}{c^2} = \frac{(6.24150909 \times 10^{-5})^2}{(2.99792458 \times 10^{8})^2} = 4.33448804 \times 10^{-26}$$

$$-\frac{N^2 e^2}{2\pi\epsilon_0 d} = \frac{(10^{23} \times 1.60217663 \times 10^{-19})^2}{2\pi \times 8.85418782 \times 10^{-12} \times 0.01} = -4.614155078 \times 10^{20} \ \frac{\text{N}}{\text{m}}$$

**Exactly the same!!!**

$$F_{1\_3} + F_4 = -\frac{N^2 e^2}{2\pi\epsilon_0 d} \frac{v^2}{c^2} = -2.0 \times 10^{-5} \ \frac{\text{N}}{\text{m}}$$

**With the standard formula**

$$F = -\frac{\mu_0 I^2}{2\pi d} = -\frac{4\pi \times 10^{-7}}{2\pi \times 0.01} = -2.0 \times 10^{-5} = \ \frac{\text{N}}{\text{m}}$$