ECE 329 – Fall 2022

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

Lecture 15

Lecture 15 – Outline

- **Electro-motive force (e.m.f.)**
- **Examples**

Reading assignment

Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:

15) Inductance – solenoid, shorted coax

Summary

Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{d}{dt} \overbrace{\int_S \mathbf{B} \cdot d\mathbf{S}}^{\text{magnetic flux } \Psi}$$

Electro-motive Force

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Psi}{dt}$$

The contour C may be moving or experiencing deformation (think motors, generators, …)

$$\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} + \oint_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}$$

$\mathbf{v}$ is the velocity of movement or deformation

Faraday's law takes the form

$$\mathcal{E} = \oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

An emf can be generated in a closed conducting loop under the following conditions:

1. **time-varying magnetic field linking a stationary loop (transformer emf)**

2. **moving loop with a time-varying area (relative to the normal component of B) in static field B (motional emf)**

3. **A moving loop in a field B which is time-varying (or inhomogeneous)**

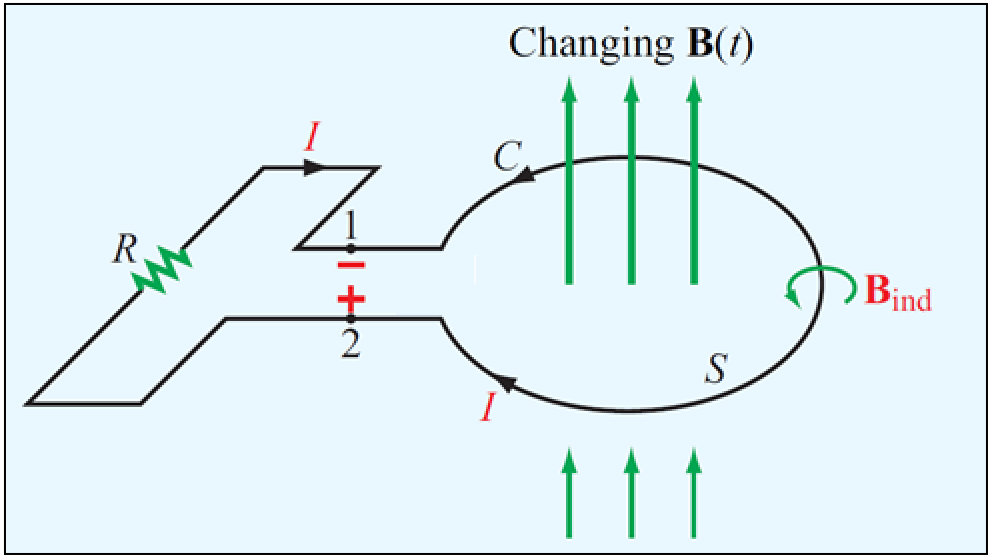
Lenz's Law

The polarity of the emf and hence the direction of the current are governed by Lenz's Law stating that the current in the loop is in a direction which opposes the change of magnetic flux producing the current.

This accounts for the negative sign on the right-hand side of Faraday's Law.

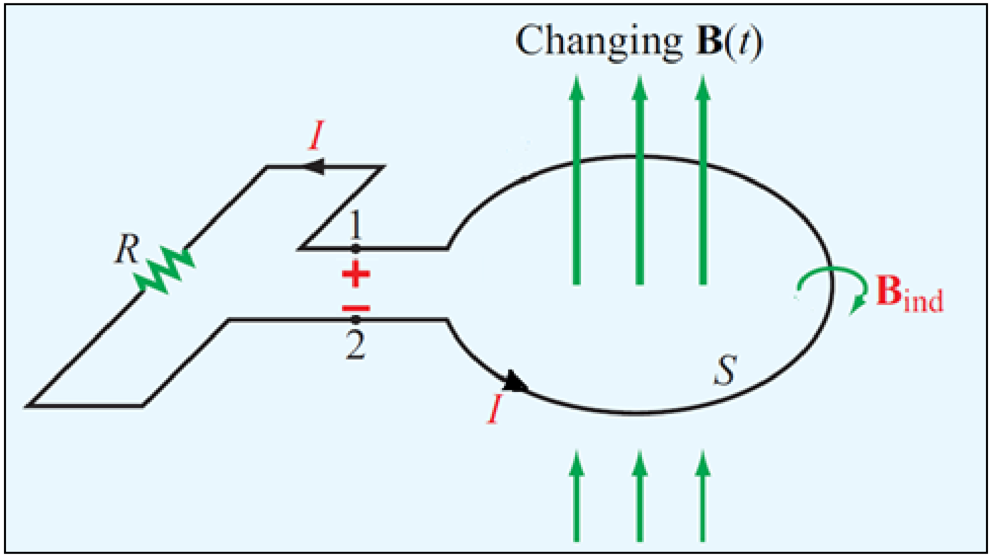
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Stationary loop in changing magnetic field



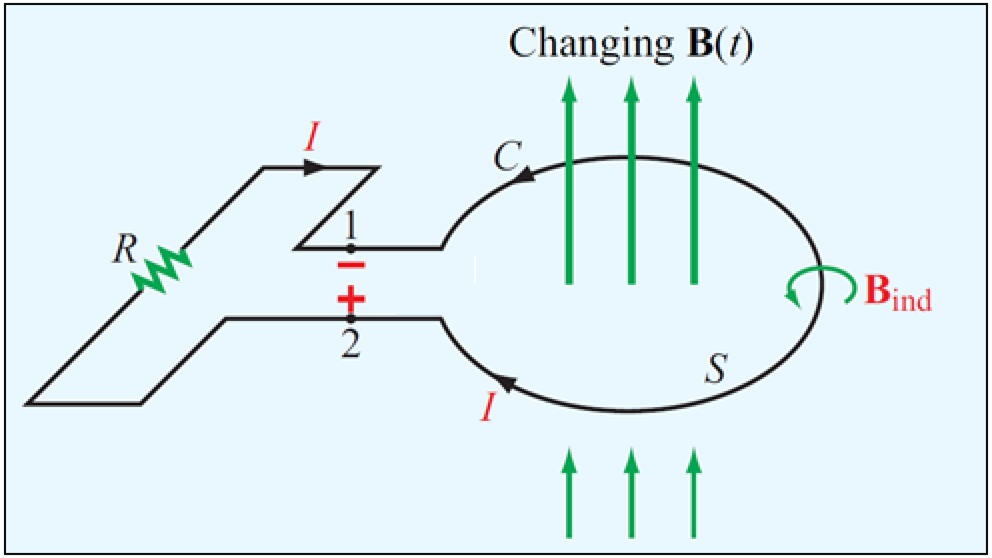
$$\frac{d\Psi}{dt} > 0$$

$$I = \frac{V_{\text{emf}}^{\text{tr}}}{R + R_{\text{i}}}$$



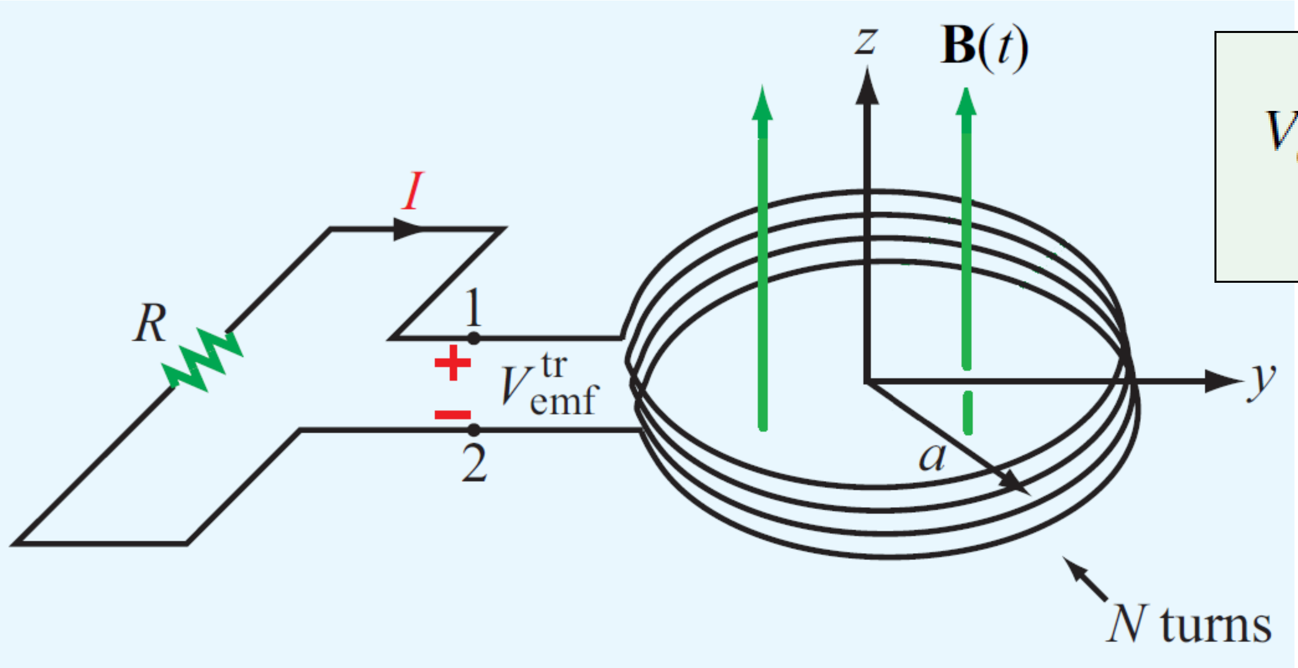
$$\frac{d\Psi}{dt} < 0$$

How could we magnify the effect?



How could we magnify the effect?

Use N turns instead of one



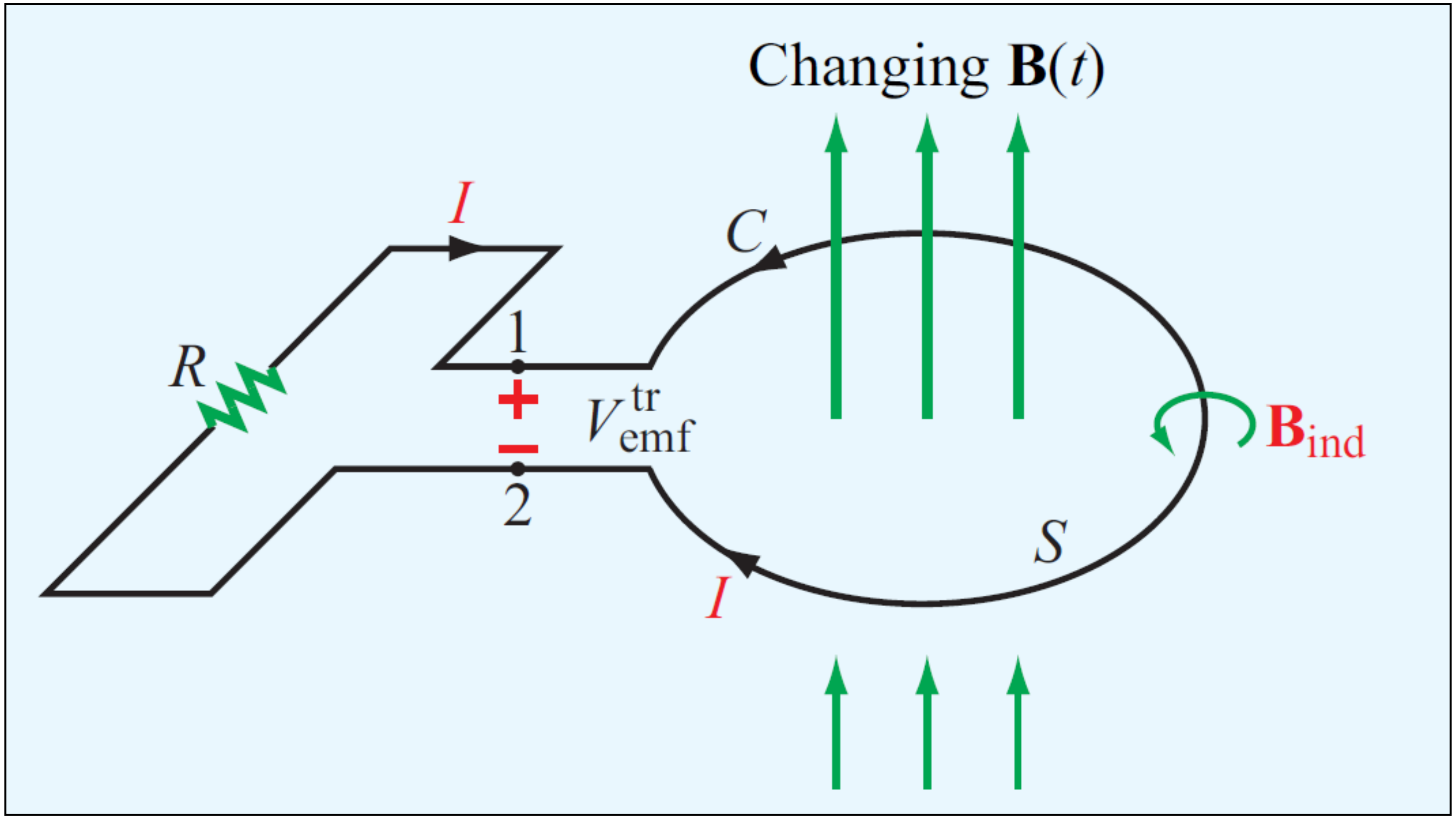
$$V_{emf}^{tr} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

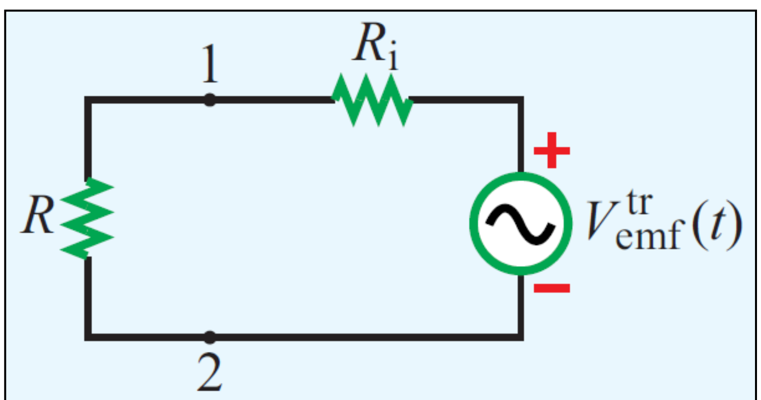
“Faraday” flashlight

magnet moves when shaking flashlight

many turns of thin copper wire

Practical implementation of stationary loop in changing magnetic field

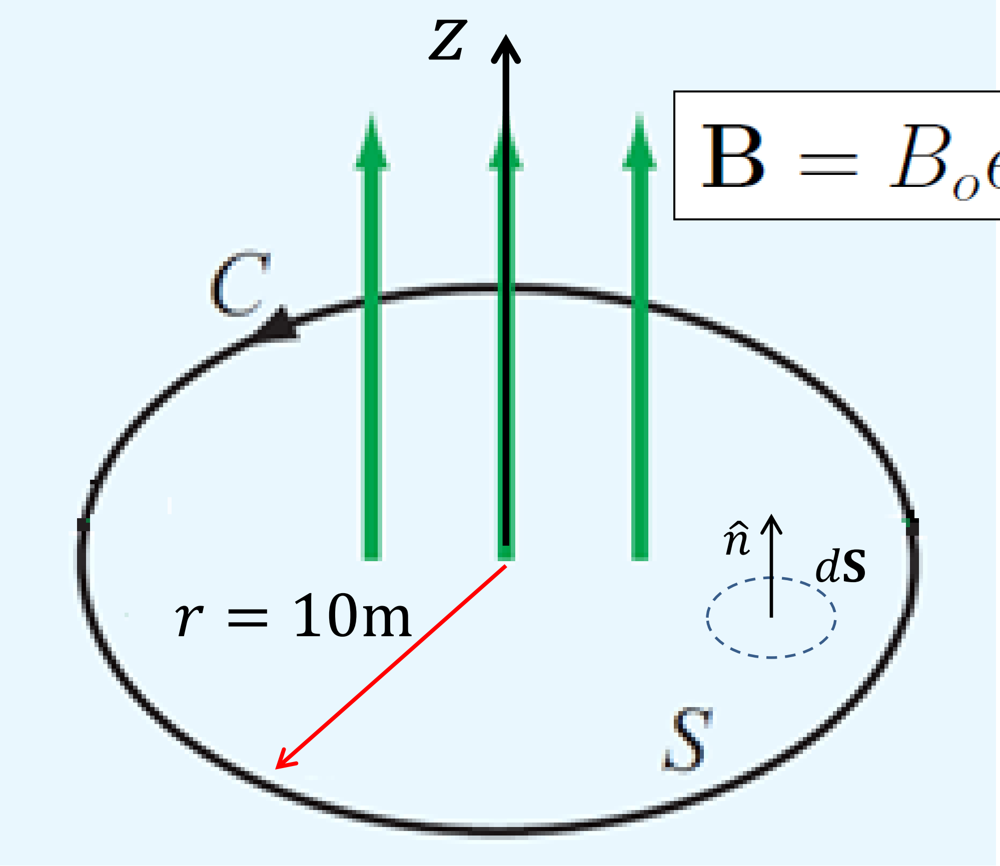




$$I = \frac{V_{\text{emf}}^{\text{tr}}}{R + R_i}$$

equivalent circuit

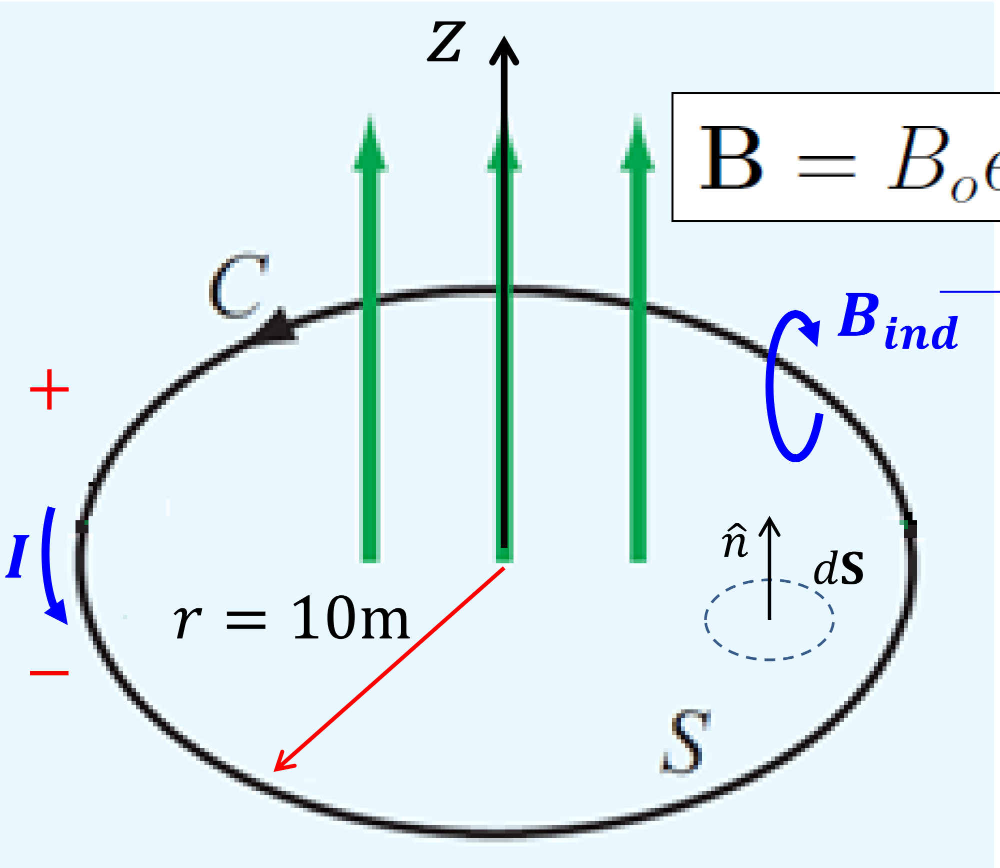
Example – Stationary circular loop of radius 10m with resistance R, counterclockwise circulation. Find emf and current.



$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \underbrace{(B_o e^{-t/\tau}\,\hat{z})}_{\text{Constant Field}} \cdot \underbrace{(\pi 10^2 \hat{z})}_{\text{Area}} = \pi 10^2 B_o e^{-t/\tau}$$

emf $$\mathcal{E} = -\frac{d\Psi}{dt} = \pi 10^2 \frac{B_o}{\tau} e^{-t/\tau}$$

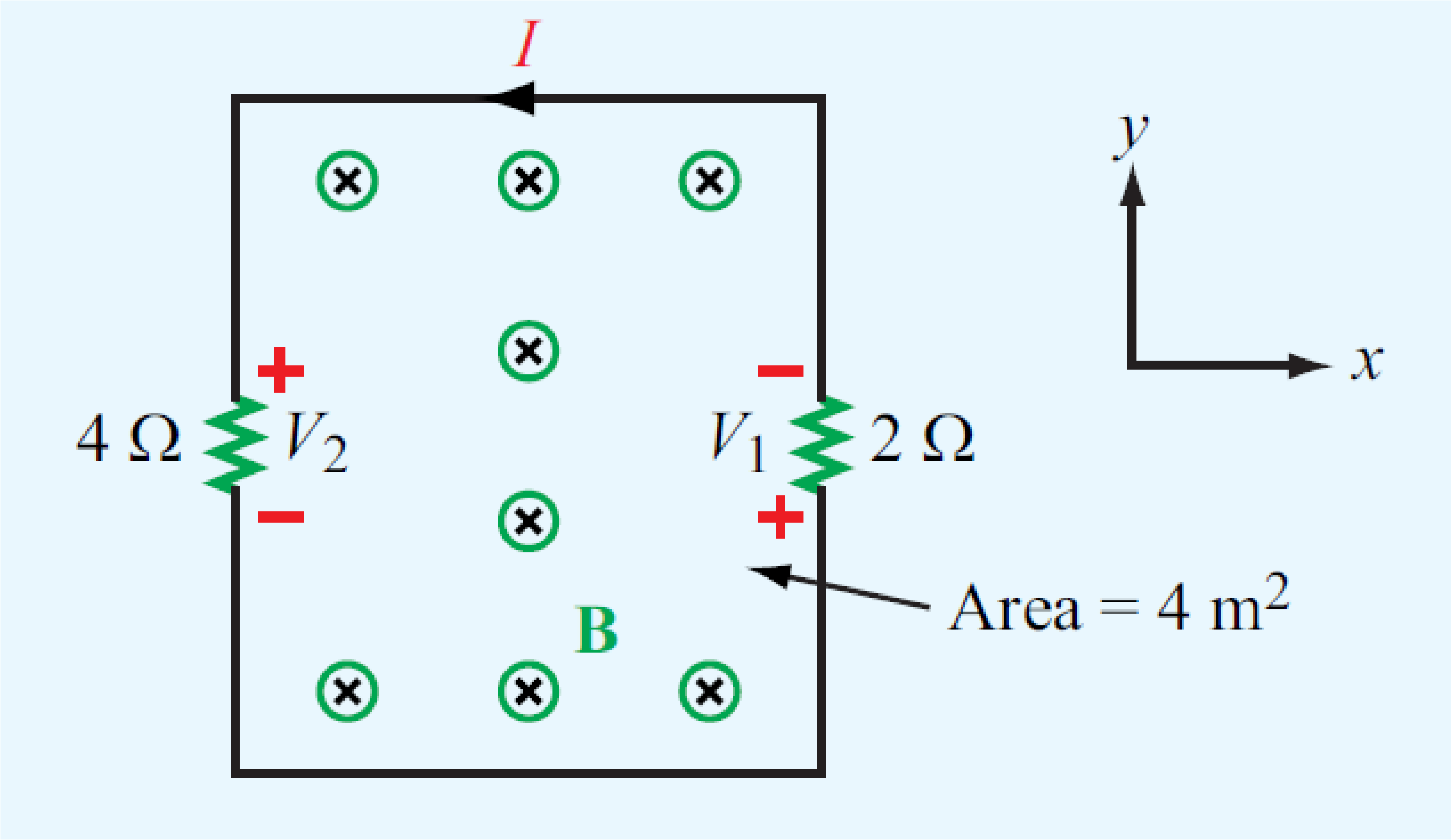
$$\boldsymbol{I} = \frac{\mathcal{E}}{\boldsymbol{R}}$$



strengthens the decaying field with negative derivative of total concatenated flux

Example – Find V_1 and V_2 across resistors (ignore resistance of wires)

$$\mathbf{B} = -\hat{\mathbf{z}}0.3t \ (\mathrm{T})$$



Tesla

$$\mathrm{T} = \frac{\mathrm{V \cdot s}}{\mathrm{m^2}} = \frac{\mathrm{N \cdot s}}{\mathrm{C \cdot m}} = \frac{\mathrm{N}}{\mathrm{A \cdot m}} = \frac{\mathrm{J}}{\mathrm{A \cdot m^2}} = \frac{\mathrm{Wb}}{\mathrm{m^2}} = \frac{\mathrm{kg}}{\mathrm{A \cdot s^2}}$$

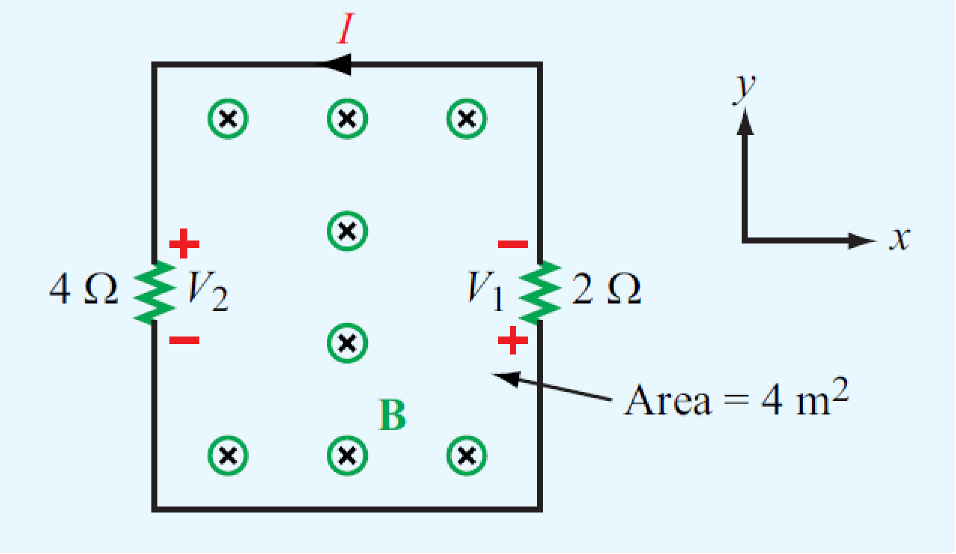
$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S (-\hat{\mathbf{z}}0.3t) \cdot \hat{\mathbf{z}}\, ds$$

$$= -0.3t \times 4 = -1.2t \qquad \text{(Wb)}$$

Weber (Wb) is the unit of magnetic flux = $(\mathbf{kg\ m^2})/(\mathbf{s^2 A})$

and the corresponding emf is

$$\mathcal{E} = -\frac{d\Psi}{dt} = 1.2 \text{ V}$$

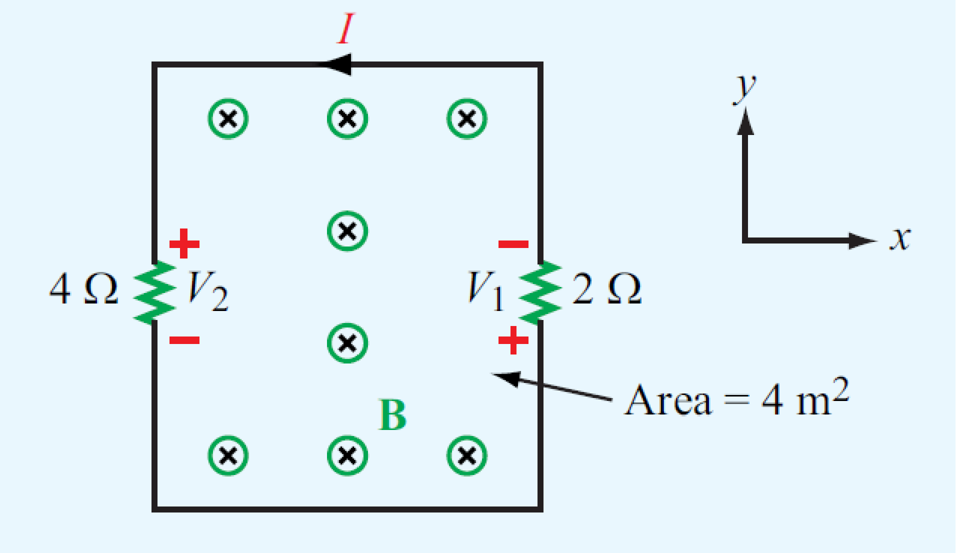


The magnetic flux through the loop is along $-z$ (into the screen) and it increases in magnitude with time t.

Lenz's law states that the induced current I should be directed so that the magnetic flux $\mathbf{B}_{\text{ind}}$ it induces counteracts the direction of change of Ψ. Therefore, the induced current flows as in the figure.

$$I = \frac{\mathcal{E}}{R_1 + R_2} = \frac{1.2}{2+4} = 0.2 \text{ A}$$

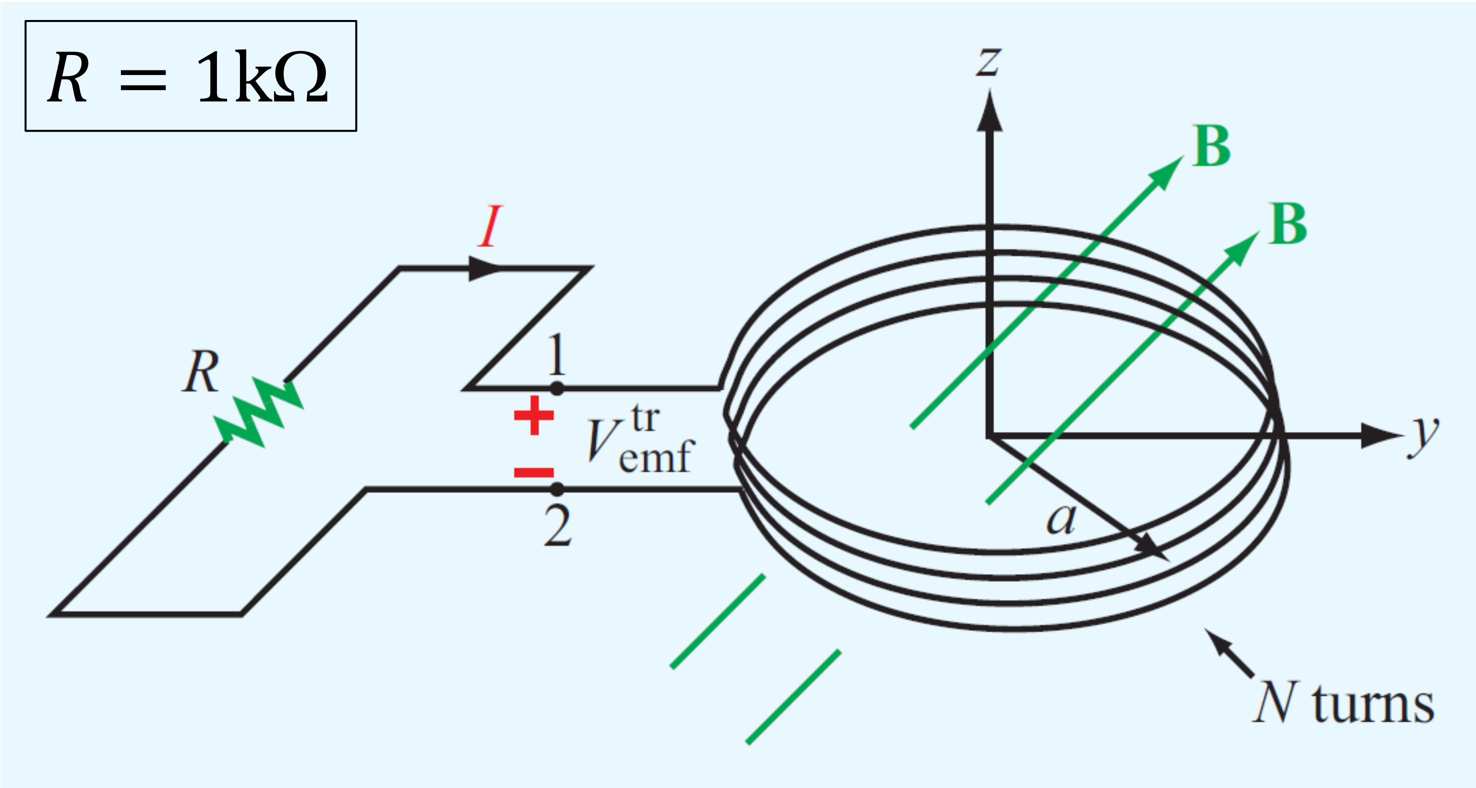
$$V_1 = I R_1 = 0.2 \times 2 = 0.4 \text{ V},$$
$$V_2 = I R_2 = 0.2 \times 4 = 0.8 \text{ V}.$$



Example – Stationary circular loop consisting of N turns of thin wire with concatenated induction field

$$\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t$$

$\boxed{N = 10}$ $\boxed{B_0 = 0.2\ \mathrm{T}}$ $\boxed{a = 10\ \mathrm{cm}}$ $\boxed{\omega = 10^3\ \mathrm{rad/s}}$



Magnetic flux concatenated with each turn

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S [B_0(\hat{\mathbf{y}}\,2 + \hat{\mathbf{z}}\,3)\sin\omega t] \cdot \hat{\mathbf{z}}\, ds = 3\pi a^2 B_0 \sin\omega t$$

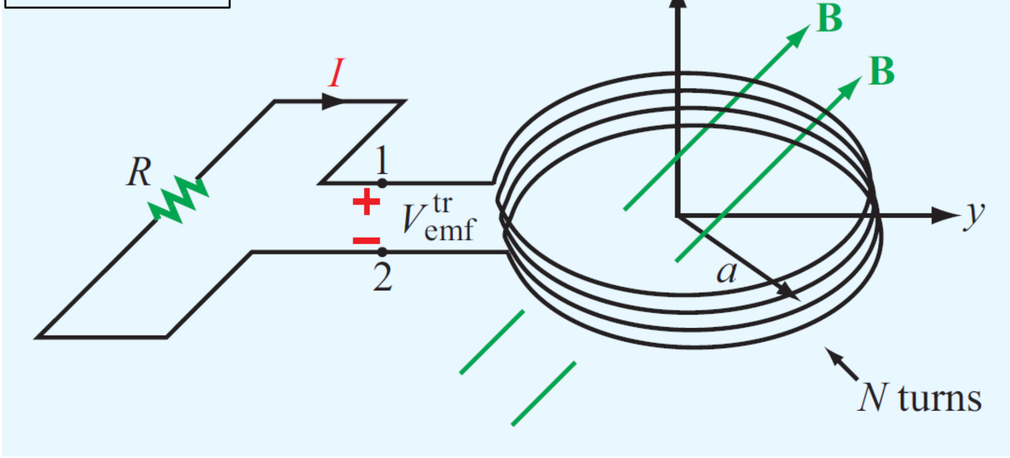
only z-component contributes to flux

Transformer emf

$$\mathcal{E} = V_{\text{emf}}^{\text{tr}} = -N\,\frac{d\Psi}{dt}$$

$$= -\frac{d}{dt}(3\pi N a^2 B_0 \sin\omega t) = -3\pi N\omega a^2 B_0 \cos\omega t$$

$R = 1\text{k}\Omega$



$N = 10$ $B_0 = 0.2\ \text{T}$

$a = 10\ \text{cm}$ $\omega = 10^3\ \text{rad/s}$

$$V_{\text{emf}}^{\text{tr}} = -188.5\cos 10^3 t\ \text{(V)}$$

At $t = 0$

$$\frac{d\Psi}{dt} = 3\pi N\omega a^2 B_0 > 0$$

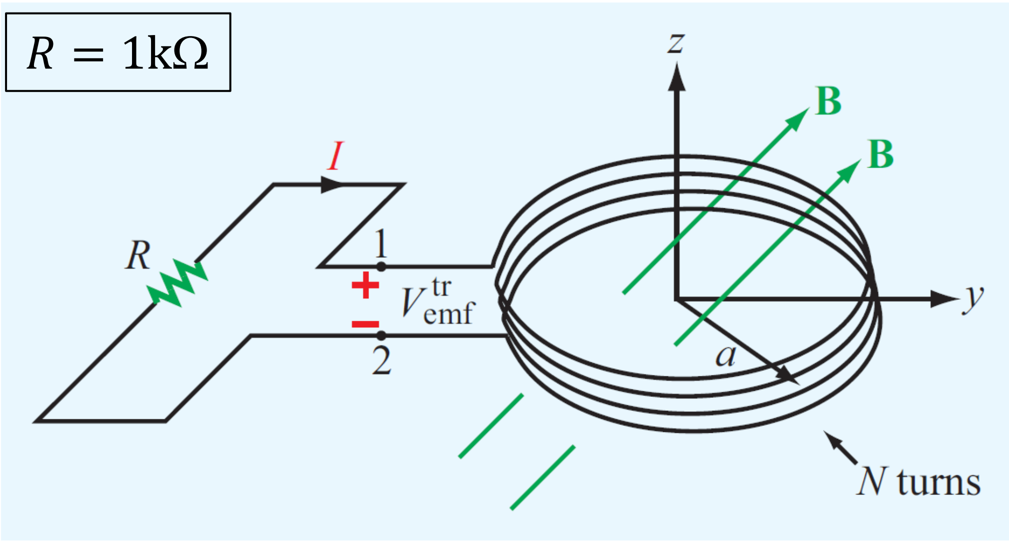
$$\boxed{V_{\text{emf}}^{\text{tr}} = -188.5 \ \ (\text{V})}$$

Flux is increasing, so induced current is as in the figure, to satisfy Lenz's law. Point 2 is at higher potential than point 1 to support the current. For the potential reference shown

$$V_{\text{emf}}^{\text{tr}} = V_1 - V_2 = -188.5 \ (\text{V})$$

The induced current is

$$I = \frac{V_2 - V_1}{R} = \frac{188.5}{10^3} \cos 10^3 t = 0.19 \cos 10^3 t \ \ (\text{A})$$



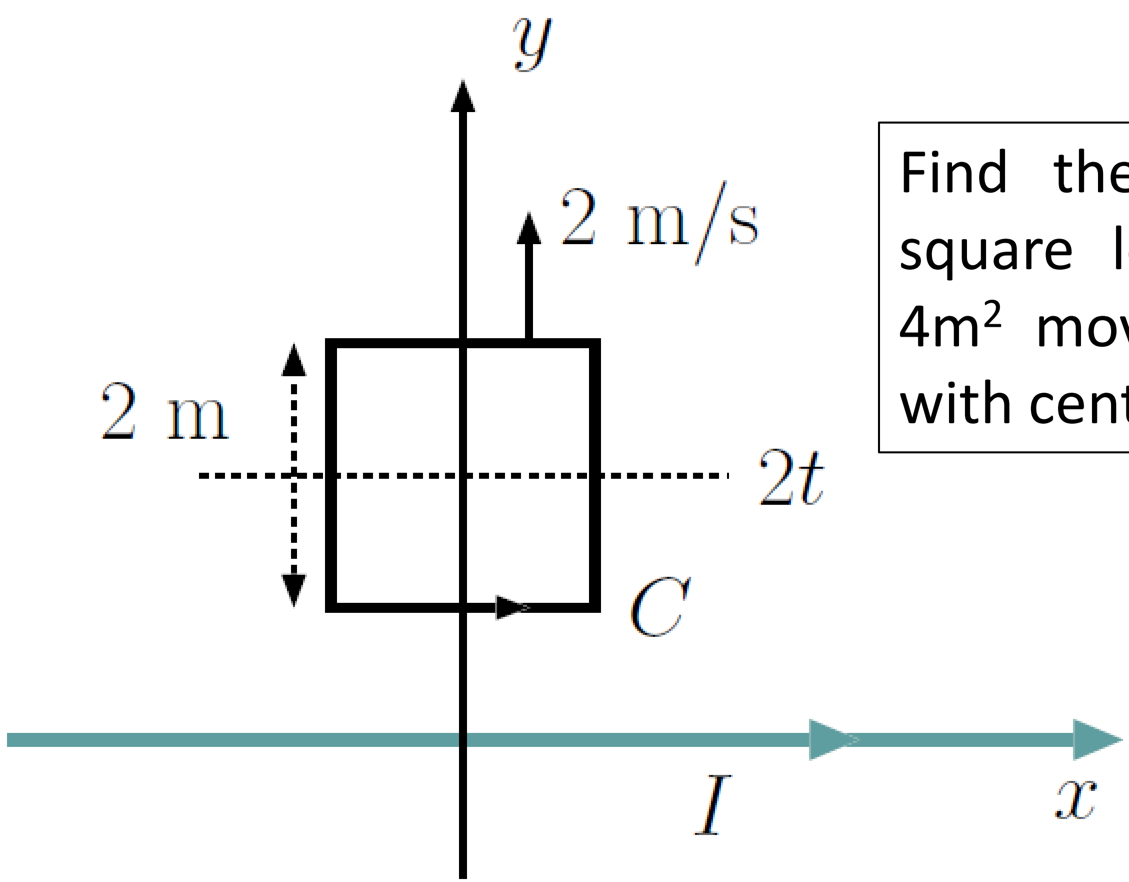
$\boxed{N = 10}$ $\boxed{B_0 = 0.2 \ \text{T}}$

$\boxed{a = 10 \ \text{cm}}$ $\boxed{\omega = 10^3 \ \text{rad/s}}$

$$V_{\text{emf}}^{\text{tr}} = -188.5 \cos 10^3 t \ \ (\text{V})$$

Example – Magnetic flux density is produced by current flowing along the *x*-axis.

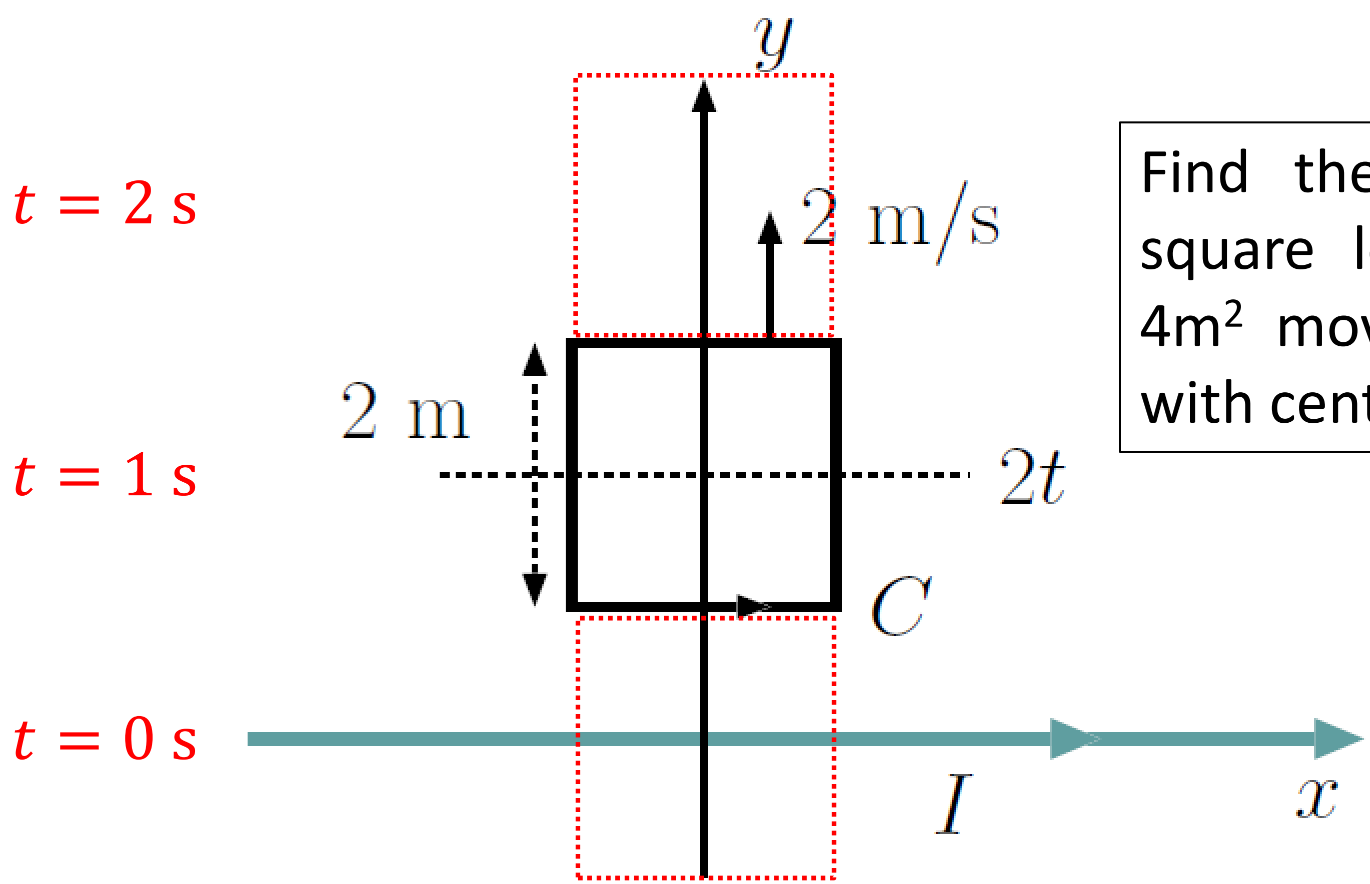
$$\mathrm{B} = \frac{\mu_o I}{2\pi r}\hat{\phi}$$



Find the emf on a square loop of area 4m^2 moving along y with center at $y = 2t$

Example – Magnetic flux density is produced by current flowing along the x-axis.

$$\mathbf{B} = \frac{\mu_o I}{2\pi r}\hat{\phi}$$



Find the emf on a square loop of area 4m^2 moving along y with center at $y = 2t$

Compute the flux through the area at any given instant t

$$\boxed{\mathrm{B} = \frac{\mu_o I}{2\pi r}\hat{\phi}}$$

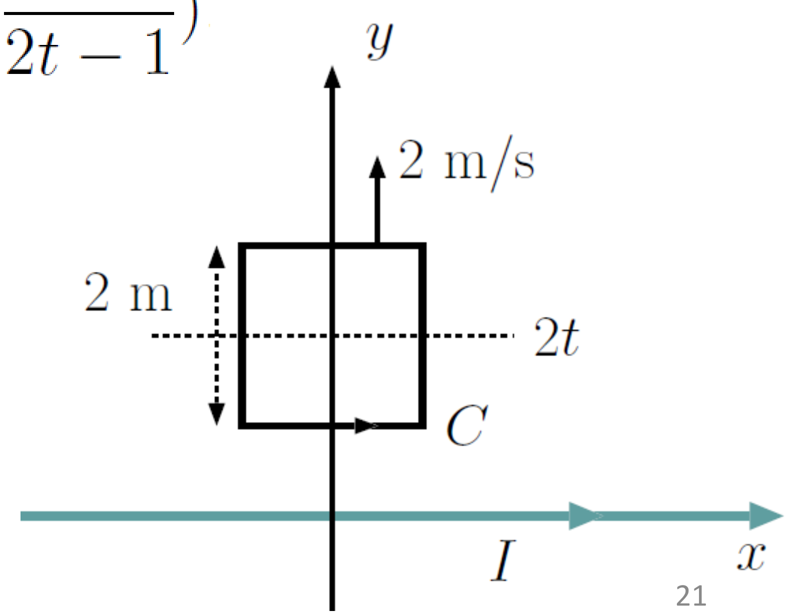
B is uniform along x and varies as $1/y$ along y.

$$\mathcal{E} = \boxed{-\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}} = \oint_C (\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}$$

$$\Psi(t) = \int_{-1}^{1} dx \int_{2t-1}^{2t+1} dy \frac{\mu_o I}{2\pi y} = \frac{\mu_o I}{\pi}\ln(\frac{2t+1}{2t-1})$$

$$\mathcal{E} = -\frac{d\Psi}{dt} = -\frac{\mu_o I}{\pi}(\frac{2t-1}{2t+1})\frac{\partial}{\partial t}(\frac{2t+1}{2t-1})$$

$$= \frac{\mu_o I}{\pi}\frac{4}{(2t+1)(2t-1)} = \frac{\mu_o I}{\pi(t^2 - \frac{1}{4})}$$



Review your basic calculus and algebra from time to time

$$\frac{d}{dt}\ln\frac{2t+1}{2t-1} = \frac{2t-1}{2t+1}\,\frac{d}{dt}\left(\frac{2t+1}{2t-1}\right)$$

$$= \frac{2t-1}{2t+1}\,\frac{(2t-1)2-(2t+1)2}{(2t-1)^2}$$

$$= -\frac{4}{(2t+1)(2t-1)}$$

Alternatively

$$\boxed{\mathbf{B} = \frac{\mu_o I}{2\pi r}\hat{\phi}}$$

B is uniform along x and varies as $1/y$ along y.

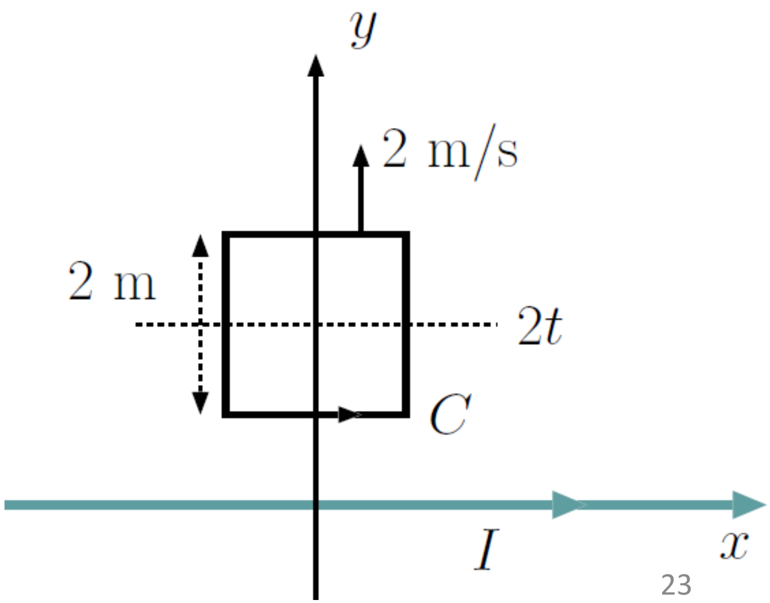
$$\mathcal{E} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} = \boxed{\oint_C (\cancel{\mathbf{E}} + \mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}}$$

$$\mathbf{v} = 2\hat{y}\ \mathrm{m/s} \qquad \mathbf{v}\times\mathbf{B} = 2\frac{\mu_o I}{2\pi r}\hat{x} \qquad d\mathbf{l} = \pm\hat{x}dx \pm\hat{y}dy$$

$$\mathcal{E} = \oint_C (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}$$

$$= 2\frac{\mu_o I}{2\pi(2t-1)}2 - 2\frac{\mu_o I}{2\pi(2t+1)}2$$

$$= \frac{\mu_o I}{\pi(t^2 - \frac{1}{4})}$$



We have defined the magnetic flux as

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(Wb)}$$

For a single loop carrying a current I, if we assume a linear relationship the ratio between flux and current is a constant. The ratio is called *self-inductance*

$$L = \frac{\Psi}{I}$$

For a solenoid with n loops, the self-inductance is

$$L \equiv \frac{n\overbrace{\Psi}^{\text{flux of a single loop}}}{I}$$

To generalize, the self-inductance is the ratio

$$L = \frac{\text{total concatened magnetic flux}}{\text{current}}$$

For a **long solenoid** **aligned along the z-axis, we can assume constant interior magnetic induction density**

$$\mathbf{B} = \mu_o I \underset{\uparrow}{N} \hat{z}$$

loops per unit length

If the solenoid has length ℓ and cross-sectional area A

$$L = \frac{n\Psi}{I} = \frac{\overbrace{N\ell}^{\text{total number of loops } n}(\mu_o I N)A}{I} = \overbrace{N^2\mu_o A\ell}^{\text{inductance depends only on geometry and material}}$$

We have also seen that variation of the flux induces *emf*

$$\mathcal{E} = -\frac{d\Psi}{dt}$$

For a solenoid with n loops

$$\mathcal{E} = -\frac{d}{dt}n\overbrace{\Psi}^{\text{flux of a single loop}} \equiv -L\frac{dI}{dt}$$

$$\mathcal{E} = -\underbrace{N^2\mu_o A\ell}_{L}\frac{dI}{dt} = -\underbrace{\frac{n^2}{\ell}\mu_o A}_{L}\frac{dI}{dt}$$

If the solenoid has resistance R and is conducting a current I then we can write

$$\mathcal{E} = -L\frac{dI}{dt}$$

$$\mathcal{E} = RI$$

$$\Rightarrow \quad \underbrace{-L\frac{dI}{dt}}_{\text{emf}} = \underbrace{RI}_{\text{voltage drop}}$$

So, the following differential equation describes the behavior of a solenoid circuit

$$\boxed{\frac{dI}{dt} + \frac{R}{L}I = 0}$$

A solenoid inductor, connected to an external circuit with a "quasi-static" (size << smallest wavelength in the signal) current I, develops a voltage drop across its terminals

$$V = L\frac{dI}{dt}$$

The instantaneous power absorbed is

$$P = VI = L\frac{dI}{dt}I = \frac{d}{dt}(\frac{1}{2}LI^2)$$

The stored energy is

$$W = \frac{1}{2}LI^2 = \frac{1}{2}N^2\mu_o A\ell I^2 = \frac{|B_z|^2}{2\mu_o}A\ell = \frac{1}{2}\mu_o|H_z|^2 A\ell$$

$$W = \frac{1}{2}\underbrace{\mu_o|H_z|^2}_{\text{stored magnetic energy per unit volume } w}\underbrace{A\ell}_{\text{volume}}$$

In general

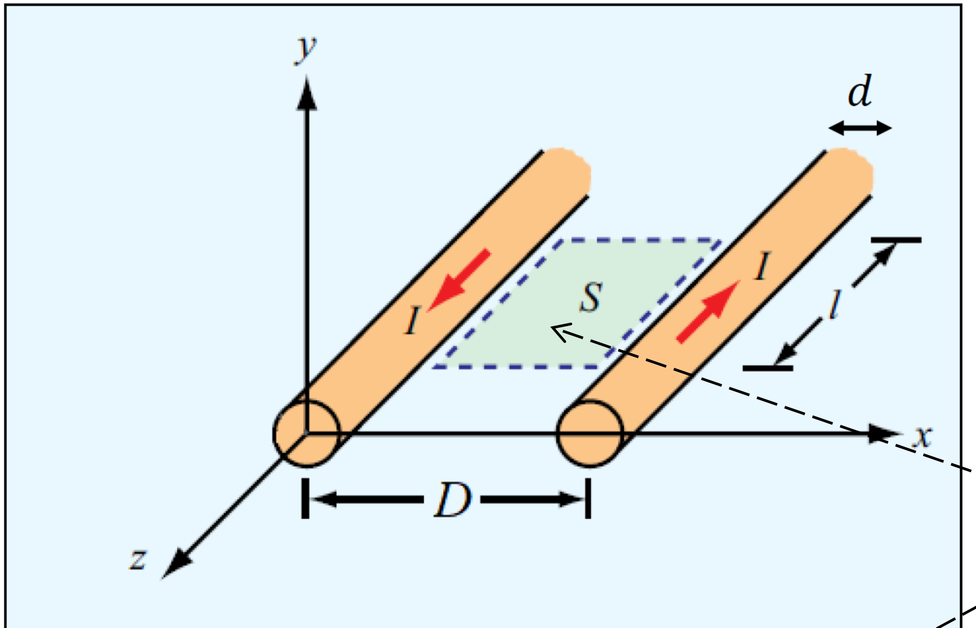
$$w = \frac{1}{2}\mu_o \mathbf{H}\cdot\mathbf{H}$$

For a material medium, we will see later that

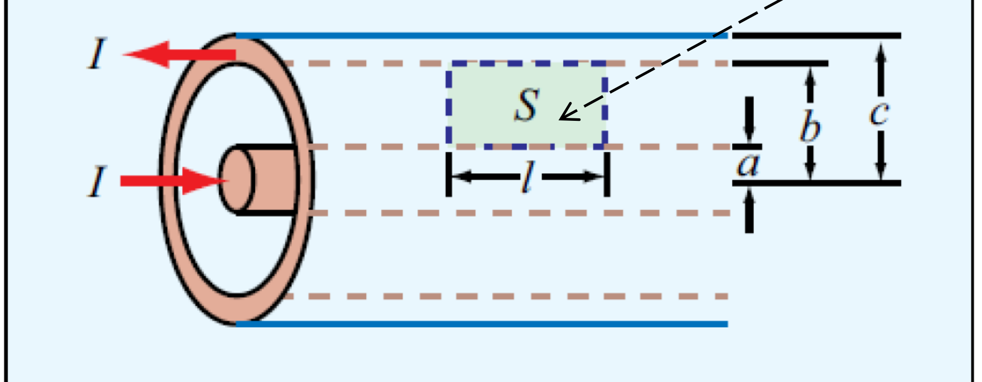
$$\mu = (1+\chi_m)\mu_o$$

magnetic susceptibility

Structures consisting of separate conductors can also couple with magnetic fields and exhibit inductance



(a) Parallel-wire transmission line

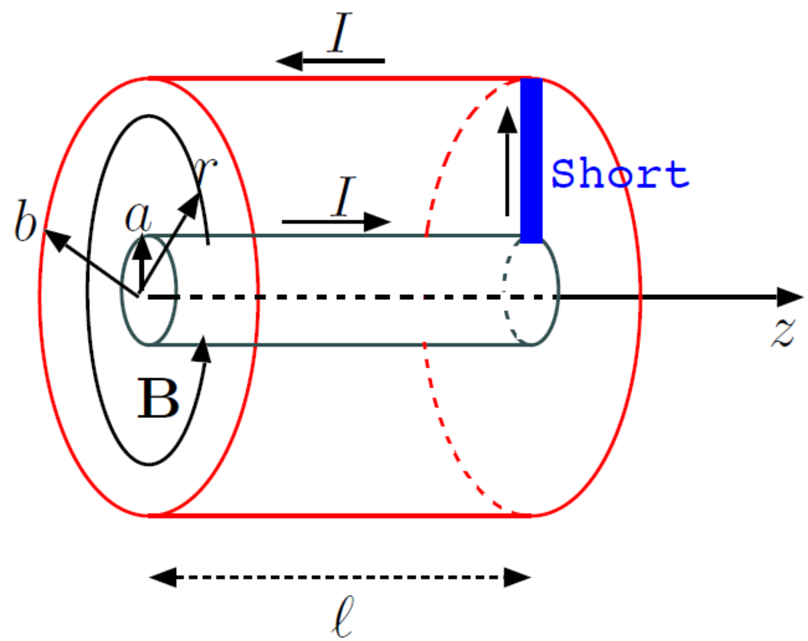


(b) Coaxial transmission line

To compute the inductance per unit length of a two-conductor transmission line, we need to determine the magnetic flux through the area *S* between the conductors.

Inductance and capacitance per unit length will be extremely important for the study of transmission lines based on configurations with multiple conductors.

Inductance of shorted coaxial cable

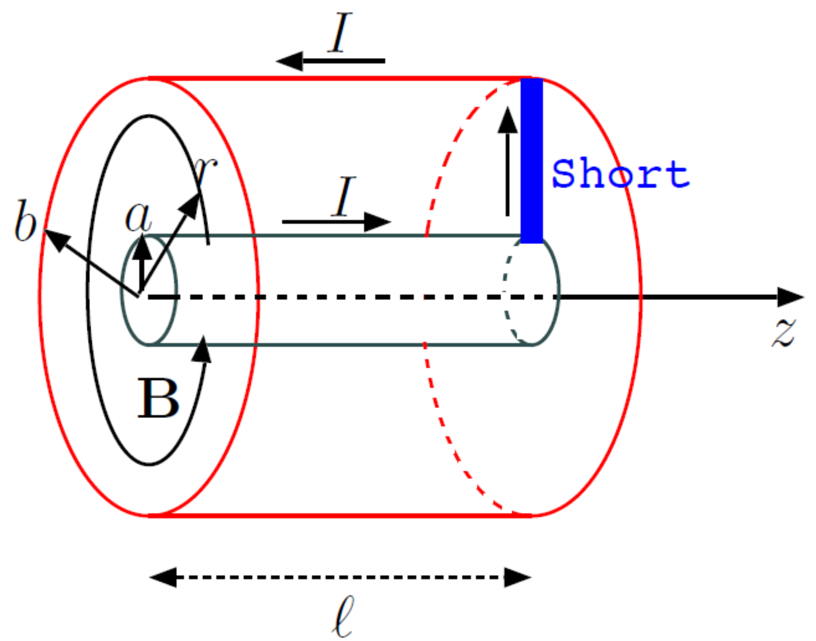


$$\ell \gg b$$

approximation of field behavior as in an infinite coaxial cable is acceptable

The magnetic field wraps around the inner conductor and is constant for a certain radius r at any azimuthal angle

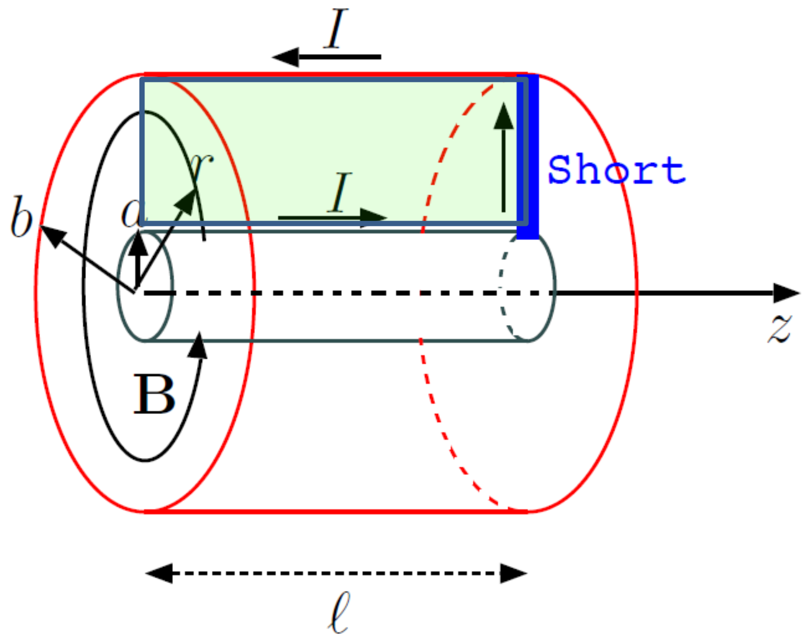
Invoking Ampere's Law



$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_o I_C$$

$\boxed{r > a}$ $\quad B_\phi 2\pi r = \mu_o I \Rightarrow B_\phi = \dfrac{\mu_o I}{2\pi r}$

From the Magnetic Flux linked with a closed current path

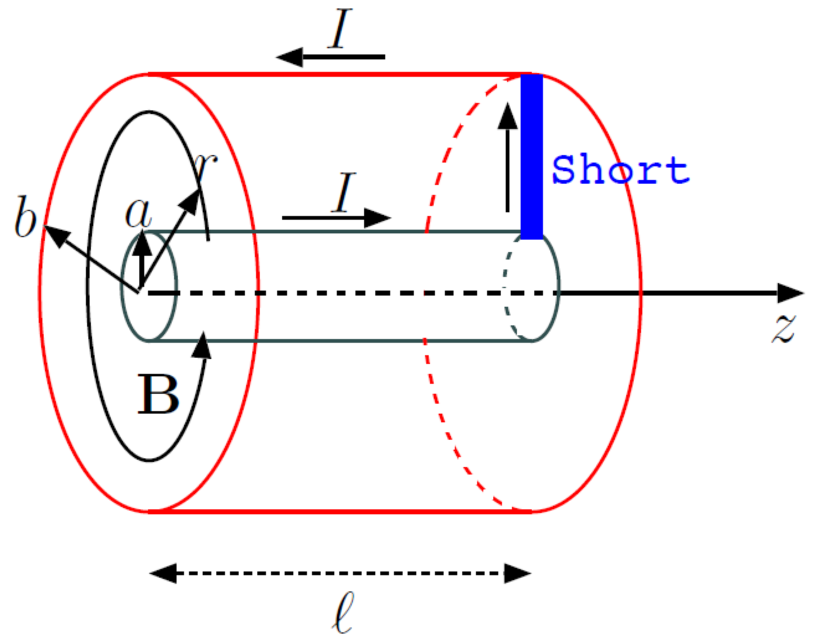


$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \ell \frac{\mu_o}{2\pi} I \int_a^b \frac{dr}{r} = \ell \frac{\mu_o}{2\pi} \ln \frac{b}{a} I$$

$$\Psi = LI$$

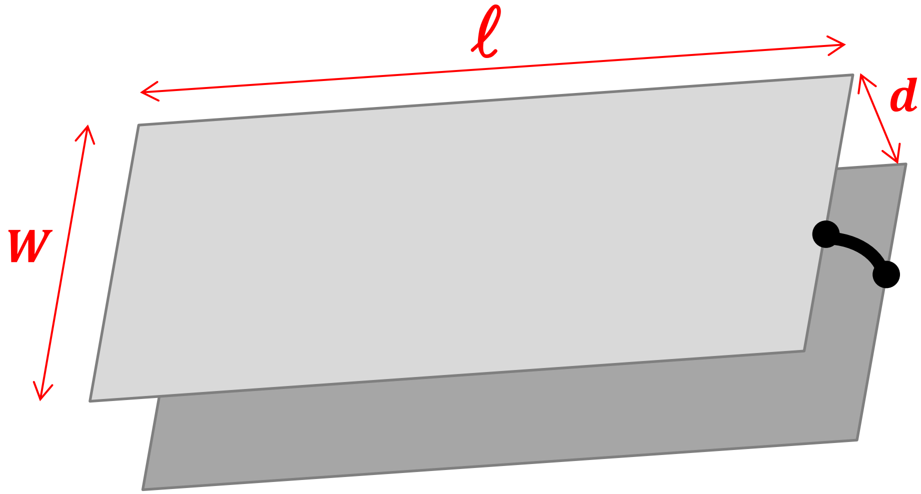
$$\boxed{L \equiv \frac{\ell \mu_o}{2\pi} \ln \frac{b}{a}}$$

Inductance and Capacitance per unit length



$$\mathcal{L} = \frac{\ln \frac{b}{a}}{2\pi}\mu_o \qquad \mathcal{C} = \frac{2\pi}{\ln \frac{b}{a}}\epsilon_o$$

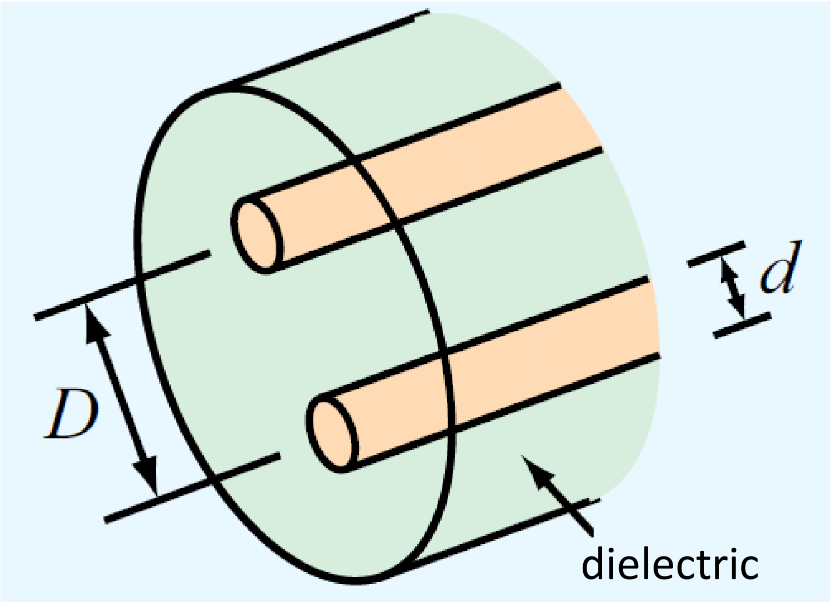
Inductance and Capacitance per unit length of Shorted Parallel Plates



$$\mathcal{L} = \frac{d}{W}\mu_o$$

$$\mathcal{C} = \frac{W}{d}\epsilon_o$$

Inductance and Capacitance per unit length of Shorted Parallel Wires



$$\mathcal{L} = \frac{\mu}{\pi} \ln\left[(D/d) + \sqrt{(D/d)^2 - 1}\right]$$

$$\mathcal{C} = \frac{\pi\epsilon}{\ln\left[(D/d) + \sqrt{(D/d)^2 - 1}\right]}$$