ECE 329 – Fall 2022

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

Lecture 13

Lecture 13 – Outline

- **Magnetostatics**
  - **Ampere's Law**
  - **Current sheet**
  - **Solenoid**
  - **Vector Potential**

Reading assignment

Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:

13) Current sheet, solenoid, vector potential and current loops

Ampere's Law

Maxwell's equations for static magnetic fields

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

differential form of Ampere's Law

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

Take the flux of the curl through an open surface

$$\int_S \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = I_C$$

Stoke's Theorem

$$\boxed{\oint_C \mathbf{H} \cdot d\mathbf{l} = I_C}$$

integral form of Ampere's Law

current that crosses the area encircled by the closed circulation path around the surface

Volumetric current density is measured in $\mathrm{A/m^2}$ but in the limit we can have current flowing on a surface of infinitesimal thickness or on a filament of infinitesimal cross-section

Volumetric representation of

$$\mathbf{J}(x,y,z) = \mathbf{J}_s(y,z)\delta(x-x_o)$$

surface current density measured in $\mathrm{A/m}$

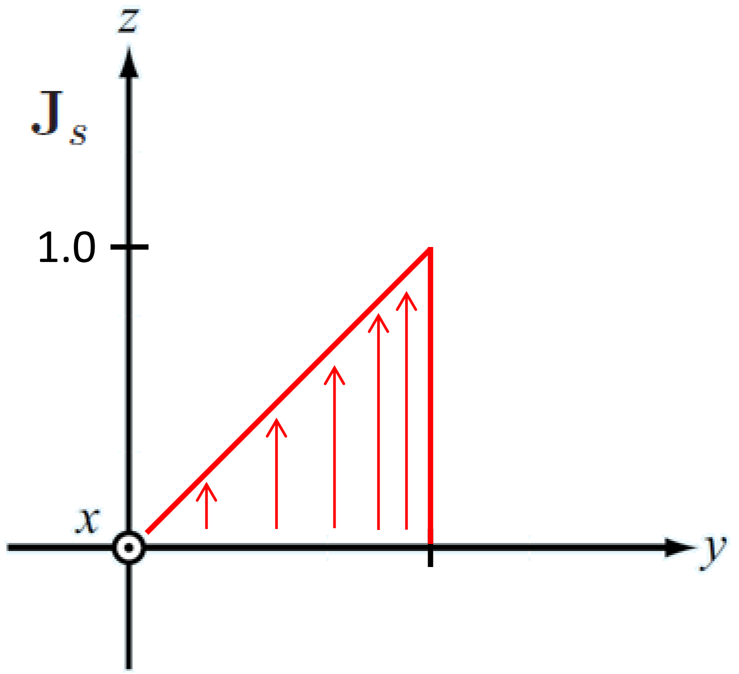
$$\mathbf{J}(x,y,z) = \hat{z}I(z)\delta(x-x_o)\delta(y-y_o)$$

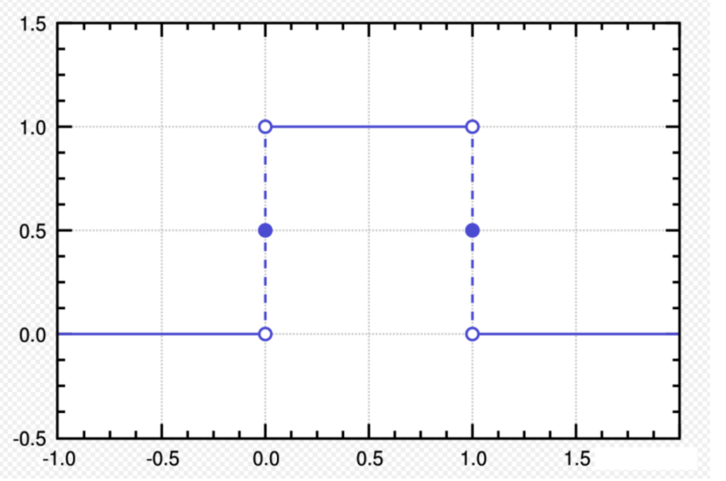
line current density along z-axis measured in A

Example – surface current density flowing on plane $x = 0$

$$\mathbf{J}_s = \hat{z} y \, \mathrm{rect}(y - 0.5) \; \mathrm{A/m}$$

Find the total current in Ampere units



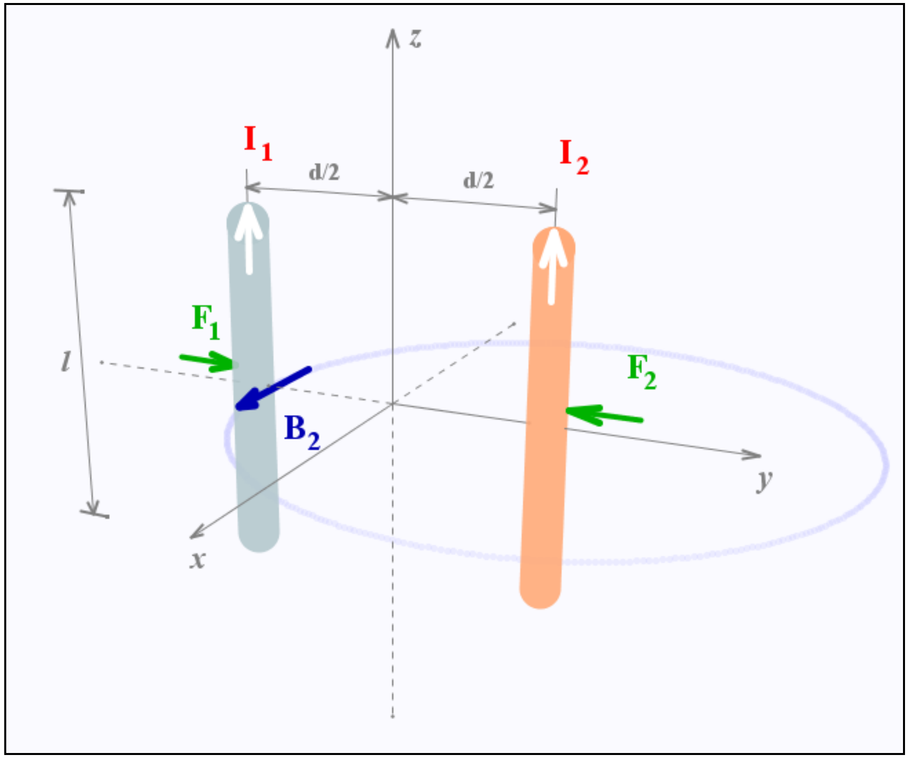


$\mathrm{rect}(y - 0.5)$

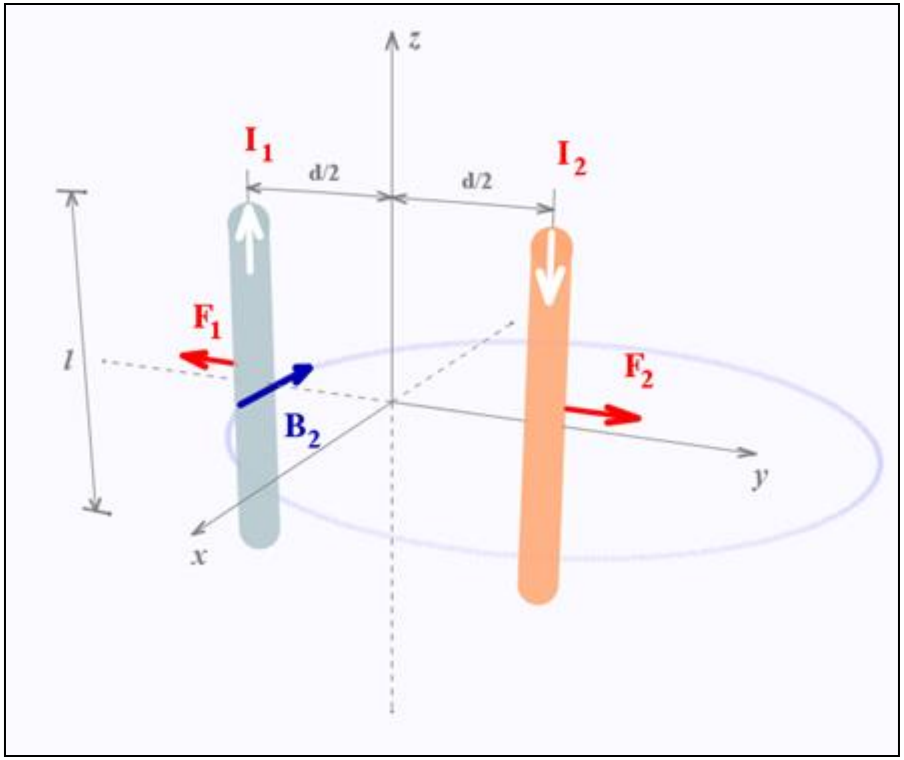
$$I = \int_{y=-\infty}^{\infty} \mathbf{J}_s \cdot \hat{z} dy = \int_{y=0}^{1} y \, dy = \frac{y^2}{2}\Big|_0^1 = \frac{1}{2} \, \mathrm{A}.$$

Example: Two parallel electrical wires carrying a current, interact with each other.

- If the wires have opposite currents, they repel each other.
- With currents in the same direction, they attract each other.



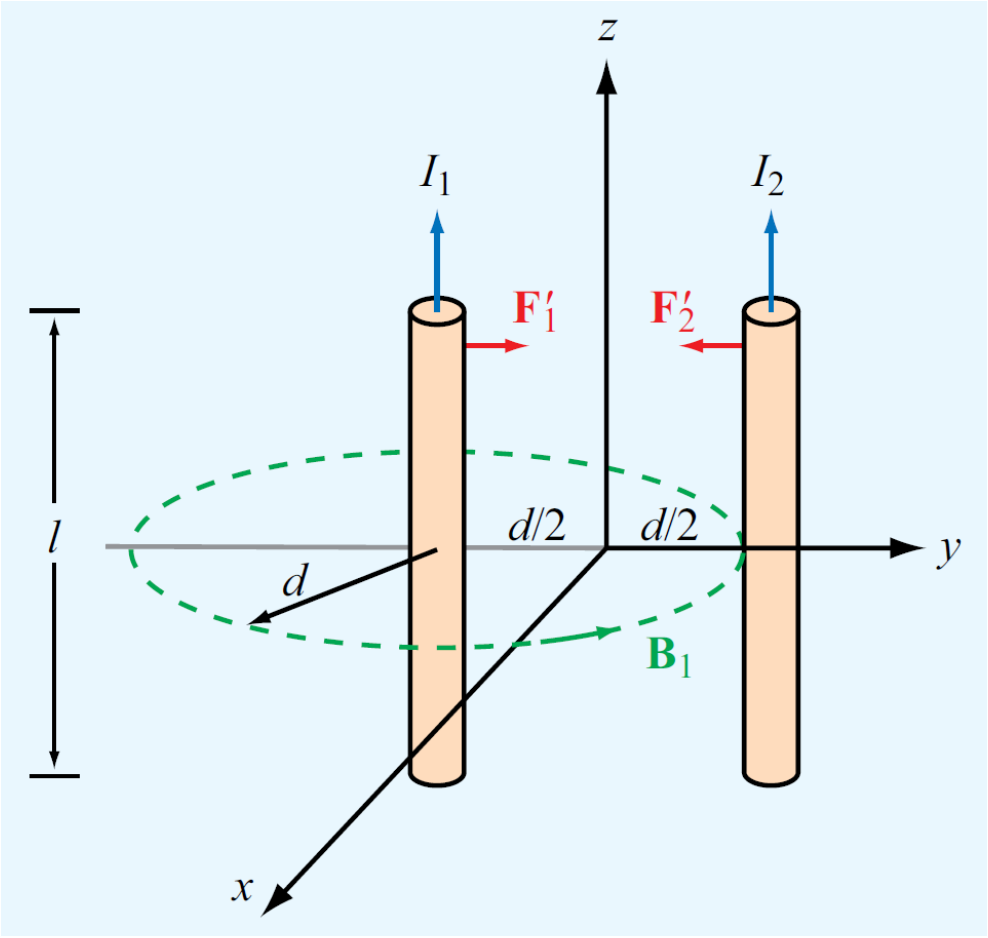
Wires attract each other with equal force



Wires repel each other with equal force

Wire 1 carries current I_1 and generates field $\mathbf{B}_1$ at distance d

$$\mathbf{B}_1 = -\hat{\mathbf{x}}\,\frac{\mu_0 I_1}{2\pi d}$$



Force exerted on a length of wire l with current I_2

$$\mathbf{F}_2 = I_2 l\hat{\mathbf{z}} \times \mathbf{B}_1$$

$$= I_2 l\hat{\mathbf{z}} \times (-\hat{\mathbf{x}})\,\frac{\mu_0 I_1}{2\pi d}$$

$$= -\hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2 l}{2\pi d}$$

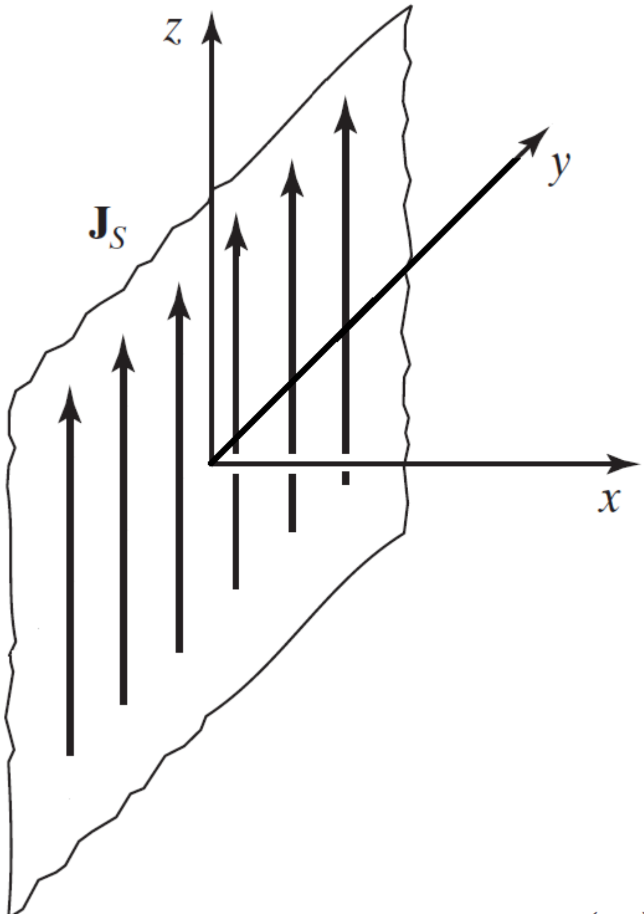
Force per unit length

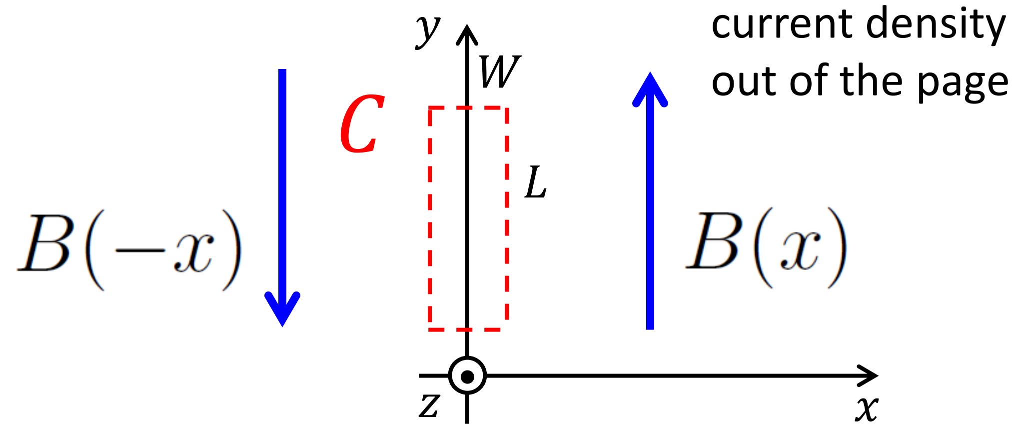
$$\mathbf{F}_2' = \frac{\mathbf{F}_2}{l} = -\hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2}{2\pi d}$$

Similarly

$$\mathbf{F}_1' = \hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2}{2\pi d}$$

Magnetic Field of infinite sheet at $x = 0$ with surface current density $\mathbf{J}_s = J_s \hat{z}$ A/m





Symmetry of the problem suggests $\mathbf{B} = \hat{y} B(x)$

odd function of x

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_o I_C$$

$$B(x)L + 0 - B(-x)L + 0 = \mu_o J_s L$$

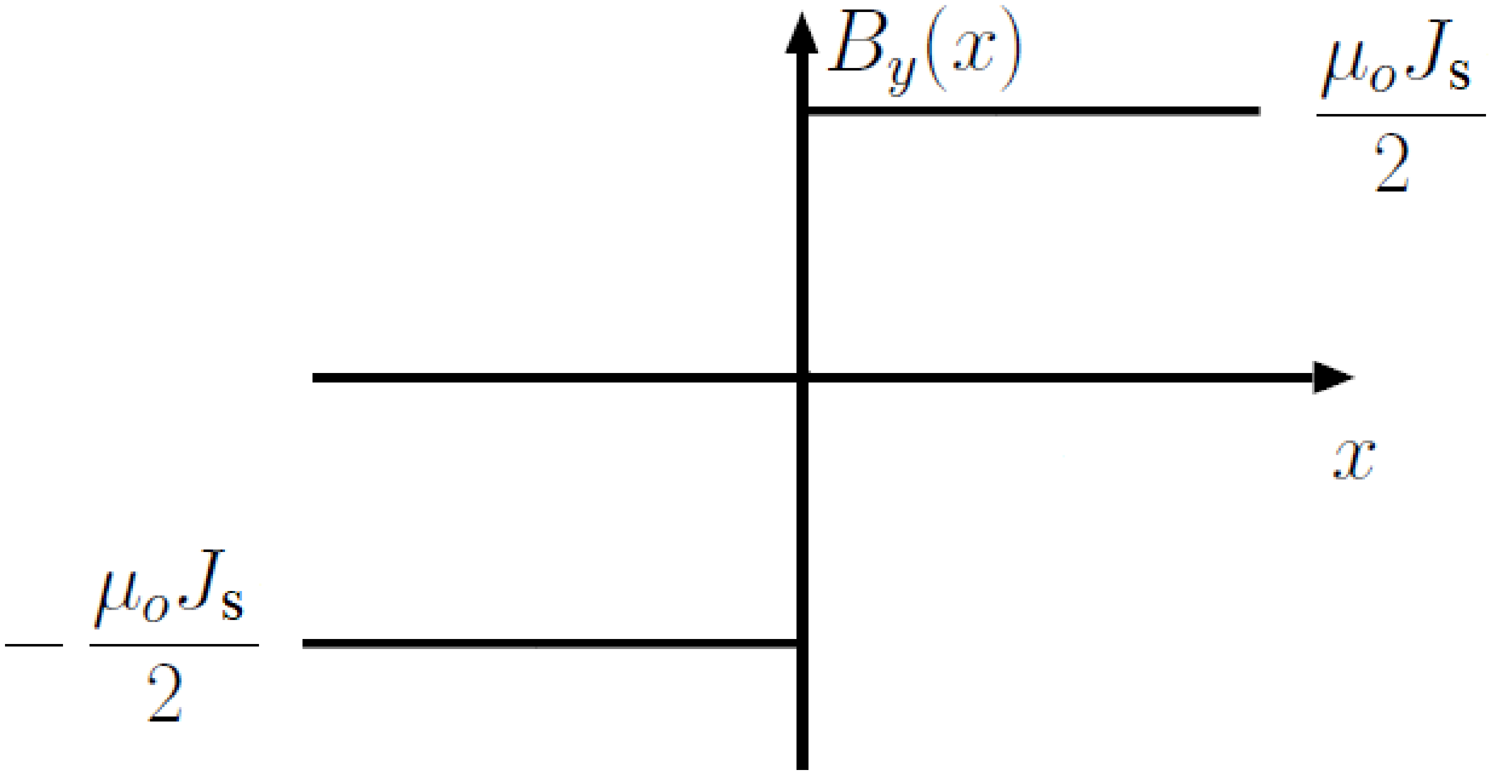
$$\boxed{B(x) = \frac{\mu_o J_s}{2}}$$

$$\boxed{\mathbf{B} = \hat{y}\frac{\mu_o J_s}{2}\mathrm{sgn}(x)}$$

$$\boxed{\mathbf{H} = \hat{y}\frac{J_s}{2}\mathrm{sgn}(x)}$$

Magnetic Field of infinite sheet at $x = 0$ with surface current density $\mathbf{J}_s = J_s\hat{z}\ \mathrm{A/m}$

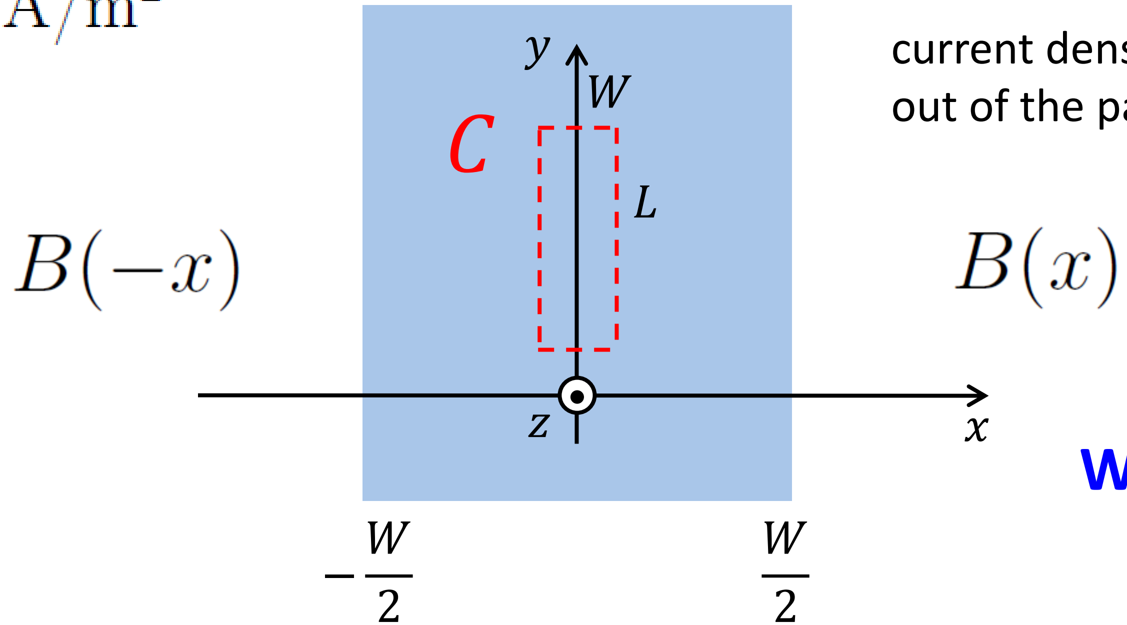
$$\boxed{\mathbf{B} = \hat{y}\frac{\mu_o J_s}{2}\mathrm{sgn}(x)} \quad \boxed{\mathbf{H} = \hat{y}\frac{J_s}{2}\mathrm{sgn}(x)}$$



Magnetic Field of infinite slab with uniform current density

$$\mathbf{J} = \hat{z} J_o \ \mathrm{A/m^2}$$

current density out of the page



We assume again

$$\mathbf{B} = \hat{y} B(x)$$

odd function of x

$$|x| < \frac{W}{2}$$

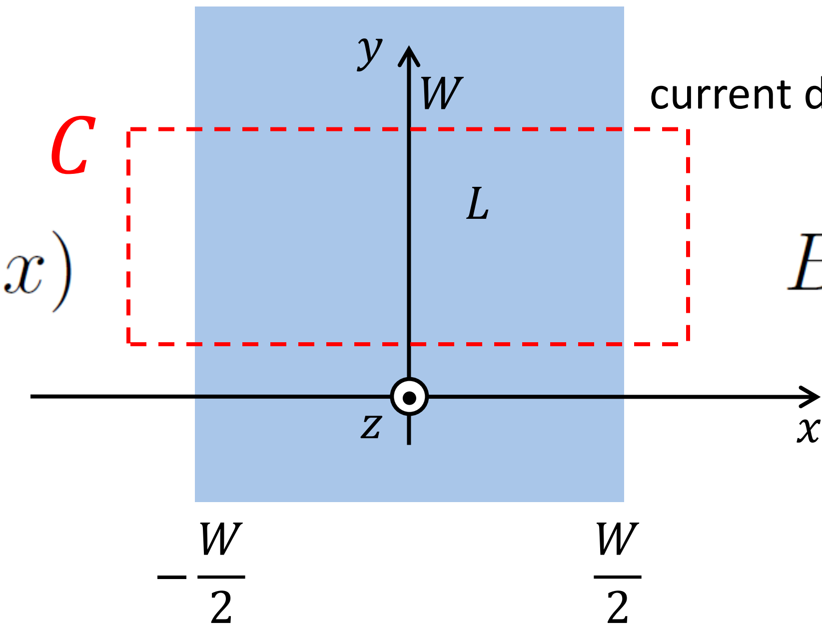
$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_o I_C$$

$$B(x)L + 0 - B(-x)L + 0 = \mu_o J_o 2xL$$

$$\boxed{B(x) = \mu_o J_o x}$$

Magnetic Field of infinite slab with uniform current density

$$\mathbf{J} = \hat{z} J_o \ \mathrm{A/m^2}$$



We assume again

$$\mathbf{B} = \hat{y} B(x)$$

odd function of x

$$|x| \geq \frac{W}{2}$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_o I_C$$

$$B(x)L + 0 - B(-x)L + 0 = \mu_o J_o W L$$

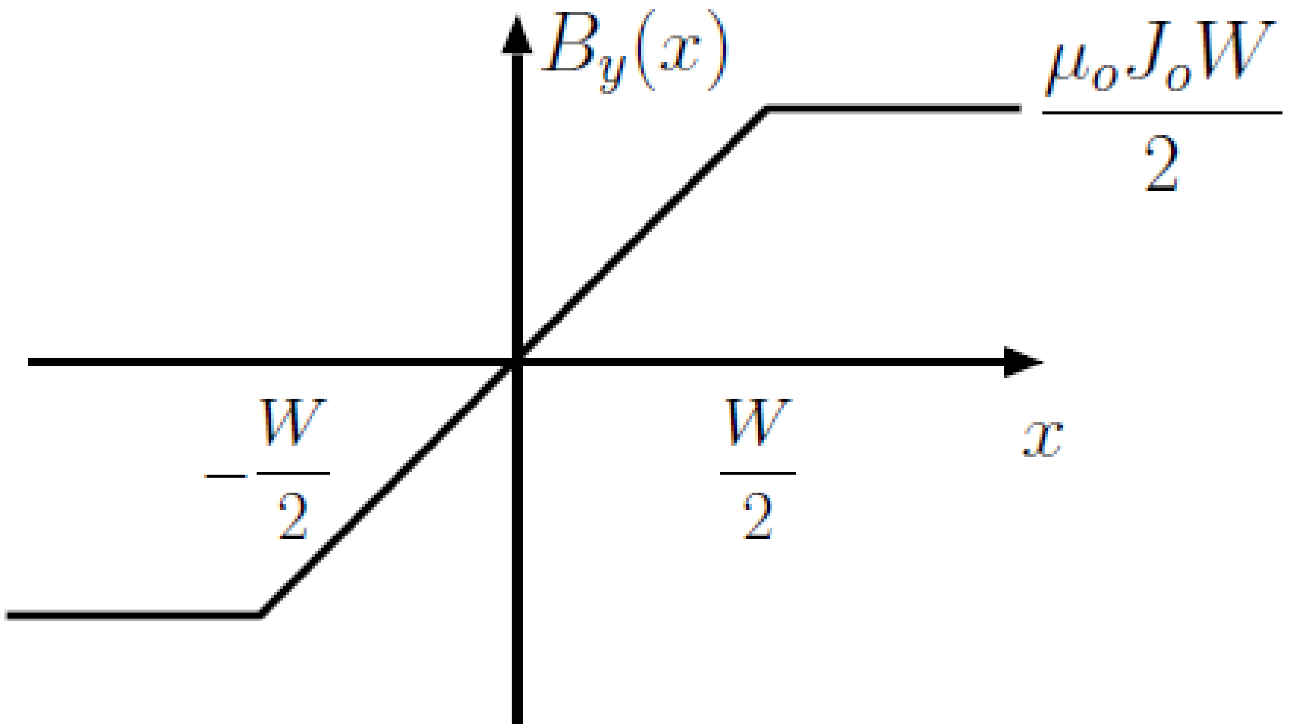
$$\boxed{B(x) = \mu_o J_o \frac{W}{2}}$$

Magnetic Field of infinite slab with uniform current density

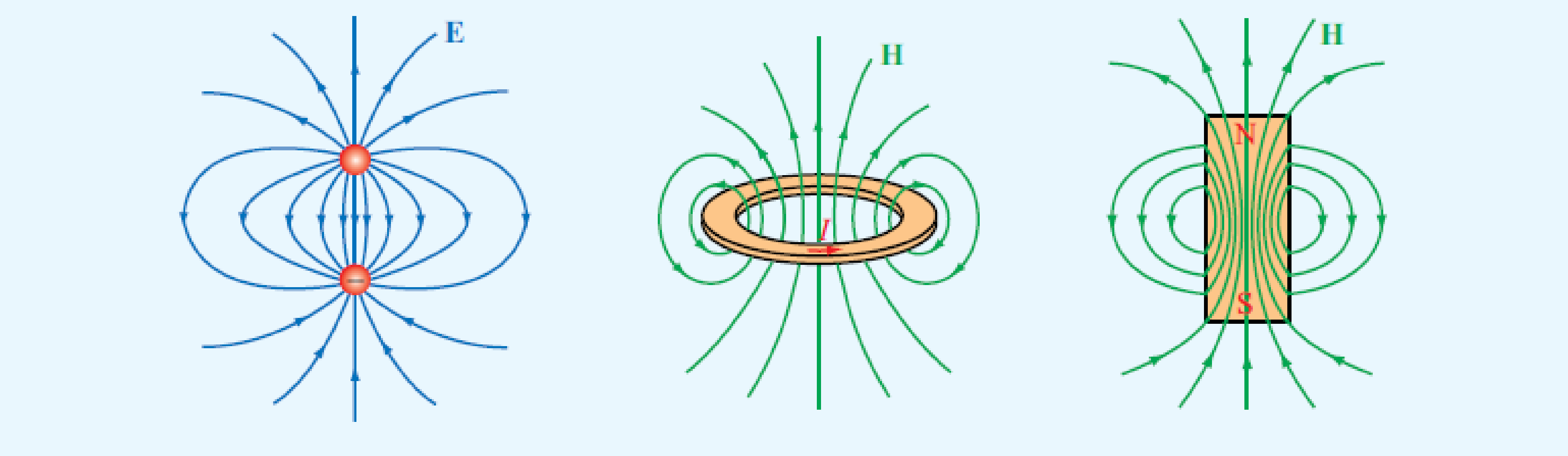
$$\mathbf{J} = \hat{z} J_o \ \mathrm{A/m^2}$$

$$\mathbf{B} = \begin{cases} \hat{y}\,\mu_o\,J_o x & |x| < \frac{W}{2} \\ \hat{y}\,\mu_o\,J_o \frac{W}{2}\,\mathrm{sgn}(x) & \text{otherwise} \end{cases}$$

$$\mathbf{H} = \begin{cases} \hat{y} J_o x & |x| < \frac{W}{2} \\ \hat{y} J_o \frac{W}{2}\,\mathrm{sgn}(x) & \text{otherwise} \end{cases}$$



Electric and Magnetic Dipoles



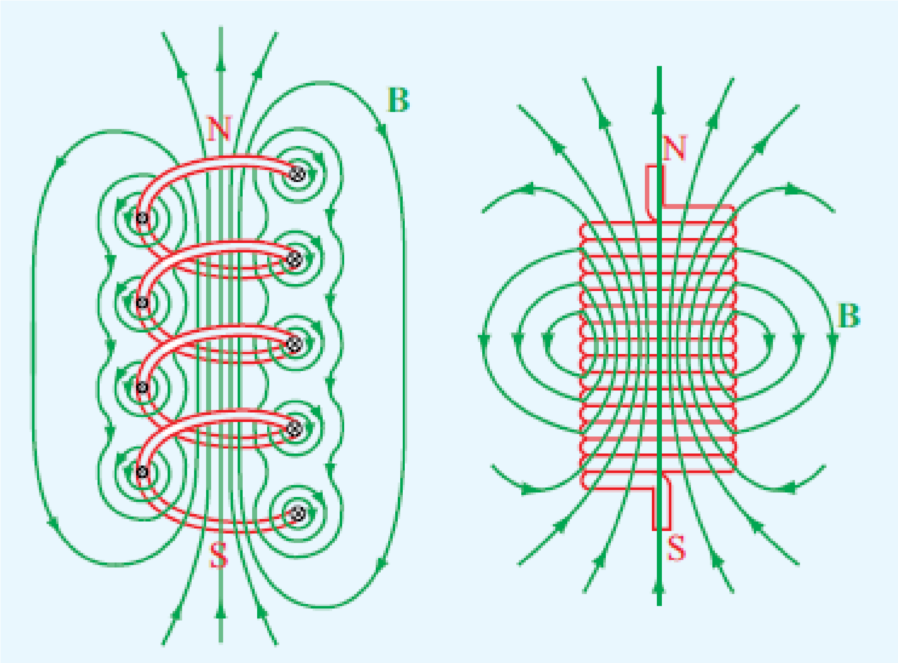
(a) Electric dipole (b) Magnetic dipole (c) Bar magnet

The hypothetical analogue of an electric point charge would be the "magnetic monopole" but elementary magnetic systems always come as "dipoles".

$\nabla \cdot \mathrm{B} = 0$ **does not have an official name but it could be called the "law of non existence of isolated monopoles".**

For analogy, it may be called "Gauss law of magnetism"

Solenoid



(a) Loosely wound solenoid
(b) Tightly wound solenoid

A tightly wound solenoid is a good way to approximate a DC surface current. If a current I is injected in the wire, it produces an effective surface current density

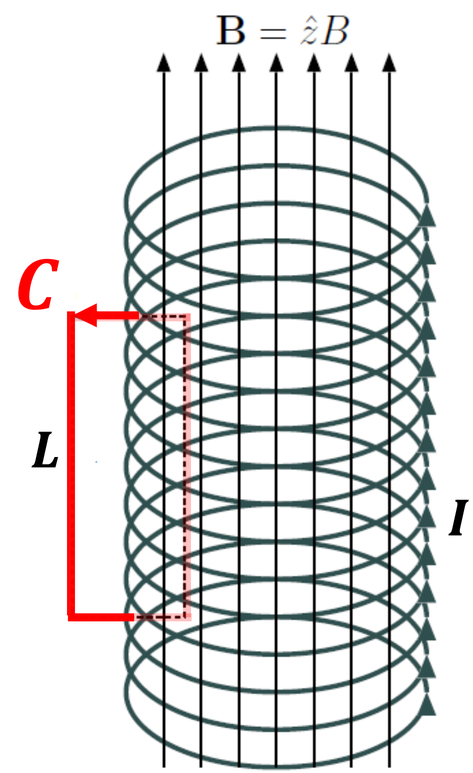
$$\mathbf{J}_s = IN\hat{\phi} \ \mathrm{A/m}$$

current loops per meter

Example – Determine the magnetic field of an infinite solenoid with N loops per unit length (counterclockwise as seen from the top), stacked in the z-direction, each carrying current I.

With tight winding, thin wires and infinite length we may assume:

- **$\mathbf{B} = 0$ outside the solenoid**
- **$\mathbf{B}$ independent of z inside the solenoid**



$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_o I_C \quad \Rightarrow \quad LB = \mu_o INL.$$

$$B = \mu_o IN$$

$$\mathbf{H} = \hat{z} IN$$

Magnetic Vector Potential

Static magnetic fields are divergence free and described by the equations

$$\begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J} \end{cases}$$

with

$$\mathbf{B} = \mu \mathbf{H}$$

$$\mu = \mu_r \mu_o$$

Since the divergence of a curl is zero, we can express mathematically a field $\mathrm{B}(x, y, z)$ **as**

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\boxed{\nabla \cdot \nabla \times \mathbf{A} = 0}$$

for any vector field $\mathbf{A} = \mathbf{A}(x, y, z)$ **which we call vector potential.**

The vector potential is somewhat analogous to the electrostatic potential V.

In electrostatics we can assign a reference $V = 0$ to any point which is convenient, to specify completely the problem.

In magnetostatics we can assign the divergence $\nabla \cdot \mathbf{A}$ to be a convenient scalar (as long as, in doing so, we respect the physical properties of the solution).

Coulomb "gauge" (choice)

The standard assignment of the vector potential divergence in magnetostatics is the Coulomb gauge:

$$\nabla \cdot \mathbf{A} = 0$$

With this choice

$$\nabla \times \mathrm{B} = \nabla \times \nabla \times \mathbf{A} = \nabla(\cancel{\nabla \cdot \mathbf{A}}) - \nabla^2 \mathbf{A} = -\nabla^2 \mathbf{A}$$

From Maxwell's equation $\nabla \times \mathbf{H} = \mathbf{J}$

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J}$$

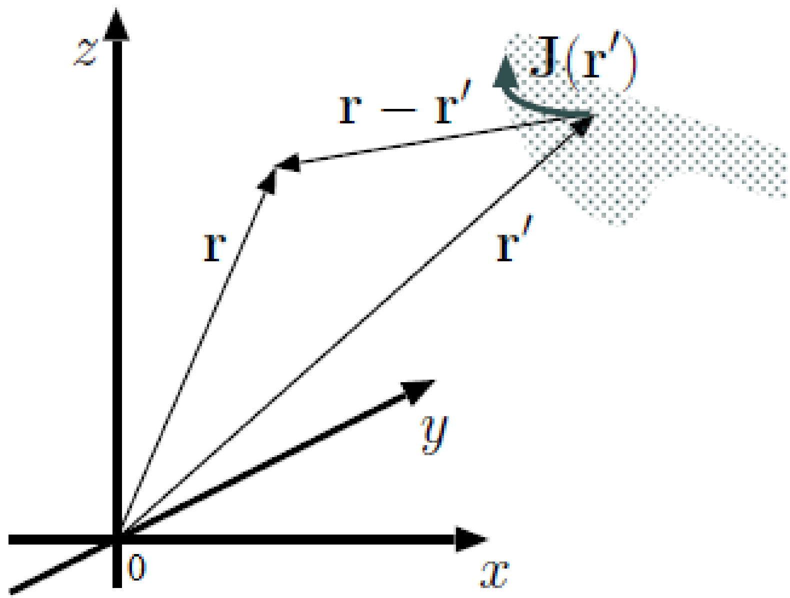
$$\boxed{\nabla^2 \mathbf{A} = -\mu_o \mathbf{J}}$$

magnetostatics version of Poisson's equation

General current density distribution

The integral approach to find the solution is similar to the one stated for the electrostatic potential in the case of a general charge distribution

$$\overbrace{\mathbf{A}(\mathbf{r})}^{\text{vector}} = \int \frac{\mu_o \overbrace{\mathbf{J}(\mathbf{r}')}^{\text{vector}}}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$



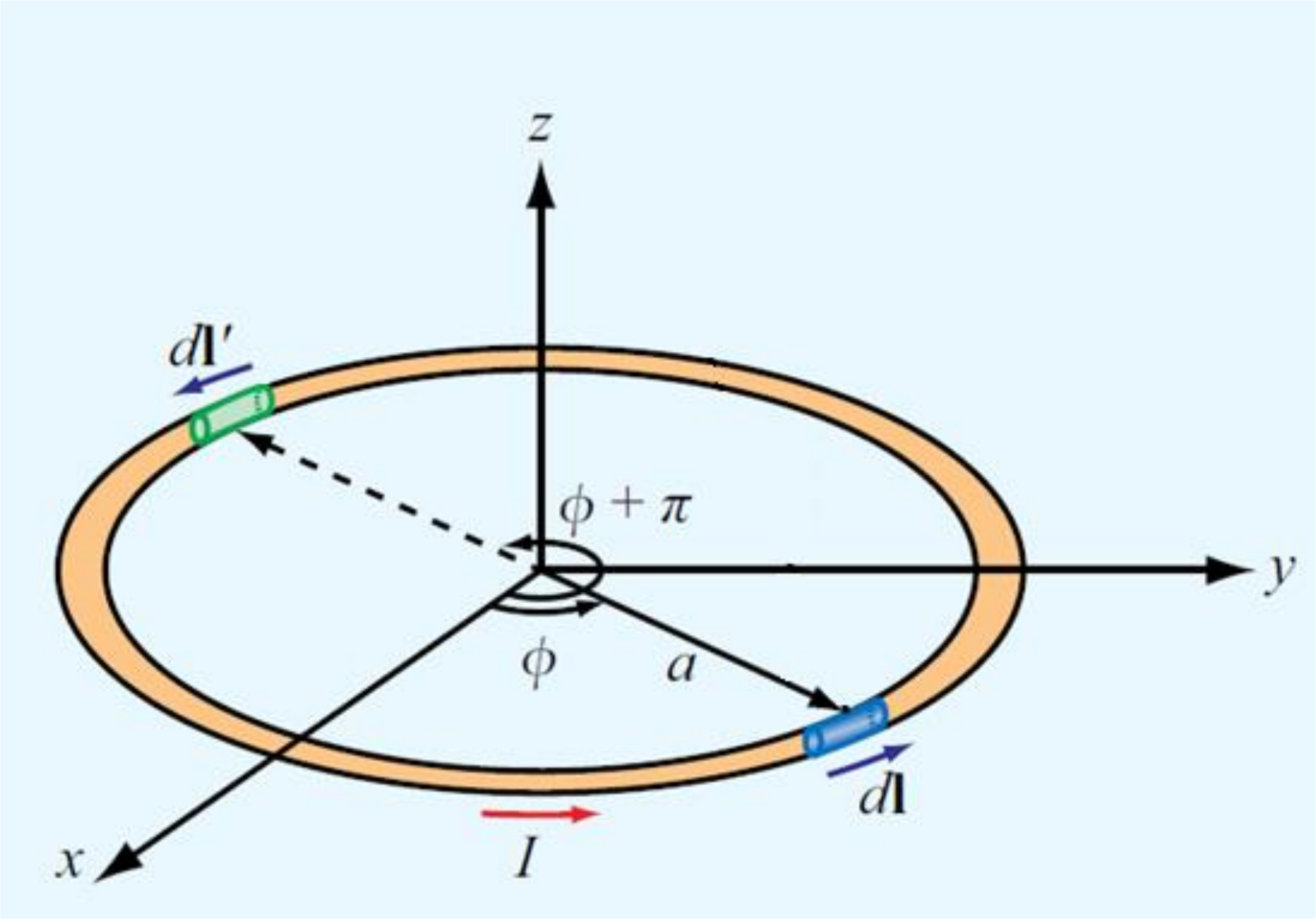
$$\boxed{\frac{\mu_o}{4\pi|\mathbf{r}-\mathbf{r}'|}}$$

Green's function for the magnetostatic problem

Once the vector potential is found, the magnetic induction is obtained by performing a curl operation

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Magnetic flux density (induction) of a single current loop



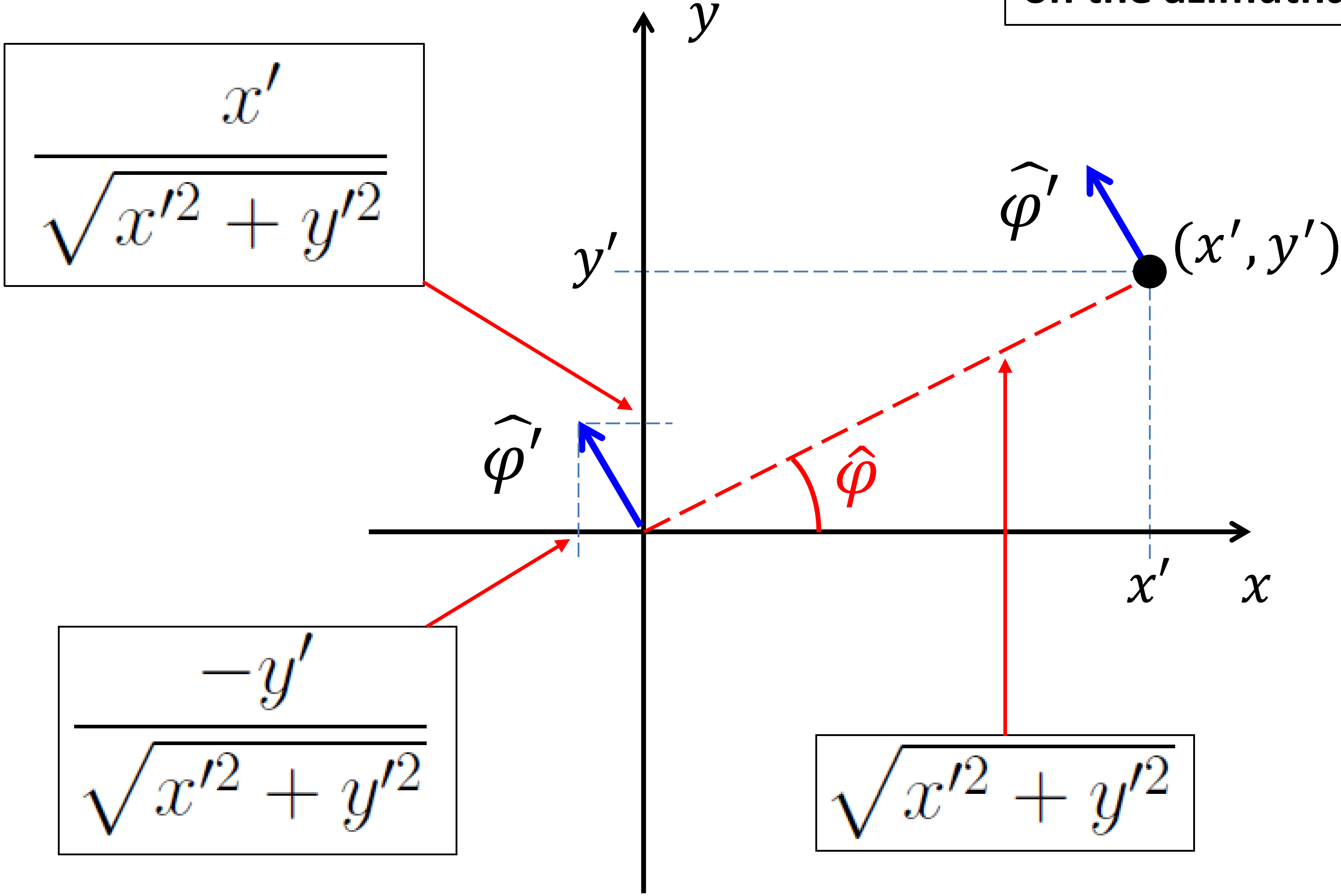
Magnetic flux density (magnetic induction) of a single current loop of radius a on the $z = 0$ plane

As a general approach we can express the current density as a volumetric distribution

$$\mathbf{J}(\mathbf{r}') = I \underbrace{\delta(z')}_{\text{delta function for } z=0} \underbrace{\delta\left(\sqrt{x'^2 + y'^2} - a\right)}_{\text{delta function for coordinates of a ring of radius } a} \underbrace{\frac{(-y', x', 0)}{\sqrt{x'^2 + y'^2}}}_{\text{unit vector for azimuthal direction } \widehat{\varphi}'}$$

We can insert this expression in the general integral solution for the vector potential to obtain the x and y components ($A_z = 0$) from which the vector $\mathbf{B}$ can be obtained by taking the curl.

on the azimuthal plane



$$\widehat{\varphi}' = \frac{(-y', x', 0)}{\sqrt{x'^2+y'^2}}$$

$$
\begin{aligned}
\mathbf{A}(\mathbf{r}) &= \frac{\mu_o I}{4\pi}\int \delta(\sqrt{x'^2+y'^2}-a)\frac{(-y',x',0)}{\sqrt{(x-x')^2+(y-y')^2+z^2}\sqrt{x'^2+y'^2}}dx'dy' \\
&= \frac{\mu_o I}{4\pi}\int \delta(r'-a)\frac{(-y',x',0)}{\sqrt{(x-x')^2+(y-y')^2+z^2}r'}r'dr'd\phi' \\
&= \frac{\mu_o I}{4\pi}\int_{-\pi}^{\pi}\frac{(-a\sin\phi',a\cos\phi',0)}{\sqrt{(x-a\cos\phi')^2+(y-a\sin\phi')^2+z^2}}d\phi' \equiv \hat{x}A_x(\mathbf{r})+\hat{y}A_y(\mathbf{r})
\end{aligned}
$$

The curl terms are, with $A_z = 0$

$$
B_x = -\frac{\partial A_y}{\partial z},\quad B_y = \frac{\partial A_x}{\partial z},\quad B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}
$$

There are no close solutions to the integral for general coordinates and integrals have to be solved numerically.

However, an analytical solution can be obtained for the axis of the loop where only $B_z \neq 0$

$$B_z = \frac{\mu_o a I}{4\pi}\int_{-\pi}^{\pi}\frac{a}{(a^2+z^2)^{3/2}}d\phi = \frac{\mu_o I a^2}{2(a^2+z^2)^{3/2}}$$

$$\boxed{|z| \gg a}$$

$$B_z \approx \frac{\mu_o I a^2}{2|z|^3}$$

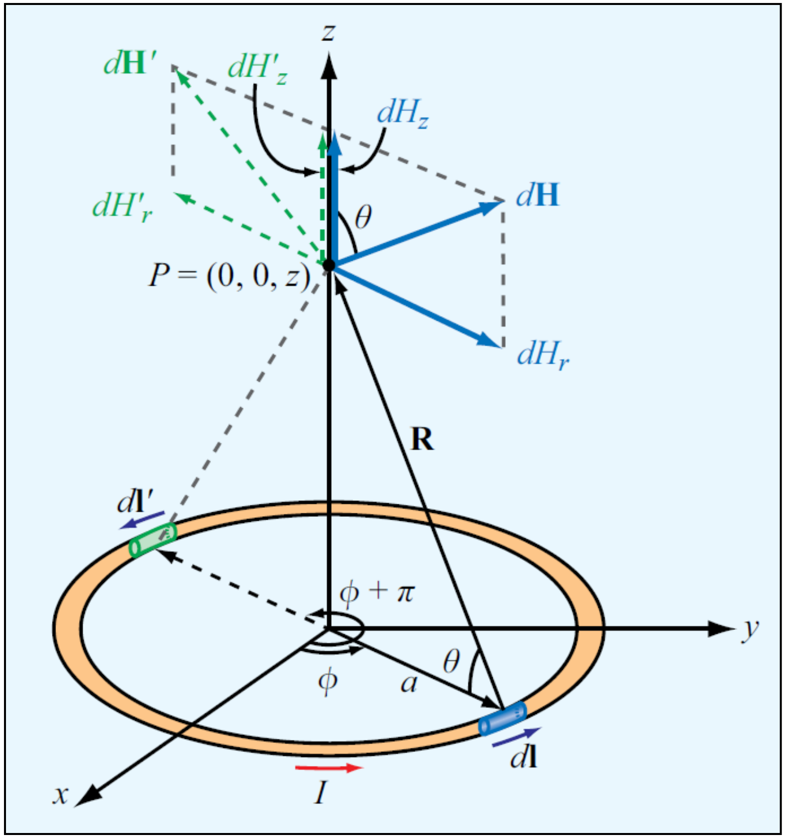
EXTRA – For this problem it is possible to perform integration of current elements along the ring to find field values along the axis (Biot-Savart Law)

$$R = \sqrt{a^2 + z^2}$$

$$dH = \frac{I\,dl}{4\pi(a^2 + z^2)}$$

Because of cancellation of components on the x-y plane, there is only H_z.

$$d\mathbf{H} = \hat{\mathbf{z}}\,\frac{I\cos\theta}{4\pi(a^2 + z^2)}\,dl$$



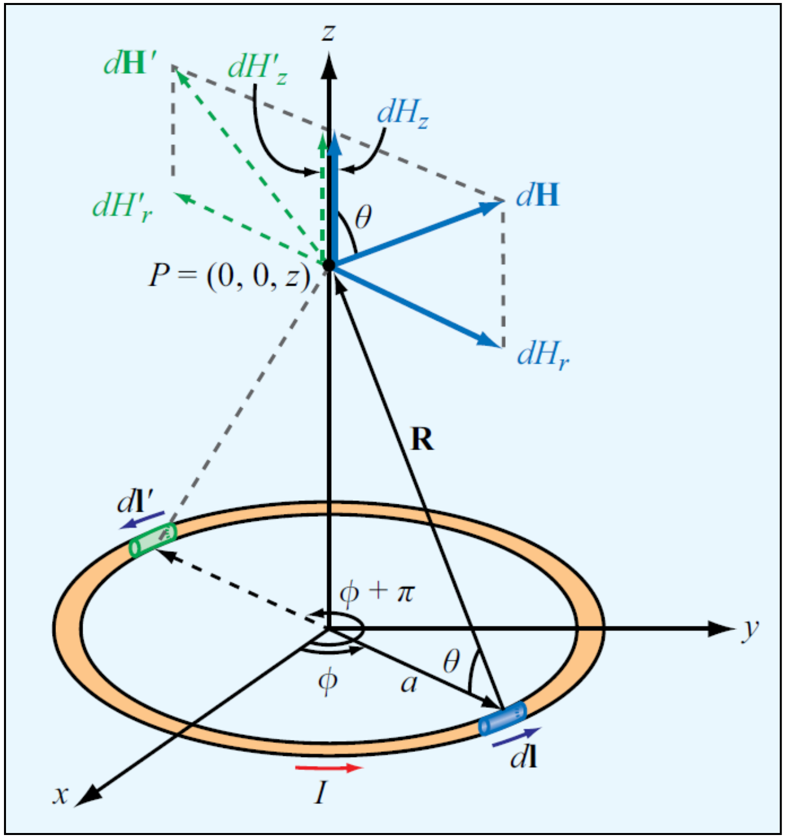
For a fixed point $P(0,0,z)$ on the axis of the loop

$$\mathbf{H} = \hat{\mathbf{z}}\, \frac{I\cos\theta}{4\pi(a^2+z^2)} \oint dl = \hat{\mathbf{z}}\, \frac{I\cos\theta}{4\pi(a^2+z^2)}(2\pi a)$$

With $\cos\theta = a/(a^2+z^2)^{1/2}$

$$\mathbf{H} = \hat{\mathbf{z}}\, \frac{Ia^2}{2(a^2+z^2)^{3/2}} \qquad \text{(A/m)}.$$

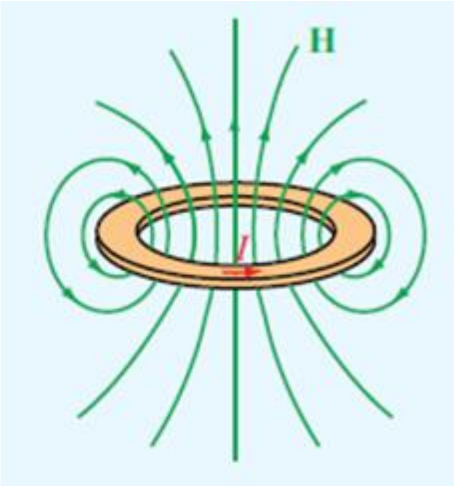
as found previously

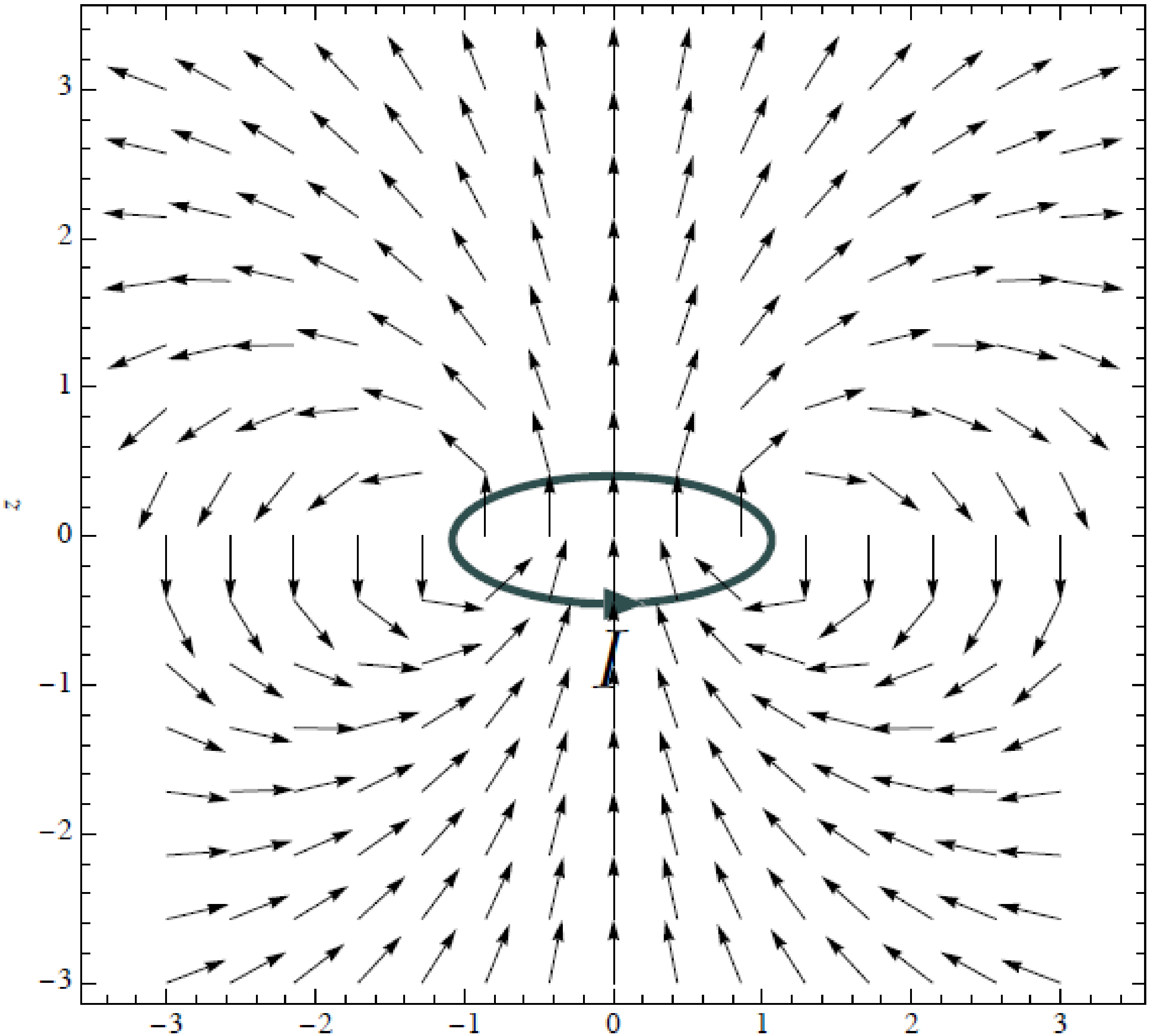


$\boxed{(z=0)}$ $\quad \mathbf{H} = \hat{\mathbf{z}}\, \dfrac{I}{2a}$

$\boxed{z^2 \gg a^2}$ $\quad \mathbf{H} = \hat{\mathbf{z}}\, \dfrac{Ia^2}{2|z|^3}$

Numerical solution





Numerical solution

$$\boxed{|z| \gg a}$$