ECE 329 – Fall 2022

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

Lecture 16

Lecture 16 – Outline

- **Inductance**
- **The Shorted Capacitor**

- **Charge conservation and the continuity equation**
- **Displacement current**

Reading assignment

Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:

- **16) Charge conservation, continuity equation, displacement current, Maxwell's equations**

We have defined the magnetic flux as

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(Wb)}$$

For a single loop carrying a current I, if we assume a linear relationship the ratio between flux and current is a constant. The ratio is called *self-inductance*

$$L = \frac{\Psi}{I}$$

For a solenoid with n loops, the self-inductance is

$$L \equiv \frac{n\overbrace{\Psi}^{\text{flux of a single loop}}}{I}$$

From Lecture 13 – Determine the magnetic field of an infinite solenoid with *N* loops per unit length (counterclockwise as seen from the top), stacked in the *z*-direction, each carrying current *I*.

With tight winding, thin wires and infinite length we may assume:

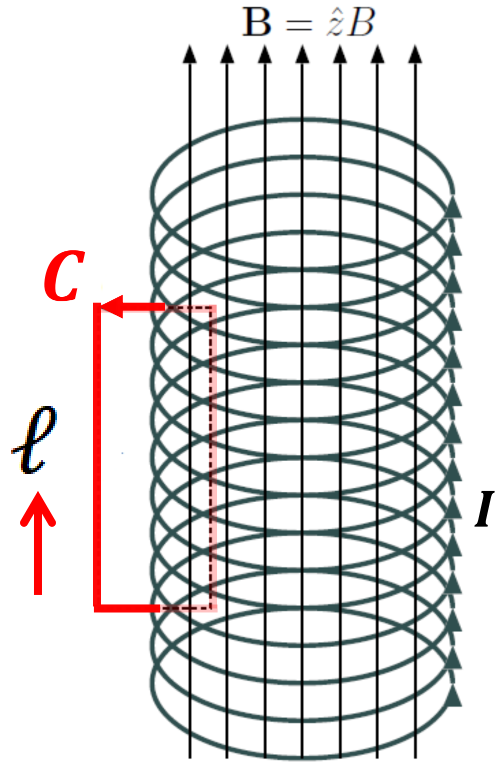
- **$\mathbf{B} = 0$ outside the solenoid**
- **$\mathbf{B}$ independent of z inside the solenoid**

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_o I_C \;\Rightarrow\; \ell B = \mu_o I N \ell$$

$$B = \mu_o I N$$

$$\mathbf{H} = \hat{z} I N$$

current loops per meter



To generalize, the self-inductance is the ratio

$$L = \frac{\text{total concatened magnetic flux}}{\text{current}}$$

For a **long solenoid** **aligned along the** z**-axis, we can assume constant interior magnetic induction density**

$$\mathbf{B} = \mu_o I \underbrace{N}_{\text{loops per unit length}} \hat{z}$$

If the solenoid has length ℓ **and cross-sectional area** A

$$L = \frac{n\Psi}{I} = \frac{\overbrace{N\ell}^{\text{total number of loops } n}(\mu_o I N)A}{I} = \overbrace{N^2 \mu_o A \ell}^{\text{inductance depends only on geometry and material}}$$

We have also seen that variation of the flux induces *emf*

$$\mathcal{E} = -\frac{d\Psi}{dt}$$

For a solenoid with n loops $\quad LI = n\overbrace{\Psi}^{\text{flux of a single loop}}$

$$\mathcal{E} = -\frac{d}{dt} n\Psi \equiv -L\frac{dI}{dt}$$

$$\mathcal{E} = -\underbrace{N^2 \mu_o A \ell}_{L} \frac{dI}{dt} = -\underbrace{\frac{n^2}{\ell} \mu_o A}_{L} \frac{dI}{dt}$$

If the solenoid has resistance R and is conducting a current I then we can write

$$\mathcal{E} = -L\frac{dI}{dt}$$

$$\mathcal{E} = RI$$

$$\Rightarrow$$

emf

voltage drop

$$-L\frac{dI}{dt} = RI$$

So, the following differential equation describes the behavior of a solenoid circuit

$$\boxed{\frac{dI}{dt} + \frac{R}{L}I = 0}$$

A solenoid inductor, connected to an external circuit with a "quasi-static" (size << smallest wavelength in the signal) current I, develops a voltage drop across its terminals

$$V = L\frac{dI}{dt}$$

The instantaneous power absorbed is

$$P = VI = L\frac{dI}{dt}I = \frac{d}{dt}(\frac{1}{2}LI^2)$$

The stored energy is

$$\boxed{B = \mu_o I N}$$

$$W = \frac{1}{2}LI^2 = \frac{1}{2}\underbrace{N^2\mu_o A\ell}_{L} I^2 = \frac{|B_z|^2}{2\mu_o}A\ell = \frac{1}{2}\mu_o|H_z|^2 A\ell$$

$$W = \frac{1}{2}\underbrace{\mu_o |H_z|^2}_{\text{stored magnetic energy per unit volume } w}\underbrace{A\ell}_{\text{volume}}$$

In general

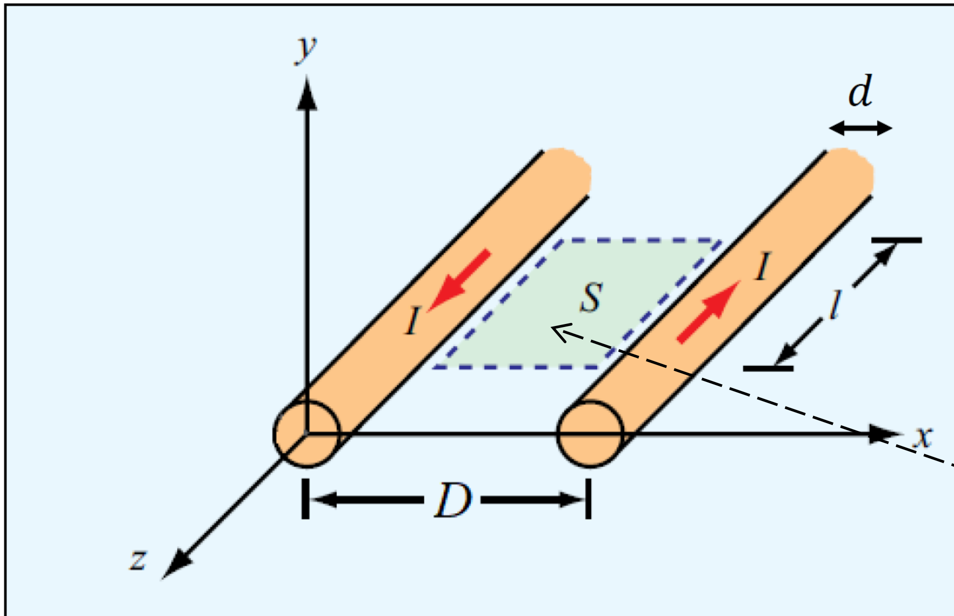
$$w = \frac{1}{2}\mu_o \mathbf{H} \cdot \mathbf{H}$$

For a material medium, we will see later that

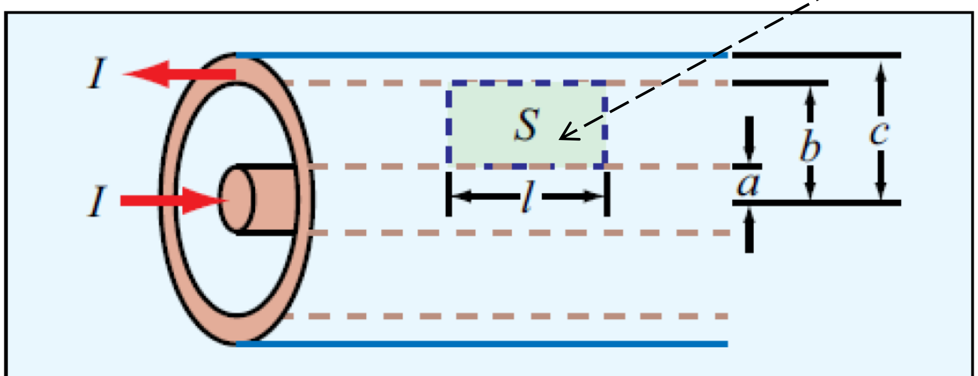
$$\mu = (1 + \chi_m)\mu_o$$

magnetic susceptibility

Structures consisting of separate conductors can also couple with magnetic fields and exhibit inductance



(a) Parallel-wire transmission line

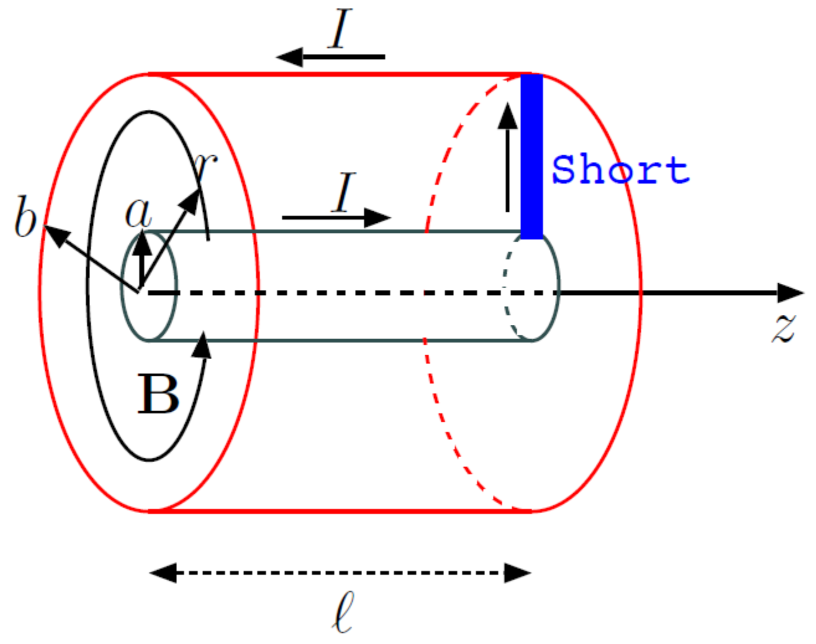


(b) Coaxial transmission line

To compute the inductance per unit length of a two-conductor transmission line, we need to determine the magnetic flux through the area *S* between the conductors.

Inductance and capacitance per unit length will be extremely important for the study of transmission lines based on configurations with multiple conductors.

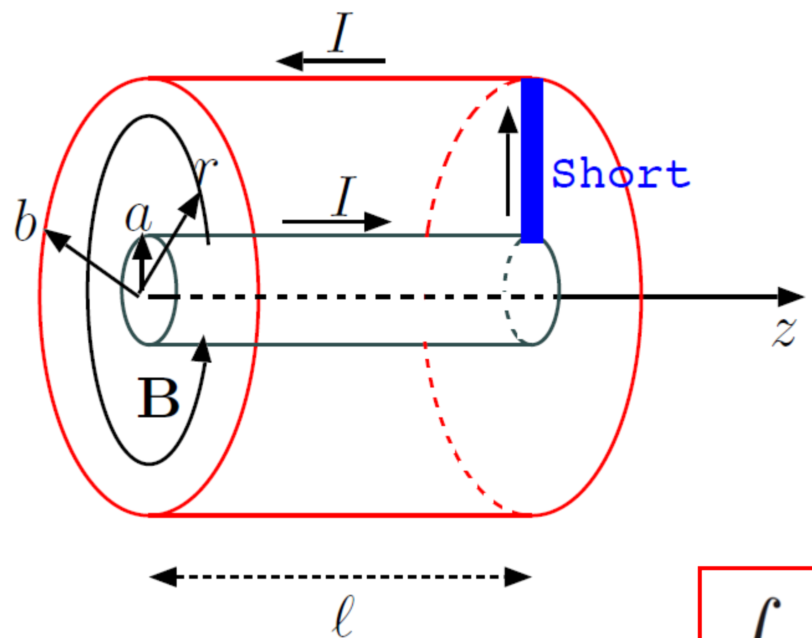
Inductance of shorted coaxial cable



$\ell \gg b$ **approximation of field behavior as in an infinite coaxial cable is acceptable**

The magnetic field wraps around the inner conductor and is constant for a certain radius r at any azimuthal angle

Invoking Ampere's Law



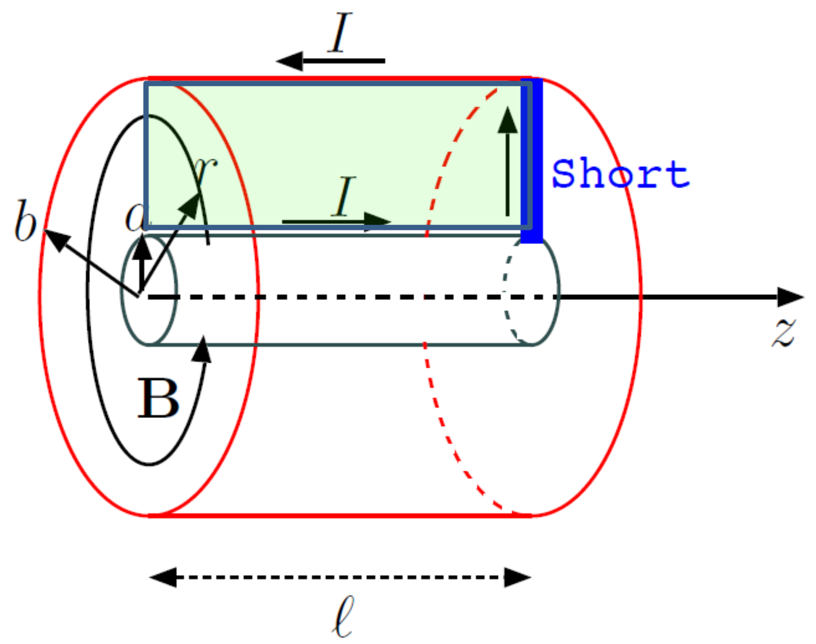
$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_o I_C$$

$$\boxed{\oint_C \mathbf{H} \cdot d\mathbf{l} = I_C}$$

integral form of Ampere's Law

$$\boxed{r > a} \quad B_\phi 2\pi r = \mu_o I \Rightarrow B_\phi = \frac{\mu_o I}{2\pi r}$$

From the Magnetic Flux linked with a closed current path

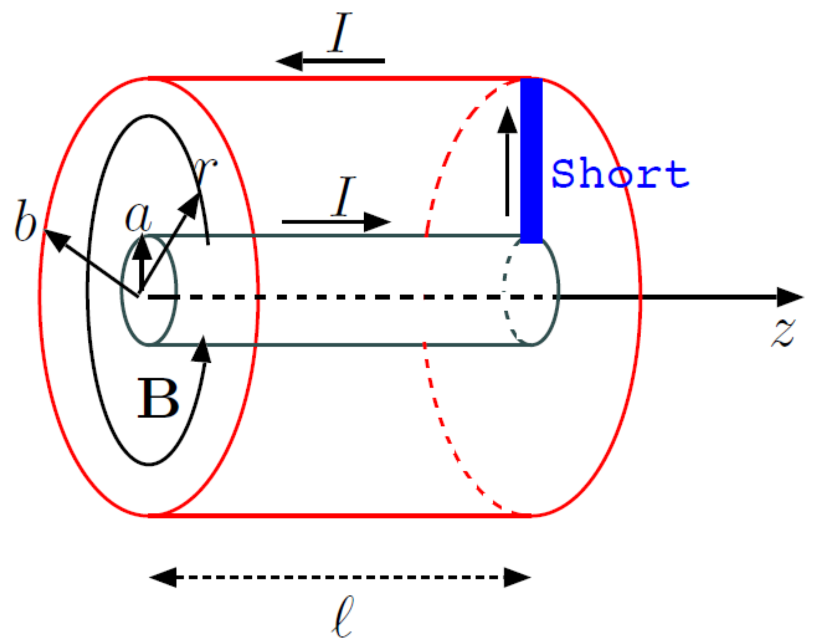


$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \ell \frac{\mu_o}{2\pi} I \int_a^b \frac{dr}{r} = \ell \frac{\mu_o}{2\pi} \ln \frac{b}{a} I$$

$$\Psi = LI$$

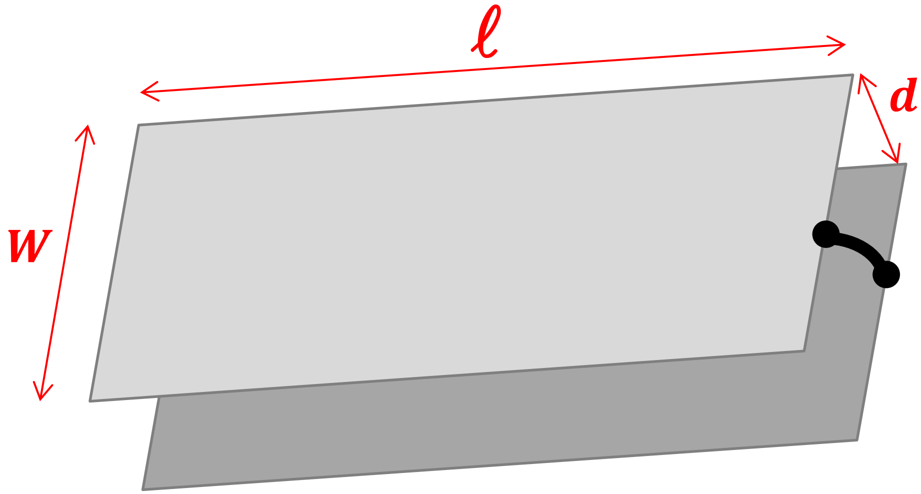
$$\boxed{L \equiv \frac{\ell \mu_o}{2\pi} \ln \frac{b}{a}}$$

Inductance and Capacitance per unit length



$$\mathcal{L} = \frac{\ln\frac{b}{a}}{2\pi}\mu_o \qquad \mathcal{C} = \frac{2\pi}{\ln\frac{b}{a}}\epsilon_o$$

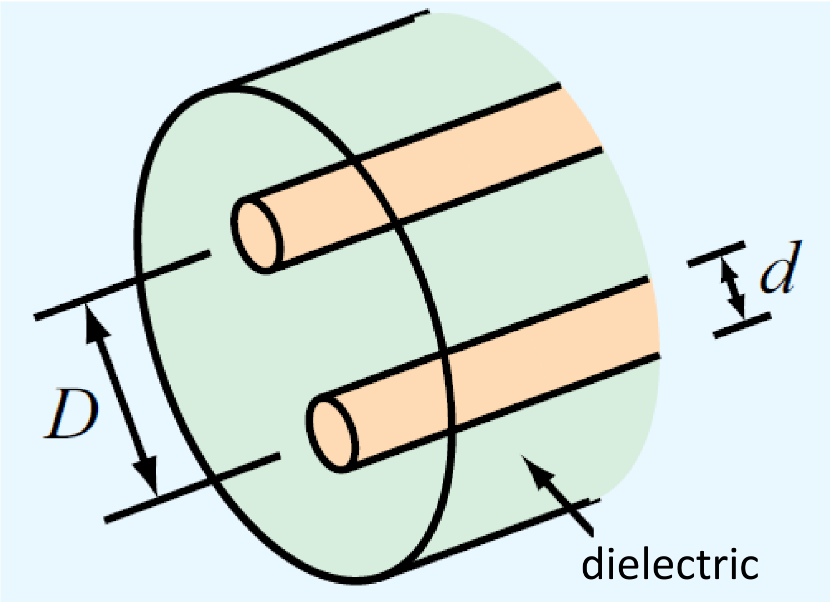
Inductance and Capacitance per unit length of Shorted Parallel Plates



$$\mathcal{L} = \frac{d}{W}\mu_o$$

$$\mathcal{C} = \frac{W}{d}\epsilon_o$$

Inductance and Capacitance per unit length of Shorted Parallel Wires



$$\mathcal{L} = \frac{\mu}{\pi} \ln\left[(D/d) + \sqrt{(D/d)^2 - 1}\right]$$

$$\mathcal{C} = \frac{\pi\epsilon}{\ln\left[(D/d) + \sqrt{(D/d)^2 - 1}\right]}$$

In general, total electric charge is conserved.

Processes which create excess charge of a given polarity in a certain region of space will create charge of opposite polarity somewhere else.

Similarly, disappearance of a charge implies that a charge of opposite polarity disappears as well. Types of phenomena found in nature are:

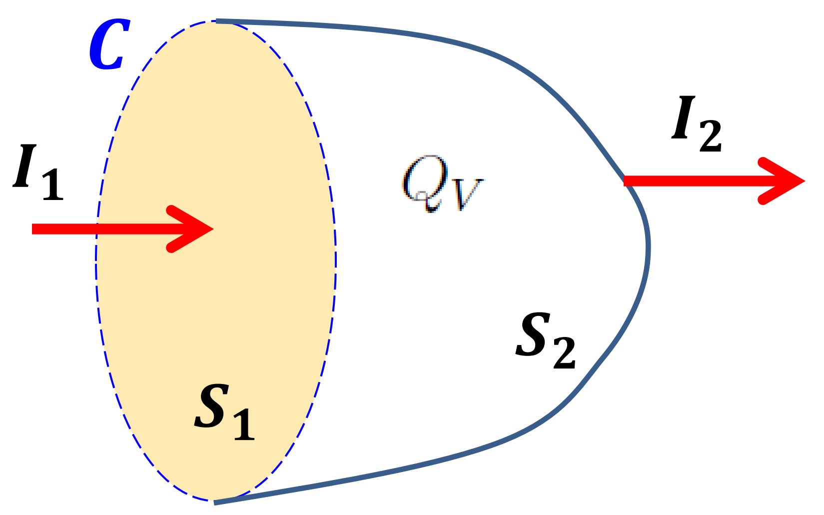
Generation – an electron leaves an atom (or a molecule) due to an external excitation (thermal, photonic or other kind of ionizing irradiation, electrical breakdown,…)
Recombination – an ion and an electron join together to produce a neutral atom (or molecule)
Annihilation – For example, a positron and an electron recombine generating a pair of gamma rays (high enerfy photons)

Transport phenomena may separate the charges and create regions of charge imbalance. This process is described by electrical current flow.

Imagine that a certain volume V is delimited by two surfaces sharing the same contour

free charge in volume V

$$Q_V = \int_V \rho dV$$



Current in

$$I_1 = \int_{S_1} \mathbf{J} \cdot d\mathbf{S}_1$$

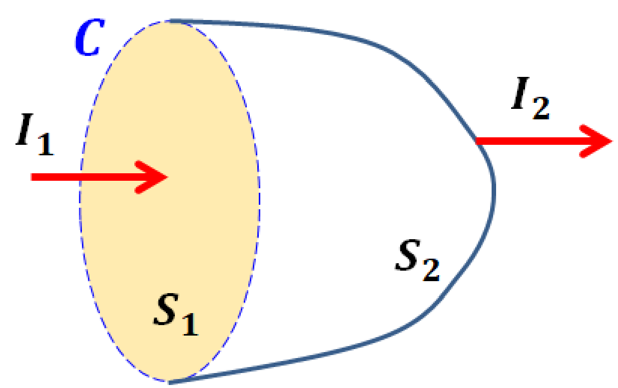
Current out

$$I_2 = \int_{S_2} \mathbf{J} \cdot d\mathbf{S}_2$$

If $I_1 \neq I_2$ the net charge Q_V contained in volume V changes at a rate

$$\frac{dQ_V}{dt} = I_1 - I_2$$

$$\frac{d}{dt}\int_V \rho dV = \int_{S_1} \mathbf{J}\cdot d\mathbf{S}_1 - \int_{S_2} \mathbf{J}\cdot d\mathbf{S}_2$$



With $\boldsymbol{S = S_1 + S_2}$ **forming a closed surface**

$$\frac{d}{dt}\int_V \rho dV = \int_{S_1} \mathbf{J}\cdot d\mathbf{S}_1 - \int_{S_2} \mathbf{J}\cdot d\mathbf{S}_2$$

$$\downarrow$$

$$\int_V \frac{\partial \rho}{\partial t} dV = -\oint_S \mathbf{J}\cdot d\mathbf{S}$$

Integral form of the continuity equation

Divergence Theorem

$$\oint_S \mathbf{J}\cdot d\mathbf{S} = \int_V \nabla\cdot\mathbf{J}\ dV$$

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \nabla \cdot \mathbf{J} \; dV$$

Differential form of the continuity equation

$$\longrightarrow \quad \boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}}$$

This is a balance equation which accounts for charge conservation when charge density varies in space due to current flow.

Displacement Current

Electrostatics	Electrodynamics
$\nabla \times \mathbf{E} = 0$ $\longrightarrow$	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$

A time-varying B induces a time-varying E through Faraday's law

Magnetostatics	Electrodynamics
$\nabla \times \mathbf{H} = \mathbf{J}$ $\longrightarrow$	?

How do we extend Ampere's law to the case of time-varying E ?

One could observe that

$$\mathbf{J} = \sigma \mathbf{E}$$

but this essentially describes a quasi-static regime where local steady-state is quickly reached.

What if charges of different polarity are separated, for instance, and also react differently to a varying electric field due to inertia?

Maxwell postulated that, to guarantee charge conservation, Ampere's law should be rewritten as

$$\nabla \times \mathbf{H} = \mathbf{J} + \underbrace{\frac{\partial \mathbf{D}}{\partial t}}_{\text{displacement current density}}$$

Verification: Take the divergence of modified Ampere's law

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\underbrace{\nabla \cdot (\nabla \times \mathbf{H})}_{\text{divergence of a curl is zero}} = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} \nabla \cdot \mathbf{D} = 0$$

Current continuity was obtained as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

and if we assume Gauss law to remain valid in non-equilibrium

$$\nabla \cdot \mathbf{D} = \rho \longrightarrow \frac{\partial}{\partial t} \nabla \cdot \mathbf{D} = \frac{\partial \rho}{\partial t}$$

Ampere's law as modified is consistent with charge conservation

Time-varying electromagnetic fields obey this set of laws

$$\nabla \cdot \mathbf{D} = \rho \qquad \text{Gauss's law}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's law}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{Ampere's law}$$

with constitutive equations

$$\mathbf{D} = \epsilon\, \mathbf{E} \qquad \mathbf{B} = \mu\, \mathbf{H}$$

This is the complete set of Maxwell's equations

$\nabla \cdot \mathbf{B} = 0$ remains valid as a consequence of Faraday's law

This can be shown by taking the divergence

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0$$

But do magnetic monopoles exist or not? They have never been observed but modern unified theories, like *String Theory* and others, all actually require the existence of magnetic monopoles… and the search continues.

For a quick summary, you can read these posts:
https://home.cern/news/opinion/experiments/observation-authentic-make-believe-monopoles
https://home.cern/news/news/physics/atlas-homes-magnetic-monopoles

Now that we have the complete set of Maxwell's equations, we can state the complete boundary conditions at interfaces

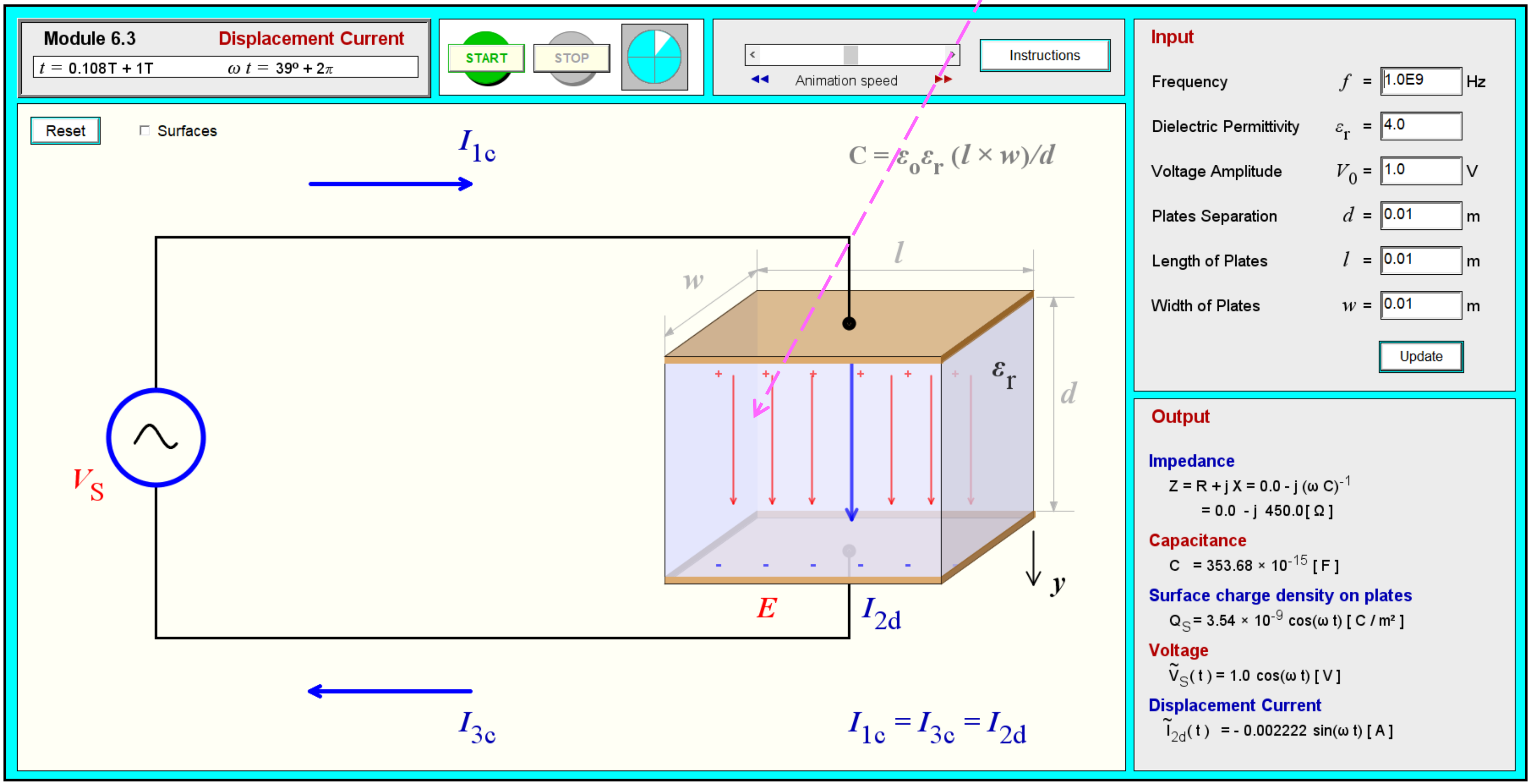
$$\begin{aligned} \hat{n} \cdot (\mathbf{D}^+ - \mathbf{D}^-) &= \rho_s \\ \hat{n} \cdot (\mathbf{B}^+ - \mathbf{B}^-) &= 0 \\ \hat{n} \times (\mathbf{E}^+ - \mathbf{E}^-) &= 0 \\ \hat{n} \times (\mathbf{H}^+ - \mathbf{H}^-) &= \mathbf{J}_\mathrm{s} \end{aligned}$$

what is this?

The parallel-plate capacitor is a good example to illustrate the physical meaning of displacement current

$$V_s(t) = V_0 \cos \omega t \quad (\mathrm{V})$$

$$\mathbf{E} = \hat{\mathbf{y}}\,\frac{V_c}{d} = \hat{\mathbf{y}}\,\frac{V_0}{d}\cos \omega t$$



The parallel-plate capacitor is a good example to illustrate the physical meaning of displacement current

$$\boxed{V_s(t) = V_0 \cos \omega t \quad (\mathrm{V})}$$

$$\boxed{C = \epsilon A/d}$$

In the perfectly conducting wires there is only conduction current and no displacement current because the fields are zero.

$$I_{1c} = C \frac{dV_C}{dt} = C \frac{d}{dt} (V_0 \cos \omega t) = -C V_0 \omega \sin \omega t$$

If the insulator in the capacitor is perfect, there is no conduction current there but only displacement current.

$$I_{2d} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} = \int_A \left[\frac{\partial}{\partial t} \left(\hat{\mathbf{y}} \frac{\epsilon V_0}{d} \cos \omega t \right) \right] \cdot (\hat{\mathbf{y}}\, ds)$$

$$= -\frac{\epsilon A}{d} V_0 \omega \sin \omega t = -C V_0 \omega \sin \omega t$$

Even though the displacement current does not transport free charges, it behaves like a real current.

$$I_{1c} = I_{2d} = I_{3c}$$

If the wires have finite conductivity σ_w then the displacement vector is not zero inside the wire and there would be also a displacement current component.

$$I_1 = I_{1c} + I_{1d} \qquad\qquad I_3 = I_{3c} + I_{3d}$$

If the insulator in the capacitor is not perfect and has a small conductivity σ_d, there would be also a conduction current besides the displacement one.

$$I_2 = I_{2c} + I_{2d}$$

In any case, the currents in wires and capacitor remain equal

$$I_1 = I_2 = I_3$$

Example – Displacement Current in a good conductor.

The conduction current flowing through a wire with conductivity $\sigma = 2 \times 10^7$ S/m and relative permittivity $\epsilon_r = 1$ is given by $I_c = 2 \sin \omega t$ (mA). If $\omega = 10^9$ rad/s, find the displacement current.

Solution: The conduction current $I_c = JA = \sigma EA$, where A is the cross section of the wire. Hence,

$$\boxed{E = \frac{I_c}{\sigma A} = \frac{2 \times 10^{-3} \sin \omega t}{2 \times 10^7 A} = \frac{1 \times 10^{-10}}{A} \sin \omega t \qquad \text{(V/m)}}$$

Application of

$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$$

with $D = \epsilon E$, leads to → next slide

Example – Displacement Current Density in a good conductor.

$$\boxed{\begin{aligned} I_\mathrm{d} &= J_\mathrm{d} A = \epsilon A \frac{\partial E}{\partial t} = \epsilon A \frac{\partial}{\partial t}\left(\frac{1 \times 10^{-10}}{A} \sin \omega t\right) \\ &= \epsilon\omega \times 10^{-10} \cos \omega t = 0.885 \times 10^{-12} \cos \omega t \quad \text{(A)} \end{aligned}}$$

where $\omega = 10^9$ rad/s and $\epsilon = \epsilon_0 = 8.85 \times 10^{-12}$ F/m.

Recall that: $\boxed{I_\mathrm{c} = 2 \sin \omega t \text{ (mA)}}$

Note that I_c and I_d are in phase quadrature (90° phase shift between them). Also, I_d is about nine orders of magnitude smaller than I_c, which is why the displacement current usually is ignored in good conductors.

Example – Displacement Current Density in a poor conductor.

A poor conductor is characterized by a conductivity $\sigma = 100$ (S/m) and permittivity $\varepsilon = 4\ \varepsilon_0$.
At what angular frequency ω is the amplitude of the conduction current density **J** equal to the amplitude of the displacement current density $\mathbf{J}_\mathrm{d}$?

$$\boxed{\text{Assume } 1\,\mathrm{A/m^2}: \quad J_c = \sigma E = \sin(\omega t)}$$

$$\boxed{E = \frac{J_c}{\sigma} = 10^{-2}\sin(\omega t)}$$

$$J_d = \varepsilon \frac{\partial E}{\partial t} = 4\varepsilon_o \omega \times 10^{-2} \times \cos(\omega t)$$

$$|J_c| = |J_d| \longrightarrow 1 = 4 \times 8.854 \times 10^{-12} \times \omega \times 10^{-2}$$

$$\boxed{\omega = \frac{1}{4 \times 8.854 \times 10^{-14}} = 2.82 \times 10^{12}\ \frac{\text{rad}}{\text{s}}}$$

$$\boxed{f = \frac{\omega}{2\pi} = 449.6\ \text{GHz}}$$