ECE 329 – Fall 2022

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

Lecture 17

Lecture 17 – Outline

- **Displacement current**
- **Magnetization of materials**
- **Diamagnetic, Paramagnetic and Ferromagnetic media**
- **Boundary Conditions**

Reading assignment

Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:

17) Magnetization current, Maxwell's equations in material media

Time-varying electromagnetic fields obey this set of laws

$$\nabla \cdot \mathbf{D} = \rho \qquad \text{Gauss's law}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's law}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{Ampere's law}$$

with constitutive equations

$$\mathbf{D} = \epsilon\, \mathbf{E} \qquad \mathbf{B} = \mu\, \mathbf{H}$$

This is the complete set of Maxwell's equations

$\nabla \cdot \mathbf{B} = 0$ remains valid as a consequence of Faraday's law

This can be shown by taking the divergence

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0$$

But do magnetic monopoles exist or not? They have never been observed but modern unified theories, like *String Theory* and others, all actually require the existence of magnetic monopoles… and the search continues.

For a quick summary, you can read these posts:

https://home.cern/news/opinion/experiments/observation-authentic-make-believe-monopoles
https://home.cern/news/news/physics/atlas-homes-magnetic-monopoles

Now that we have the complete set of Maxwell's equations, we can state the complete boundary conditions at interfaces

$$\hat{n} \cdot (\mathbf{D}^+ - \mathbf{D}^-) = \rho_s$$

$$\hat{n} \cdot (\mathbf{B}^+ - \mathbf{B}^-) = 0$$

$$\hat{n} \times (\mathbf{E}^+ - \mathbf{E}^-) = 0$$

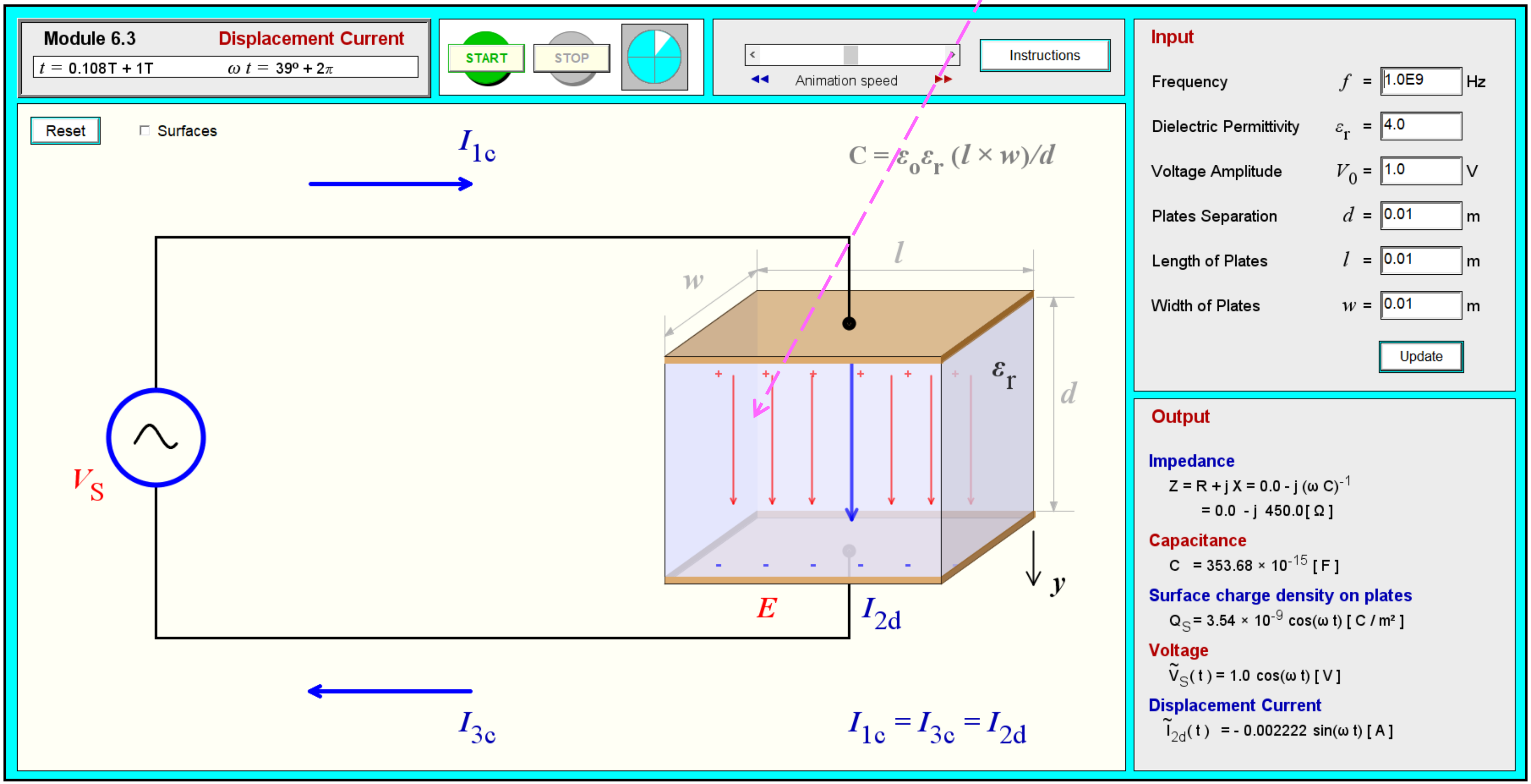
$$\hat{n} \times (\mathbf{H}^+ - \mathbf{H}^-) = \mathbf{J}_\mathbf{s}$$

what is this?

The parallel-plate capacitor is a good example to illustrate the physical meaning of displacement current

$$V_s(t) = V_0 \cos \omega t \quad \text{(V)}$$

$$\mathbf{E} = \hat{\mathbf{y}}\,\frac{V_c}{d} = \hat{\mathbf{y}}\,\frac{V_0}{d}\cos \omega t$$



The parallel-plate capacitor is a good example to illustrate the physical meaning of displacement current

$$\boxed{V_s(t) = V_0 \cos \omega t \quad (\mathrm{V})}$$

$$\boxed{C = \epsilon A/d}$$

In the perfectly conducting wires there is only conduction current and no displacement current because the fields are zero.

$$I_{1c} = C\,\frac{dV_C}{dt} = C\,\frac{d}{dt}\,(V_0 \cos \omega t) = -C V_0 \omega \sin \omega t$$

If the insulator in the capacitor is perfect, there is no conduction current there but only displacement current.

$$I_{2d} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} = \int_A \left[\frac{\partial}{\partial t}\left(\hat{\mathbf{y}}\,\frac{\epsilon V_0}{d} \cos \omega t\right)\right] \cdot (\hat{\mathbf{y}}\; ds)$$

$$= -\frac{\epsilon A}{d} V_0 \omega \sin \omega t = -C V_0 \omega \sin \omega t$$

Even though the displacement current does not transport free charges, it behaves like a real current.

$$I_{1c} = I_{2d} = I_{3c}$$

If the wires have finite conductivity σ_w then the displacement vector is not zero inside the wire and there would be also a displacement current component.

$$I_1 = I_{1c} + I_{1d} \qquad\qquad I_3 = I_{3c} + I_{3d}$$

If the insulator in the capacitor is not perfect and has a small conductivity σ_d, there would be also a conduction current besides the displacement one.

$$I_2 = I_{2c} + I_{2d}$$

In any case, the currents in wires and capacitor remain equal

$$I_1 = I_2 = I_3$$

Example – Displacement Current in a good conductor.

The conduction current flowing through a wire with conductivity $\sigma = 2 \times 10^7$ S/m and relative permittivity $\epsilon_r = 1$ is given by $I_c = 2 \sin \omega t$ (mA). If $\omega = 10^9$ rad/s, find the displacement current.

Solution: The conduction current $I_c = JA = \sigma EA$, where A is the cross section of the wire. Hence,

$$\boxed{E = \frac{I_c}{\sigma A} = \frac{2 \times 10^{-3} \sin \omega t}{2 \times 10^7 A} = \frac{1 \times 10^{-10}}{A} \sin \omega t \qquad \text{(V/m)}}$$

Application of

$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$$

with $D = \epsilon E$, leads to $\longrightarrow$ next slide

Example – Displacement Current Density in a good conductor.

$$\boxed{\begin{aligned} I_{\mathrm{d}} &= J_{\mathrm{d}} A = \epsilon A \frac{\partial E}{\partial t} = \epsilon \cancel{A} \frac{\partial}{\partial t} \left(\frac{1 \times 10^{-10}}{\cancel{A}} \sin \omega t \right) \\ &= \epsilon \omega \times 10^{-10} \cos \omega t = 0.885 \times 10^{-12} \cos \omega t \quad \text{(A)} \end{aligned}}$$

where $\omega = 10^9$ rad/s and $\epsilon = \epsilon_0 = 8.85 \times 10^{-12}$ F/m.

Recall that: $\boxed{I_{\mathrm{c}} = 2 \sin \omega t \text{ (mA)}}$

Note that I_{c} and I_{d} are in phase quadrature (90° phase shift between them). Also, I_{d} is about nine orders of magnitude smaller than I_{c}, which is why the displacement current usually is ignored in good conductors.

Example – Displacement Current Density in a poor conductor.

A poor conductor is characterized by a conductivity $\sigma = 100$ (S/m) and permittivity $\varepsilon = 4\,\varepsilon_0$.
At what angular frequency ω is the amplitude of the conduction current density **J** equal to the amplitude of the displacement current density $\mathbf{J}_d$?

$$\boxed{\text{Assume } 1\text{A/m}^2\text{:} \quad J_c = \sigma E = \sin(\omega t)}$$

$$\boxed{E = \frac{J_c}{\sigma} = 10^{-2}\sin(\omega t)}$$

$$J_d = \varepsilon \frac{\partial E}{\partial t} = 4\varepsilon_o \omega \times 10^{-2} \times \cos(\omega t)$$

$$|J_c| = |J_d| \longrightarrow 1 = 4 \times 8.854 \times 10^{-12} \times \omega \times 10^{-2}$$

$$\boxed{\omega = \frac{1}{4 \times 8.854 \times 10^{-14}} = 2.82 \times 10^{12}\,\frac{\text{rad}}{\text{s}}}$$

$$\boxed{f = \frac{\omega}{2\pi} = 449.6 \text{ GHz}}$$

Macroscopic Maxwell's equations for time-varying EM fields

$$\nabla \cdot \mathbf{D} = \rho \qquad \text{Gauss's law}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's law}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{Ampere's law}$$

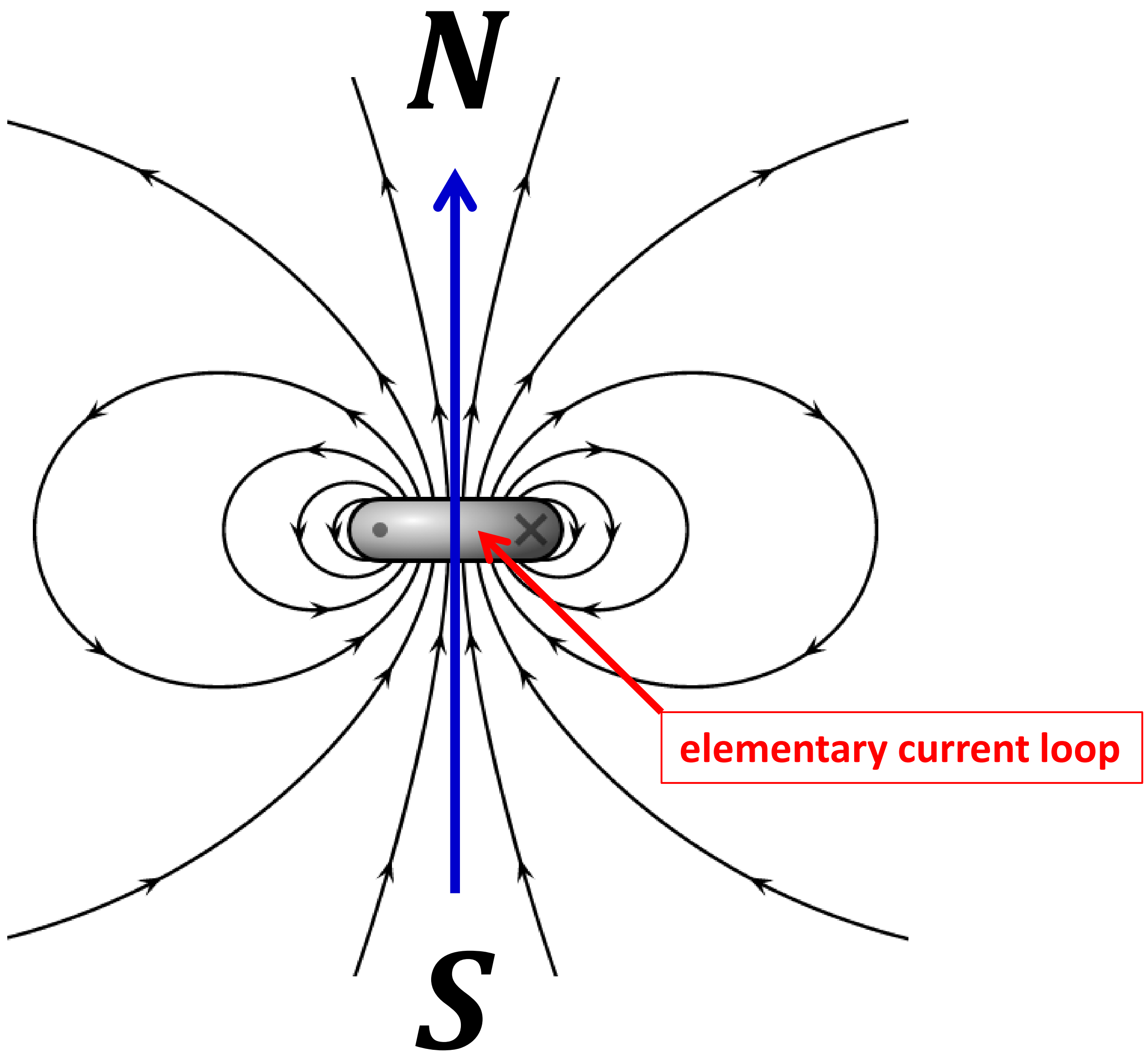
due to free charge carriers only

with constitutive equations

material permeability

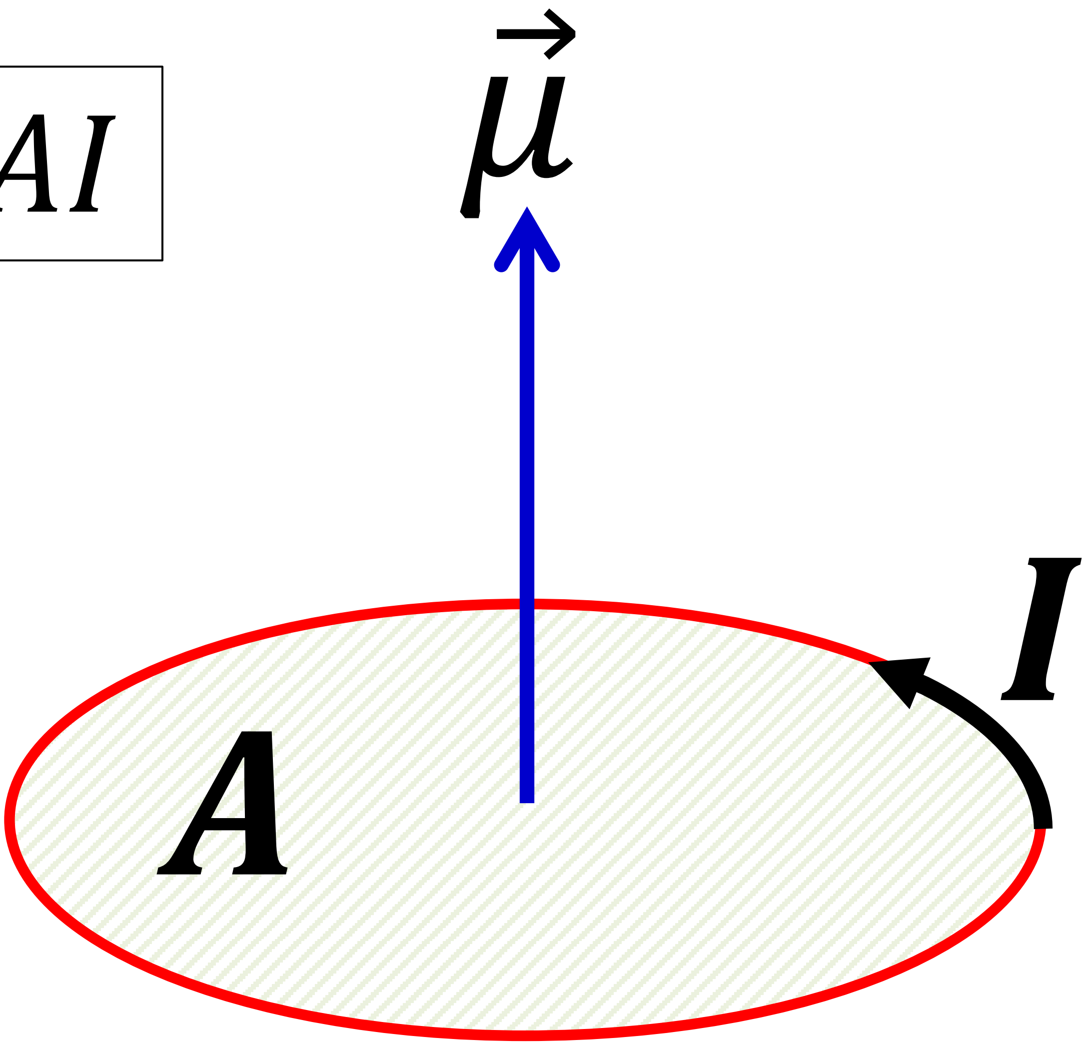
$$\mathbf{D} = \epsilon\,\mathbf{E} \qquad \mathbf{B} = \mu\,\mathbf{H}$$

Elementary magnetic dipole



“Amperian” definition of magnetic moment

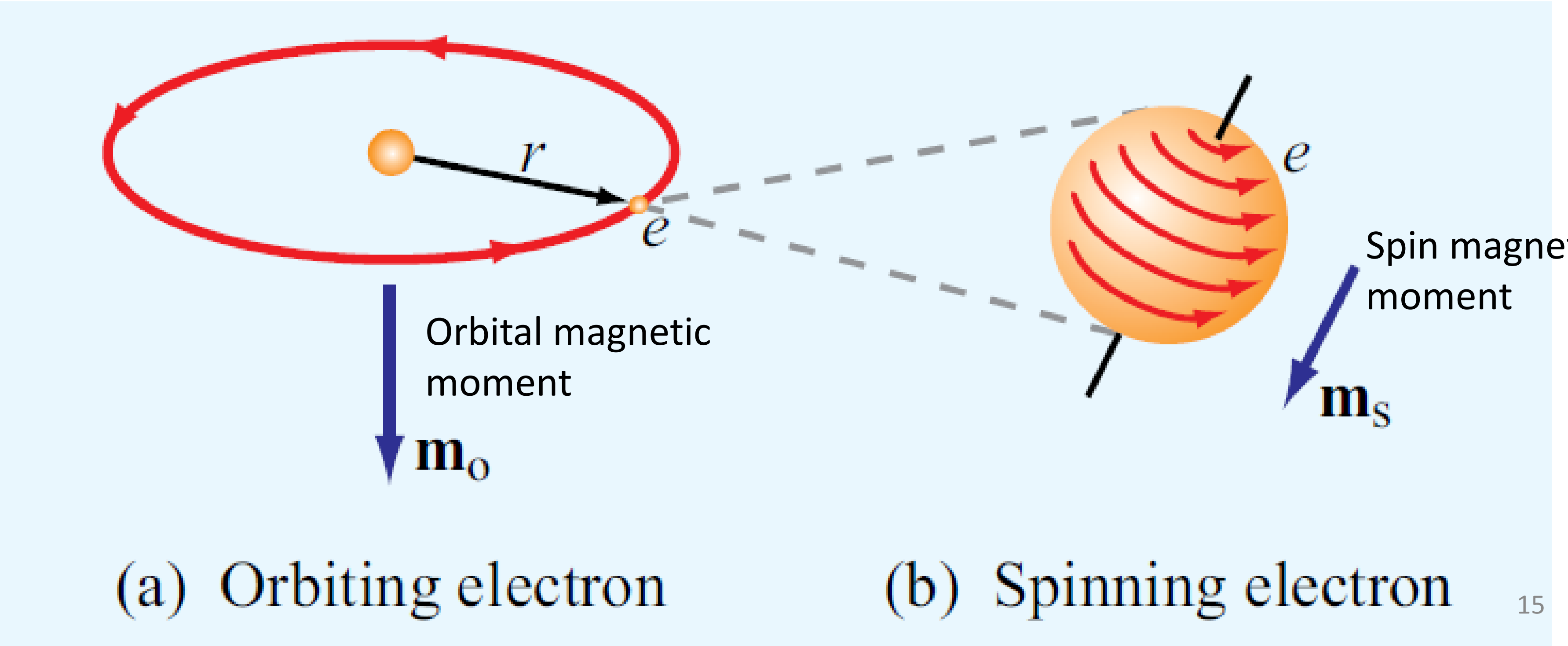
$$\boxed{\vec{\mu} = AI}$$



Besides currents in material media related to polarization of bound charge, we have divergence-free currents due to closed loop orbits.

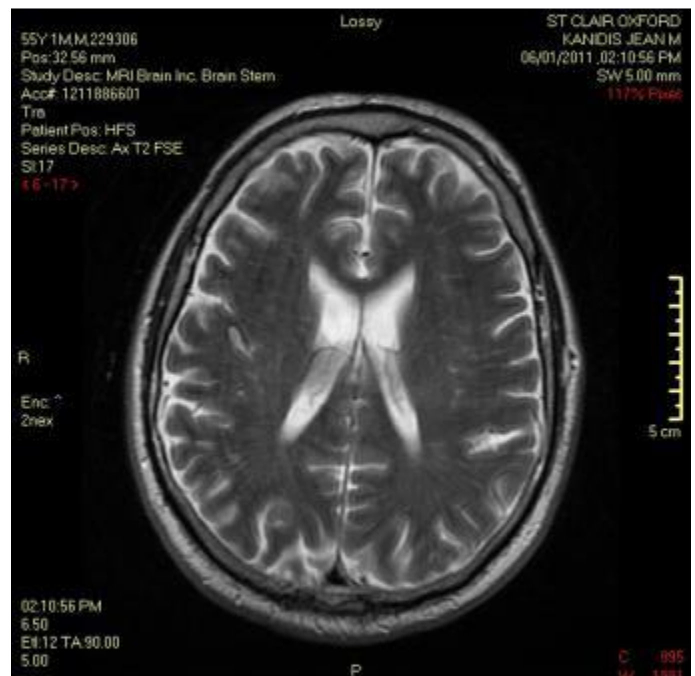
Magnetization in a material is due to atomic scale current loops associated with:

1. orbital motions of the electrons around the nucleus and of protons inside the nucleus
2. electron spin



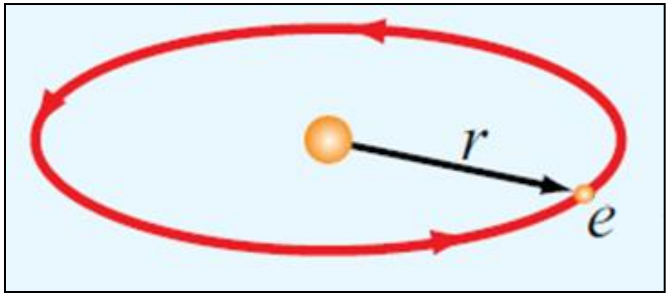
The total orbital and spin magnetic moment of an atom is dominated by the sum of the magnetic moments of its electrons since the magnetic moment due to proton motion typically is three orders of magnitude smaller.

However, in Magnetic Resonance Imaging (MRI) the hydrogen nuclei (protons) are excited by strong fixed magnetic fields and radio pulses, particularly to map tissues rich in water and fat.



If we describe the motion of electrons with the simple "orbital" model of the atom by Bohr, circular orbits of radius r with constant speed u have one revolution over a period

$$T = 2\pi r / u$$



The orbit is a tiny loop with current

$$I = -\frac{e}{T} = -\frac{eu}{2\pi r}$$

The associated *orbital magnetic moment* has magnitude

$$m_{\mathrm{o}} = IA = \left(-\frac{eu}{2\pi r}\right)(\pi r^2) = -\frac{eur}{2}$$

$$= -\frac{eur}{2}\frac{m_{\mathrm{e}}}{m_{\mathrm{e}}} = -\left(\frac{e}{2m_{\mathrm{e}}}\right) m_{\mathrm{e}} ur$$

electron mass

Orbital magnetic moment magnitude

$$m_{\rm o} = -\left(\frac{e}{2m_{\rm e}}\right)\underbrace{m_{\rm e}ur}_{\text{electron angular momentum}}$$

According to quantum mechanics, the electron angular momentum is quantized as a multiple of the reduced Planck constant

$$m_{\rm o} = -\left(\frac{e}{2m_{\rm e}}\right)n\hbar \qquad n = 0, 1, 2, \ldots$$

The minimum (non-zero) orbital magnetic moment is

$$m_{\rm o} = -\frac{e\hbar}{2m_{\rm e}}$$

Spin magnetic moment **has a magnitude, predicted from quantum mechanics, which is the same as the minimum orbital magnetic moment**

$$m_s = -\frac{e\hbar}{2m_e}$$

This is the same as the minimum orbital magnetic moment.

Electrons in an atom exist in pairs with opposite spin directions. In an atom with even number of electrons, the spin magnetic moment cancels out.

So, an atom needs an odd number of electrons to have non-zero spin magnetic moment.

The magnetic behavior of a material is governed by the interaction of the magnetic dipole moments of its atoms with an external magnetic field.

In analogy with electrical polarization

$$\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M} = \mu_0(\mathbf{H} + \mathbf{M})$$

where the *magnetization vector* $\mathbf{M}$ is the vector sum of the magnetic dipole moments of the atoms contained in a volume of material.

In most magnetic materials there is a linear relationship

$$\mathbf{M} = \underbrace{\chi_{\rm m}}_{\text{magnetic susceptibility}}\mathbf{H}$$

Therefore, we have

$$\mathbf{B} = \mu_0(\mathbf{H} + \chi_{\mathrm{m}}\mathbf{H}) = \underbrace{\mu_0\underbrace{(1 + \chi_{\mathrm{m}})}_{\mu_r}}_{\mu}\mathbf{H}$$

$$\boxed{\mathbf{B} = \mu\mathbf{H}}$$

μ is the magnetic permeability of the material with units of Henry/meter (H/m).

$$\boxed{\mathrm{H} = \frac{\mathrm{V{\cdot}s}}{\mathrm{A}} = \Omega{\cdot}\mathrm{s}}$$

Materials are classified in terms of magnetic behavior as

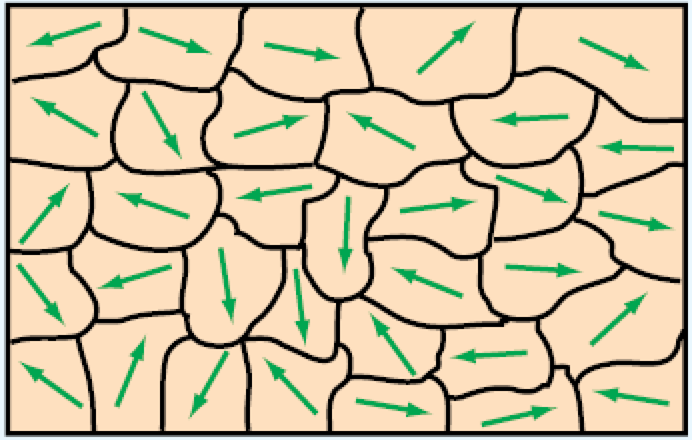
- **Diamagnetic:** **These materials are characterized by small negative susceptibility. They do not have permanent magnetization $\mathbf{M}$.**

- **Paramagnetic:** **These materials are characterized by small positive susceptibility. They have permanent magnetization $\mathbf{M}$. An applied magnetic field aligns the spin angular momentum. This happens for materials with unfilled inner electron shells.**

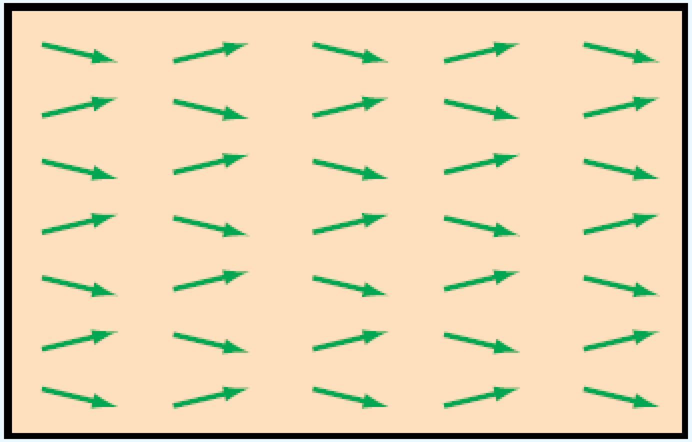
- **Ferromagnetic:** **These materials are characterized by very high susceptibility. The magnetic moments tend to align quite readily along the direction of an external magnetic field and maintain some residual magnetism after the applied field is removed.**

Ferromagnetic materials have microscopic magnetized domains (volume ~ 10^{-10} m^3) containing on the order of 10^{19} atoms which have magnetic moments permanently aligned with each other.

In the absence of an external magnetic field, the orientations of these domains are random. When a magnetic field is applied, the domains partially align.



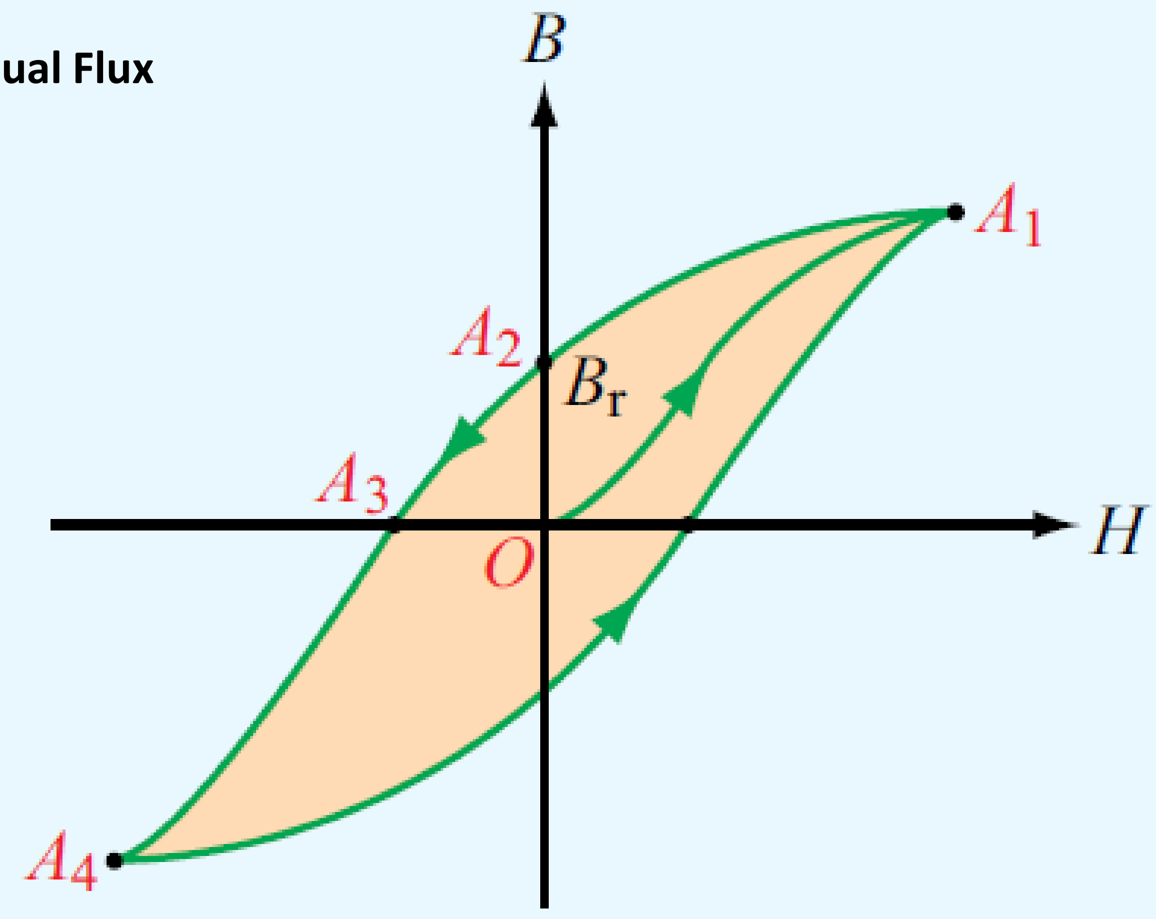
(a) Unmagnetized domains



(b) Magnetized domains

Ferromagnetic materials exhibit hysteresis due to residual magnetism when an applied field is removed

B_r = Residual Flux



Typical hysteresis curve for a ferromagnetic material.

Properties of magnetic materials

	Diamagnetism	Paramagnetism	Ferromagnetism
Permanent magnetic dipole moment	No	Yes, but weak	Yes, and strong
Primary magnetization mechanism	Electron orbital magnetic moment	Electron spin magnetic moment	Magnetized domains
Common substances	Bismuth, copper, diamond, gold, lead, mercury, silver, silicon	Aluminum, calcium, chromium, magnesium, niobium, platinum, tungsten	Iron, nickel, cobalt
Typical value of χ_m **Typical value of μ_r**	$\approx -10^{-5}$ ≈ 1	$\approx 10^{-5}$ ≈ 1	$\lvert\chi_m\rvert \gg 1$ and hysteretic $\lvert\mu_r\rvert \gg 1$ and hysteretic

Macroscopic Model

It is impractical to apply directly the microscopic Maxwell equations to material, so we extend the macroscopic model introduced earlier

charge density

$$\rho = \rho_f - \nabla \cdot \mathbf{P}$$

free charges

polarization (bound charges)

magnetization (bound charges)

current density

$$\mathbf{J} = \mathbf{J}_f + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}$$

Introduce the model into

$$\boxed{\begin{array}{rclr} \nabla \cdot \epsilon_o \mathbf{E} &=& \rho & \text{Gauss' law} \\ \nabla \times \dfrac{1}{\mu_o}\mathbf{B} &=& \mathbf{J} + \dfrac{\partial \epsilon_o \mathbf{E}}{\partial t} & \text{Ampere's law} \end{array}}$$

$$\nabla \cdot \overbrace{(\epsilon_o \mathbf{E} + \mathbf{P})}^{\mathbf{D} = \epsilon \mathbf{E}} = \rho_f$$

$$\nabla \times \underbrace{\left(\frac{1}{\mu_o}\mathbf{B} - \mathbf{M}\right)}_{\mathbf{H} = \frac{1}{\mu}\mathbf{B}} = \mathbf{J}_f + \frac{\partial}{\partial t}\underbrace{(\epsilon_o \mathbf{E} + \mathbf{P})}_{\mathbf{D} = \epsilon \mathbf{E}}$$

Bound charge density and current satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

with

$$\rho = \rho_b = -\nabla \cdot \mathbf{P}$$

$$\mathbf{J} = \mathbf{J}_b = \frac{\partial \mathbf{P}}{\partial t}$$

$$\boxed{\frac{\partial \rho_b}{\partial t} + \nabla \cdot \mathbf{J}_b = \frac{\partial}{\partial t}(-\nabla \cdot \mathbf{P}) + \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = 0}$$

$$\frac{\partial \rho_b}{\partial t} + \nabla \cdot \mathbf{J}_b = \frac{\partial}{\partial t}(-\nabla \cdot \mathbf{P}) + \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = 0$$

$$\frac{\partial \rho_b}{\partial t} + \nabla \cdot \mathbf{J}_b = \underbrace{\frac{\partial}{\partial t}(-\nabla \cdot \mathbf{P}) + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{P})}_{\text{it is really equal to zero!}} = 0$$

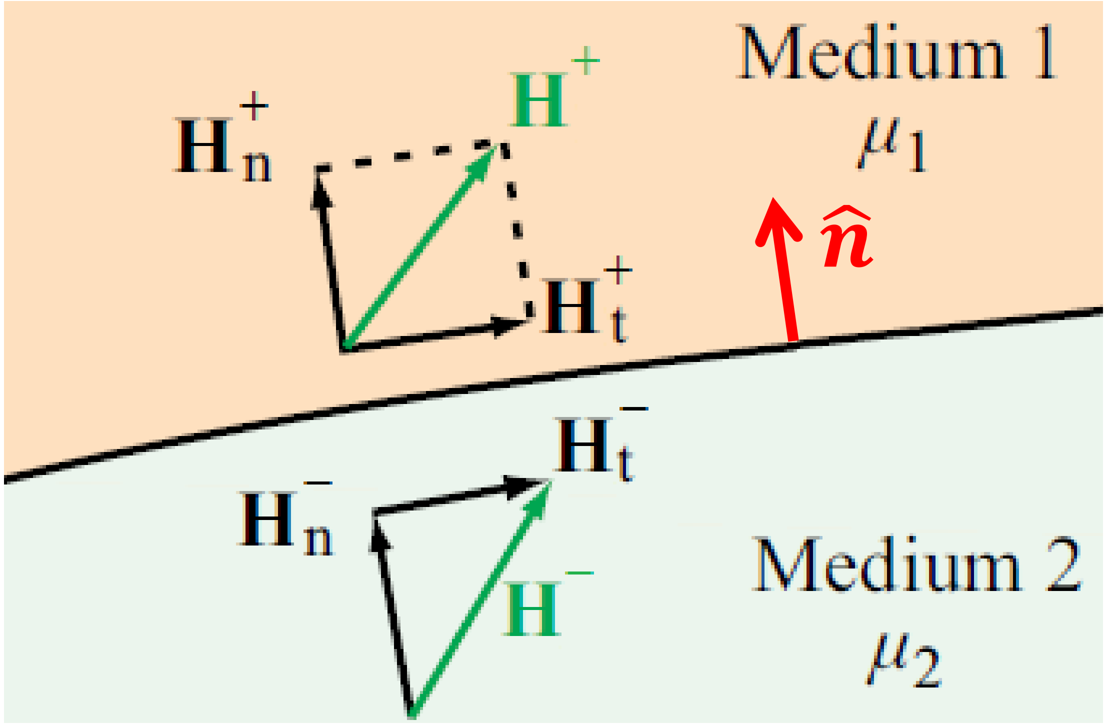
The same equation is satisfied by a density of the form

$$\mathbf{J}_b = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}$$

since

$$\nabla \cdot \mathbf{J}_b = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} + \underbrace{\nabla \cdot (\nabla \times \mathbf{M})}_{= 0} = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t}$$

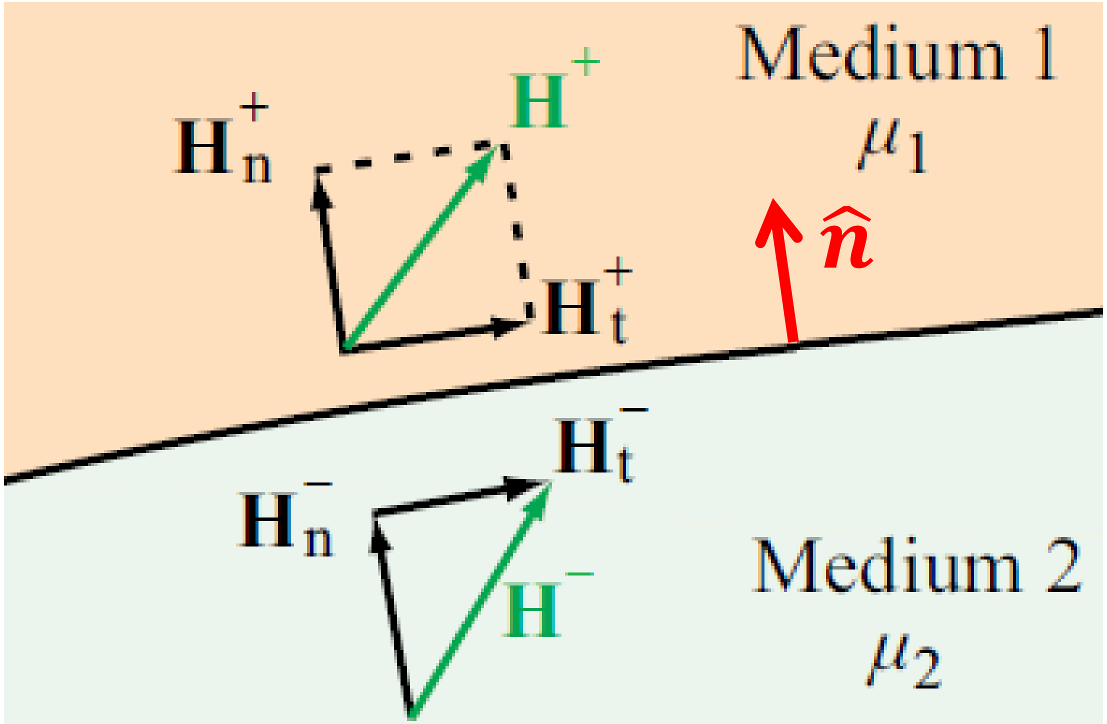
Magnetic Boundary Conditions



$$\boxed{B_n^+ = B_n^-}$$

$$\boxed{\mu_1 H_n^+ = \mu_2 H_n^-}$$

Magnetic Boundary Conditions



Surface current $\boxed{\mathbf{J}_s = 0}$

$$\boxed{H_t^+ = H_t^-}$$

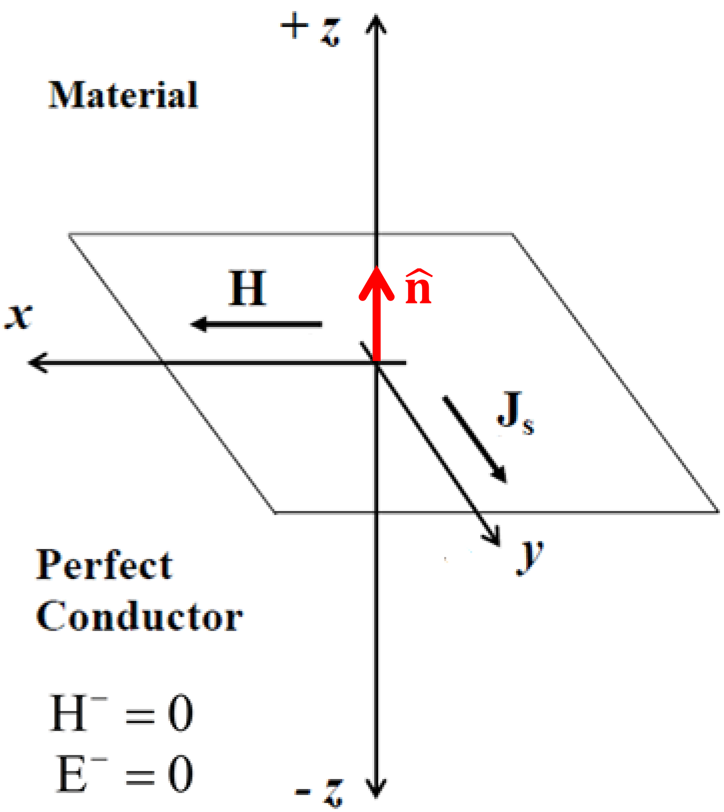
Surface current $\boxed{\mathbf{J}_s \neq 0}$

$$\boxed{\hat{\mathrm{n}} \times \left(\mathrm{H}^+ - \mathrm{H}^- \right) = \mathrm{J}_S}$$

Surface current $\boxed{\mathbf{J}_s \neq 0}$

$$\boxed{\hat{\mathrm{n}} \times \left(\mathrm{H}^{+} - \mathrm{H}^{-}\right) = \mathrm{J}_S}$$

This boundary condition in practice applies mainly to the cases of perfect conductor surface and sheet of current.



$$\mathbf{J}_S = J_{So} \cos(\omega t)\, \hat{\mathbf{y}}$$

$$\hat{\mathrm{n}} \times \left(\mathrm{H}^{+} - \cancel{\mathrm{H}^{-}}\right) = \mathrm{J}_S$$

$$\hat{z} \times H_x^{+} = \mathrm{J}_S\ \hat{\mathbf{y}}$$

On a perfect conductor surface, the tangent electric field and the normal magnetic field are always <u>zero</u>.

Practice this Right Hand Rule