ECE 329 – Fall 2022

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

Lecture 18

Lecture 18 – Outline

- **Boundary Conditions: Surface Current**
- **Wave Equation in source free media**
- **Transverse Electromagnetic (TEM) Plane Waves**

Reading assignment

Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:

18) Wave equation and plane TEM waves in source-free media

Macroscopic Model

It is impractical to apply directly the microscopic Maxwell equations to material, so we extend the macroscopic model introduced earlier

charge density

$$\rho = \rho_f - \nabla \cdot \mathbf{P}$$

free charges	**polarization (bound charges)**	**magnetization (bound charges)**

current density

$$\mathbf{J} = \mathbf{J}_f + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}$$

Introduce the model into

$$\boxed{\begin{array}{rclr} \nabla \cdot \epsilon_o \mathbf{E} & = & \rho & \text{Gauss' law} \\ \nabla \times \dfrac{1}{\mu_o} \mathbf{B} & = & \mathbf{J} + \dfrac{\partial \epsilon_o \mathbf{E}}{\partial t} & \text{Ampere's law} \end{array}}$$

$$\nabla \cdot (\overbrace{\epsilon_o \mathbf{E} + \mathbf{P}}^{\mathbf{D} = \epsilon \mathbf{E}}) = \rho_f$$

$$\nabla \times (\underbrace{\frac{1}{\mu_o} \mathbf{B} - \mathbf{M}}_{\mathbf{H} = \frac{1}{\mu} \mathbf{B}}) = \mathbf{J}_f + \frac{\partial}{\partial t} (\underbrace{\epsilon_o \mathbf{E} + \mathbf{P}}_{\mathbf{D} = \epsilon \mathbf{E}})$$

Bound charge density and current satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

with

$$\rho = \rho_b = -\nabla \cdot \mathbf{P}$$

$$\mathbf{J} = \mathbf{J}_b = \frac{\partial \mathbf{P}}{\partial t}$$

$$\boxed{\frac{\partial \rho_b}{\partial t} + \nabla \cdot \mathbf{J}_b = \frac{\partial}{\partial t}(-\nabla \cdot \mathbf{P}) + \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = 0}$$

$$\frac{\partial \rho_b}{\partial t} + \nabla \cdot \mathbf{J}_b = \frac{\partial}{\partial t}(-\nabla \cdot \mathbf{P}) + \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = 0$$

$$\frac{\partial \rho_b}{\partial t} + \nabla \cdot \mathbf{J}_b = \underbrace{\frac{\partial}{\partial t}(-\nabla \cdot \mathbf{P}) + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{P})}_{\text{it is really equal to zero!}} = 0$$

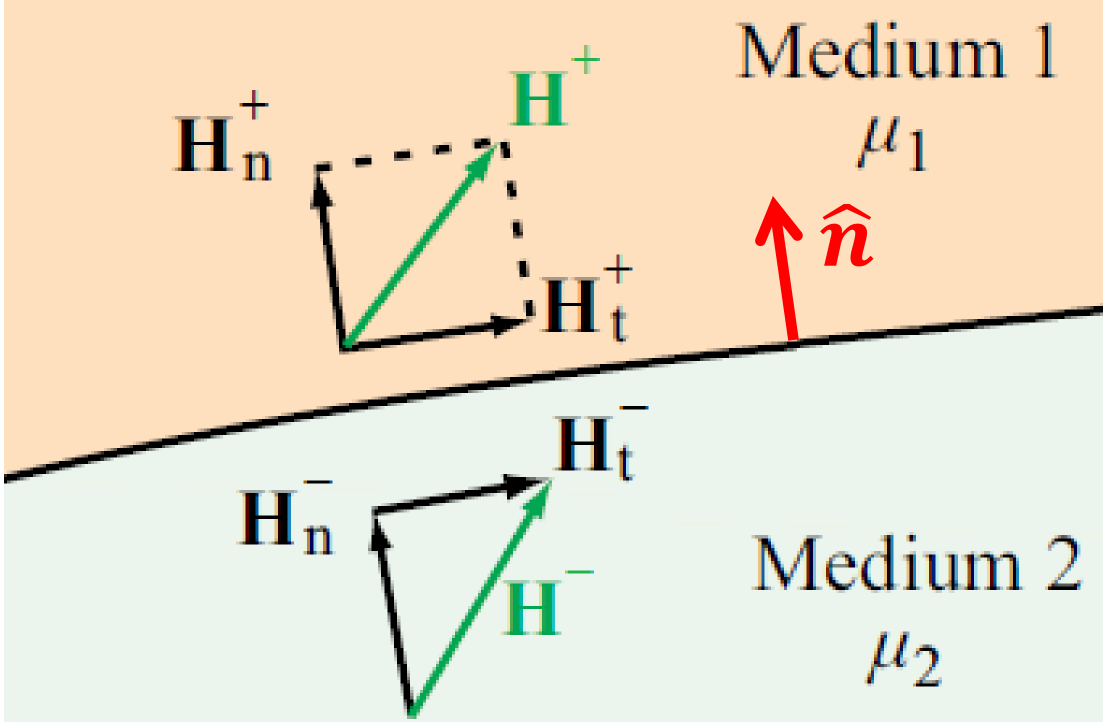
The same equation is satisfied by a current density of the form

$$\mathbf{J}_b = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}$$

since

$$\nabla \cdot \mathbf{J}_b = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} + \underbrace{\nabla \cdot (\nabla \times \mathbf{M})}_{=0} = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t}$$

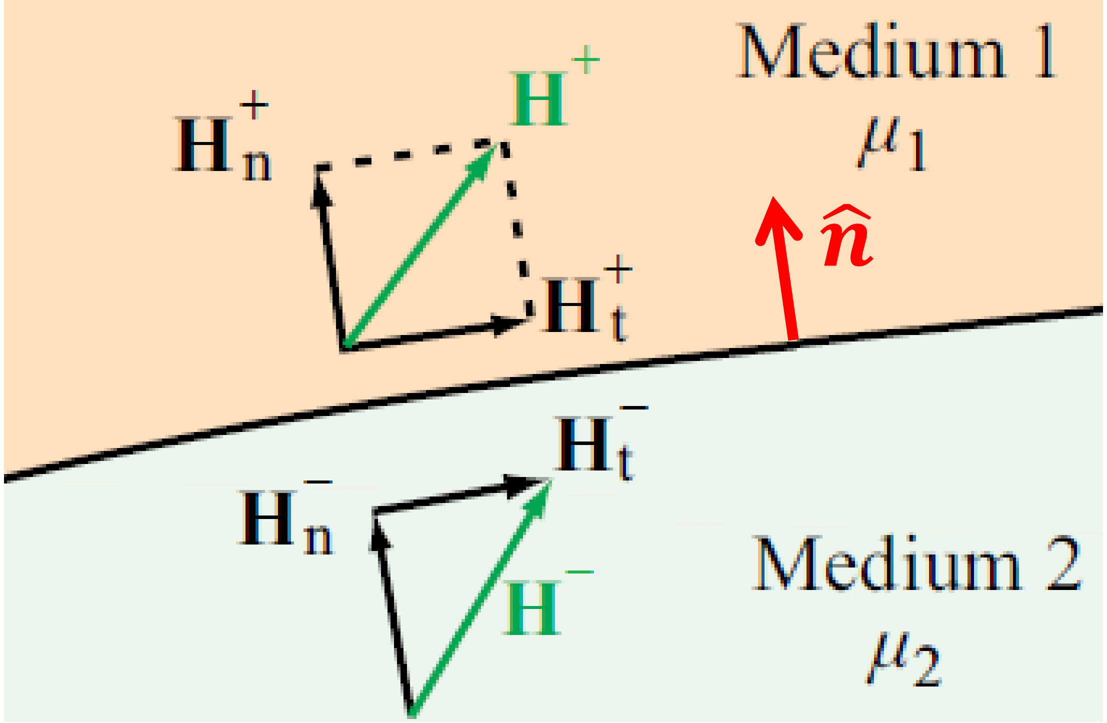
Magnetic Boundary Conditions



$$\boxed{B_n^+ = B_n^-}$$

$$\boxed{\mu_1 H_n^+ = \mu_2 H_n^-}$$

Magnetic Boundary Conditions



Surface current $\boxed{\mathbf{J}_s = 0}$

$$\boxed{H_t^+ = H_t^-}$$

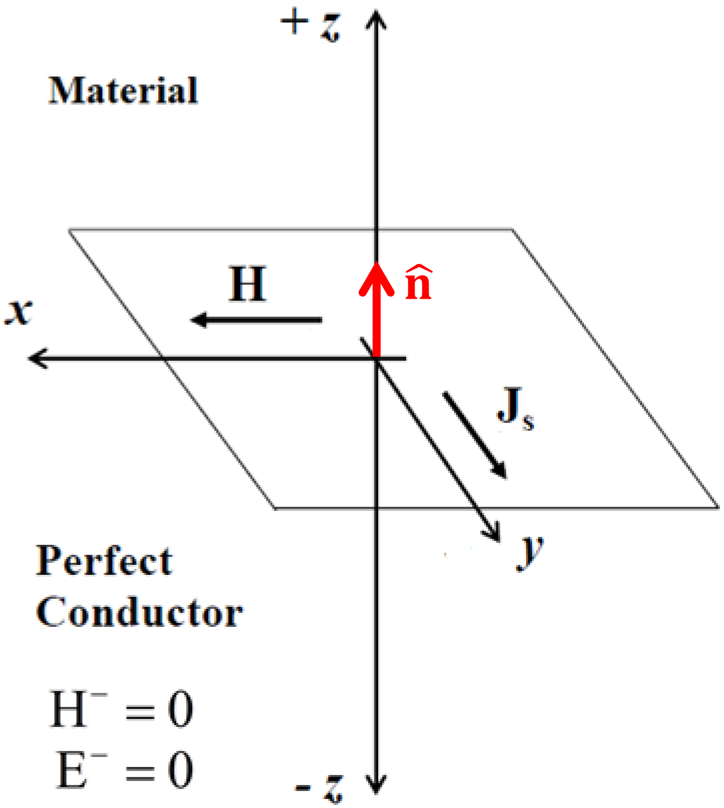
Surface current $\boxed{\mathbf{J}_s \neq 0}$

$$\boxed{\hat{\mathrm{n}} \times \left(\mathrm{H}^+ - \mathrm{H}^-\right) = \mathrm{J}_S}$$

Surface current $\boxed{\mathbf{J}_s \neq 0}$

$$\boxed{\hat{\mathrm{n}} \times \left(\mathrm{H}^{+} - \mathrm{H}^{-}\right) = \mathrm{J}_S}$$

This boundary condition in practice applies mainly to the cases of perfect conductor surface and sheet of current.



$$\mathrm{H}^{-} = 0$$
$$\mathrm{E}^{-} = 0$$

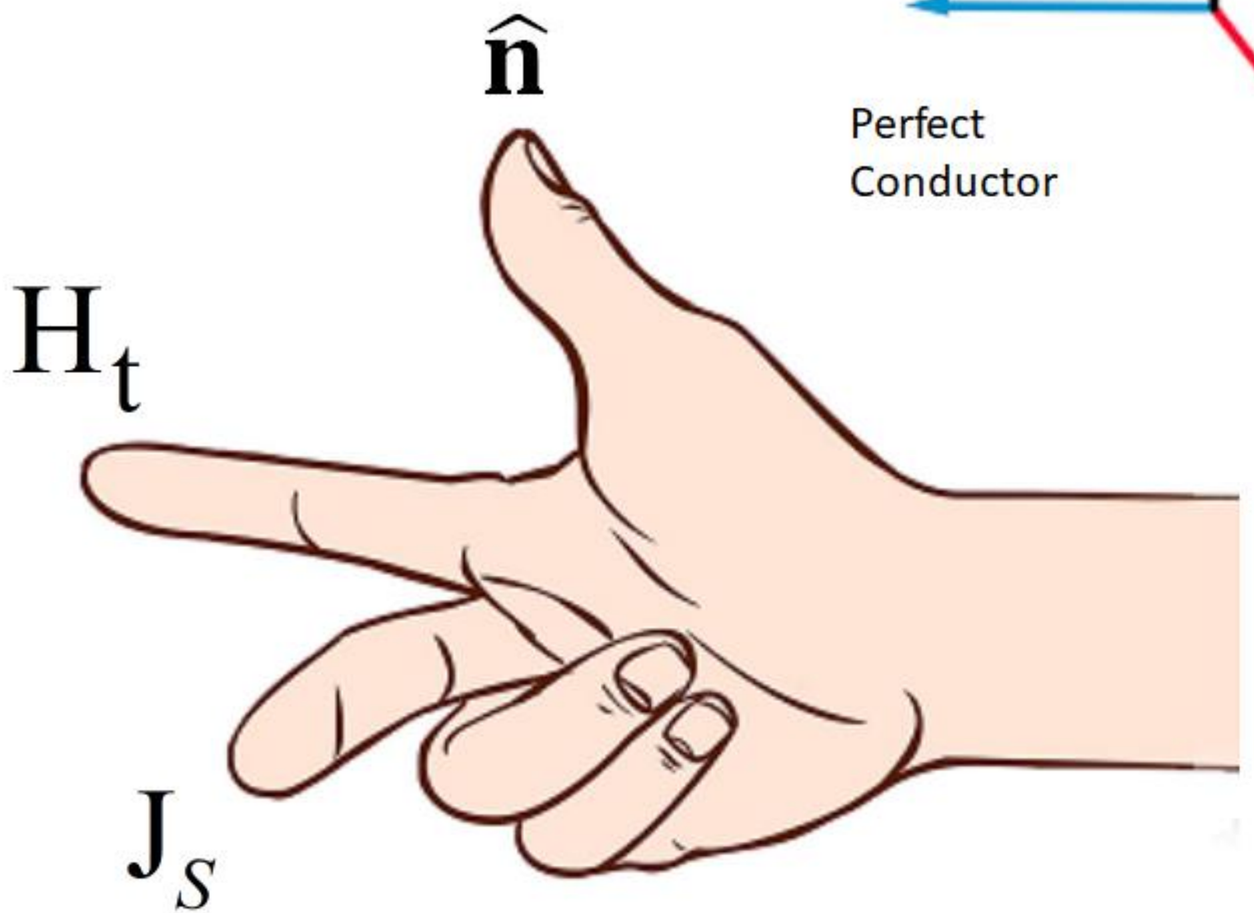
$$\mathbf{J}_s = J_{so} \cos(\omega t)\, \hat{\mathbf{y}}$$

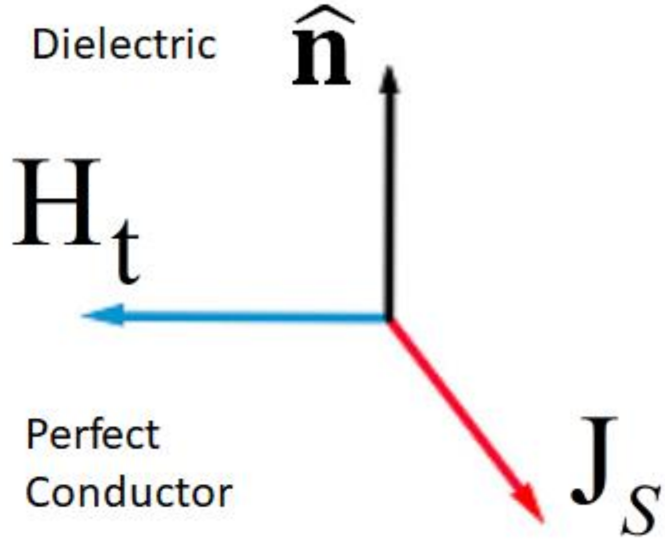
$$\hat{\mathrm{n}} \times \left(\mathrm{H}^{+} - \cancel{\mathrm{H}^{-}}\right) = \mathrm{J}_S$$

$$\hat{z} \times H_x^{+} = \mathrm{J}_S\ \hat{\mathbf{y}}$$

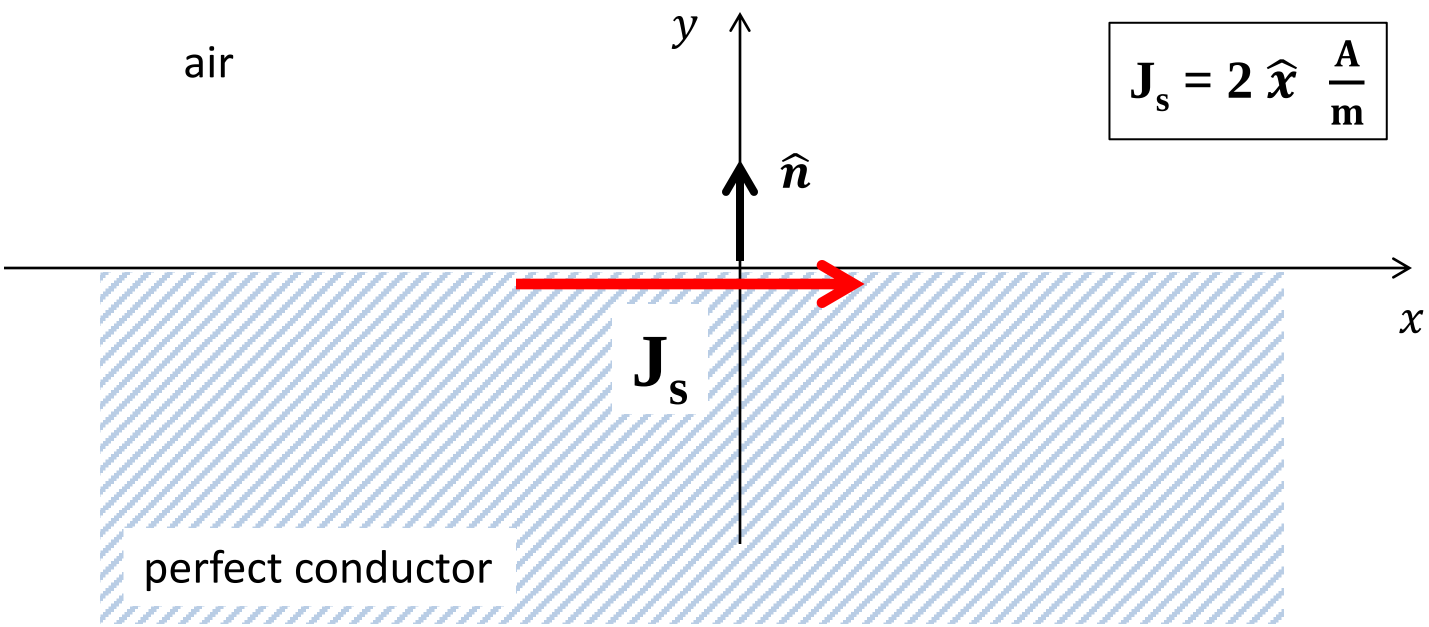
On a perfect conductor surface, the tangent electric field and the normal magnetic field are always <u>zero</u>.

Practice this Right Hand Rule



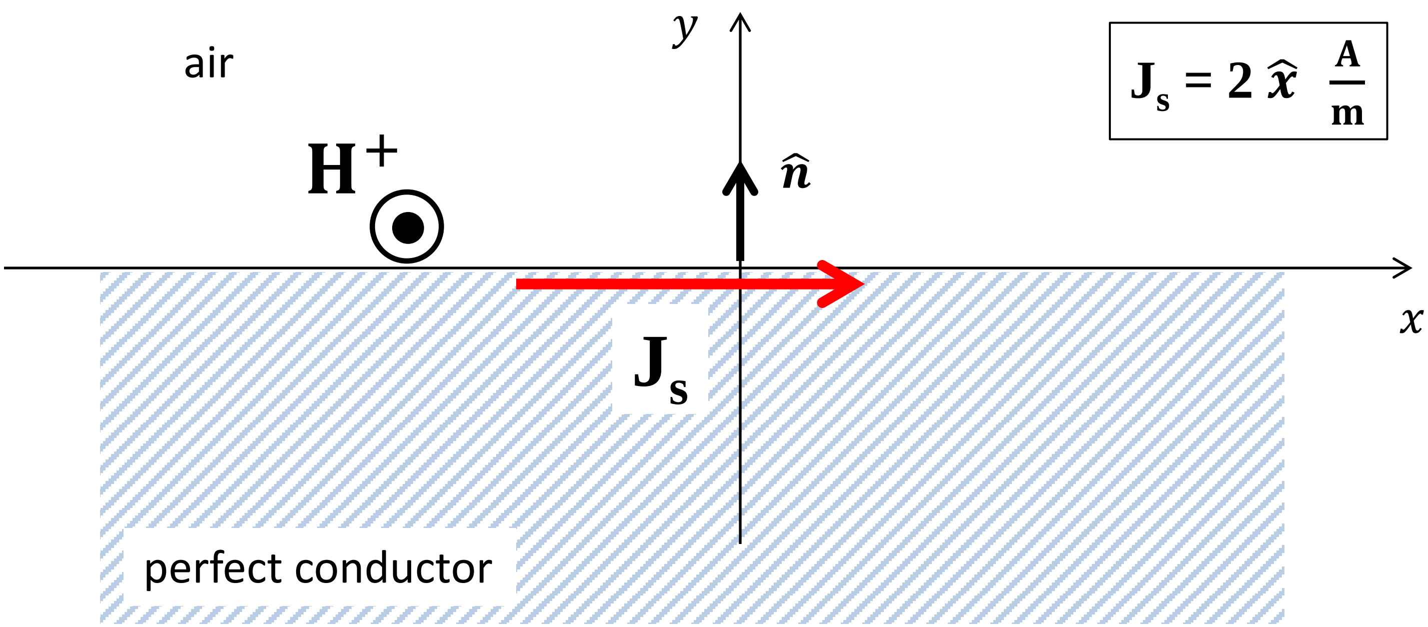


Consider a surface current density on a perfect conductor



What is the tangential magnetic field $\mathbf{H}^{+}$ in air just above the perfect conductor surface?

Consider a surface current density on a perfect conductor



What is the tangential magnetic field $\mathbf{H}^+$ in air just above the perfect conductor surface?

$$\hat{\boldsymbol{n}} \times (\mathbf{H}^+ - \cancel{\mathbf{H}^-}) = 2\hat{\boldsymbol{x}}$$

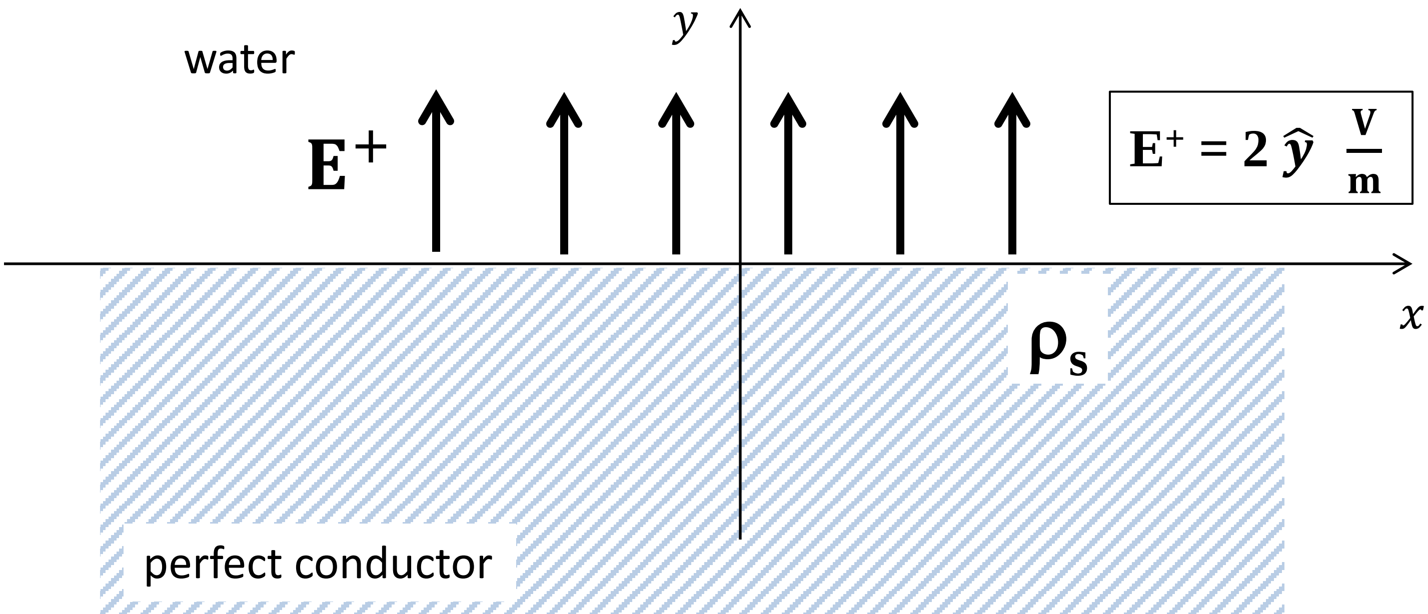
$$\hat{\boldsymbol{y}} \times \mathbf{H}_t^+ = 2\hat{\boldsymbol{x}}$$

$$\mathbf{H}^- = 0 \quad \textbf{(perfect conductor)}$$

$$\mathbf{H}_n^+ = 0$$

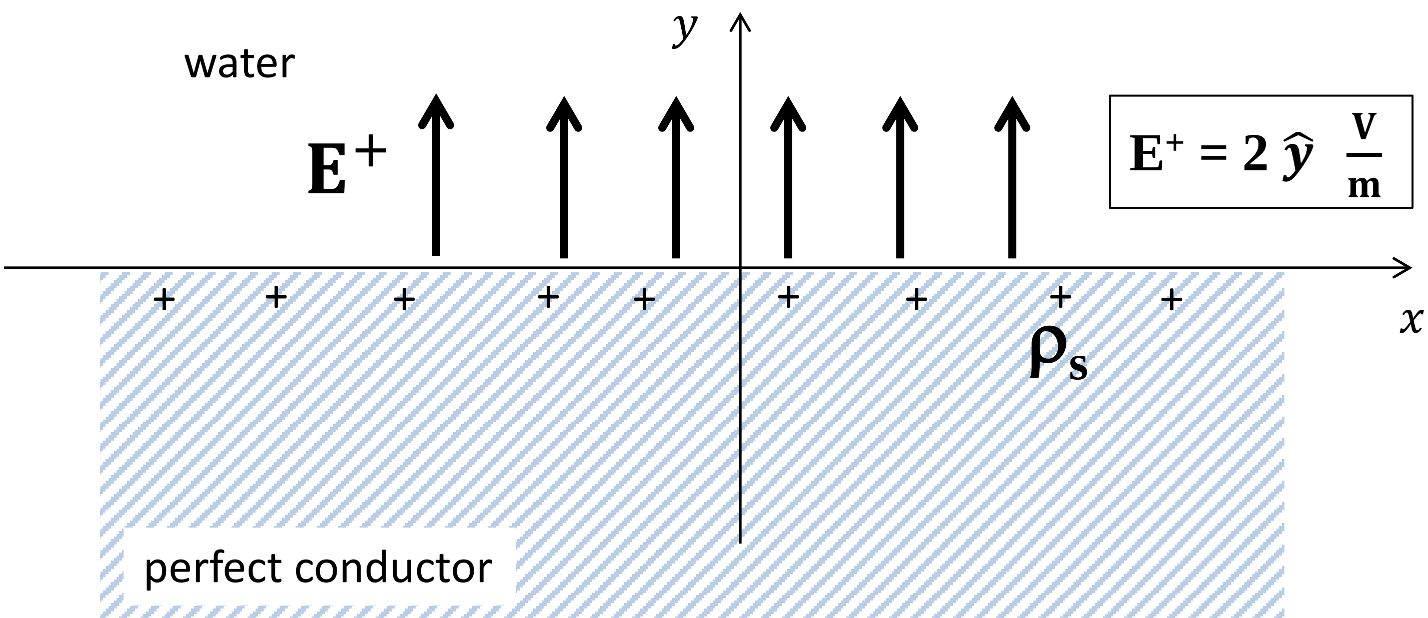
$$\boxed{\mathbf{H}_t^+ = 2\hat{\boldsymbol{z}}}$$

The electric field on the surface of a perfect conductor is 2 V/m.



What is the surface charge density on the perfect conductor?

The electric field on the surface of a perfect conductor is 2 V/m.



What is the surface charge density on the perfect conductor?

$$\rho_s = D_n^+ = \epsilon_{water} E_n^+$$

$$\rho_s = 80 \times 8.854 \times 10^{-12} \times 2$$

$$\boxed{\rho_s = 1.417 \times 10^{-9} \ \frac{C}{m^2}}$$

$$E^- = 0 \quad \text{(perfect conductor)}$$
$$D_n^- = 0$$
$$E_t^+ = 0$$

Electromagnetic Waves

Once again, here is the full set of Maxwell's equations

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho && \text{Gauss's law} \\
\nabla \cdot \mathbf{B} &= 0 && \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && \text{Faraday's law} \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} && \text{Ampere's law}
\end{aligned}$$

with constitutive equations

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \qquad \mathbf{B} = \mu \mathbf{H}$$

We will consider homogeneous and non-conducting media (that is, constant permittivity ε and constant permeability μ, zero conductivity σ)

Solutions of time-varying Maxwell's Equations

In the static regimes, static distributions of charge ρ are the sources of electrostatic fields and constant current densities $\mathbf{J}$ are the sources of magnetostatic fields.

In the time-varying regime, distributions of time-varying charge ρ and current density $\mathbf{J}$ are the sources of time-varying electromagnetic fields. These charges and current densities obey the continuity equation

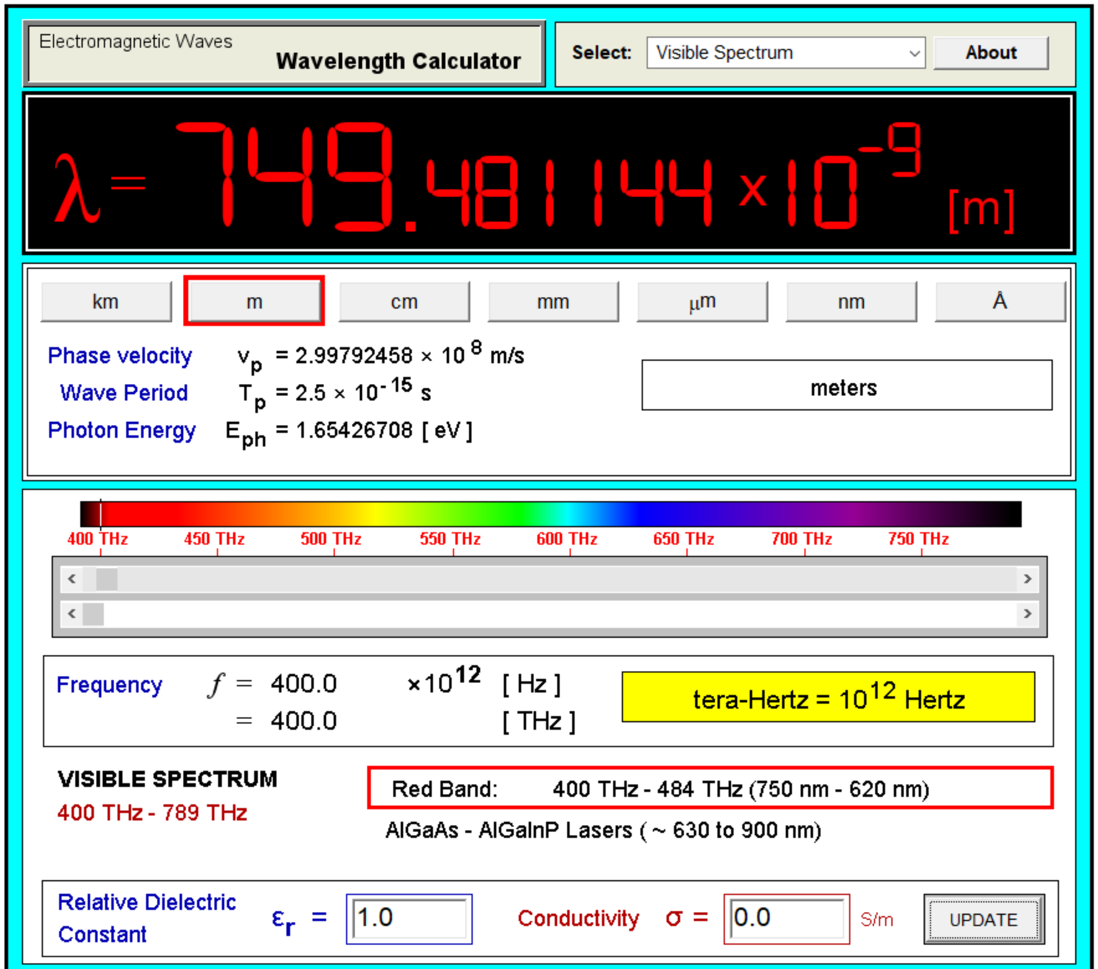
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Solutions of the time-varying problem exist away from the sources, in the so-called "source-free medium" where

$$\rho = \mathbf{J} = 0.$$

Solutions of time-varying Maxwell's Equations are in the form of "electromagnetic waves".

These waves exist in a very broad "spectrum" of frequencies. Explore the spectrum with the Wavelength Calculator application:



Source free homogeneous medium with

$$\boxed{\begin{aligned}\epsilon &= \text{constant}\\ \mu &= \text{constant}\end{aligned}} \qquad \boxed{\rho = \mathbf{J} = 0.}$$

Maxwell's equations simplify to

$$\left.\begin{aligned}\nabla \cdot \mathbf{E} &= 0\\ \nabla \cdot \mathbf{H} &= 0\end{aligned}\right\} \quad \text{Divergence free}$$

$$\left.\begin{aligned}\nabla \times \mathbf{E} &= -\mu\frac{\partial \mathbf{H}}{\partial t}\\ \nabla \times \mathbf{H} &= \epsilon\frac{\partial \mathbf{E}}{\partial t}\end{aligned}\right\} \quad \text{Curls} \neq 0 \text{ couple fields into each other}$$

Curl coupled equations are not easy to solve analytically, so we seek a different form. We observe that

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\cancel{\nabla \cdot \mathbf{E}}) - \nabla^2 \mathbf{E}$$

$$\boxed{\nabla \cdot \mathbf{E} = 0}$$

$$\nabla \times [\nabla \times \mathbf{E}] = \nabla \times [-\mu \frac{\partial \mathbf{H}}{\partial t}]$$

$$\boxed{-\nabla^2 \mathbf{E} = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H}}$$

$$\boxed{-\nabla^2\mathbf{E} = -\mu\frac{\partial}{\partial t}\nabla\times\mathbf{H}}$$

Substitute the curl of H with the right-hand side of Ampere's law

$$-\nabla^2\mathbf{E} = -\mu\frac{\partial}{\partial t}\epsilon\frac{\partial\mathbf{E}}{\partial t}$$

We obtain a form of the *D'Alembert* wave equation

$$\boxed{\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2}}$$

$$\boxed{\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}}$$

3D vector wave equation

Written explicitly:

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

We look for an x-polarized solution propagating along z

$$\mathbf{E} = \hat{x} E_x(z, t)$$

$$\boxed{\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2}}$$

1D scalar wave equation

$$\boxed{\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2}}$$

This equation is satisfied by elementary solutions:

$$E_x = \cos(\omega(t - \sqrt{\mu\epsilon} z))$$

$$E_x = \cos(\omega(t + \sqrt{\mu\epsilon} z))$$

which represen x-polarized periodic field solutions with oscillation angular frequency ω. We can write them as

$$\boxed{E_x = \cos(\omega(t \mp \frac{z}{v}))}$$

where $v \equiv \frac{1}{\sqrt{\mu\epsilon}}$ **has dimensions of velocity (m/s)**

We can obtain the most standard wave representation:

$$\boxed{E_x = \cos(\omega(t - \sqrt{\mu\epsilon} z))}$$

$$E_x = \cos((\omega t - \omega\sqrt{\mu\epsilon}\, z))$$

$$= \cos((\omega t - 2\pi f \sqrt{\mu\epsilon}\, z))$$

$$= \cos((\omega t - 2\pi \frac{f}{v}\, z))$$

$$= \cos((\omega t - \frac{2\pi}{\lambda}\, z))$$

$$= \cos((\omega t - \beta z))$$

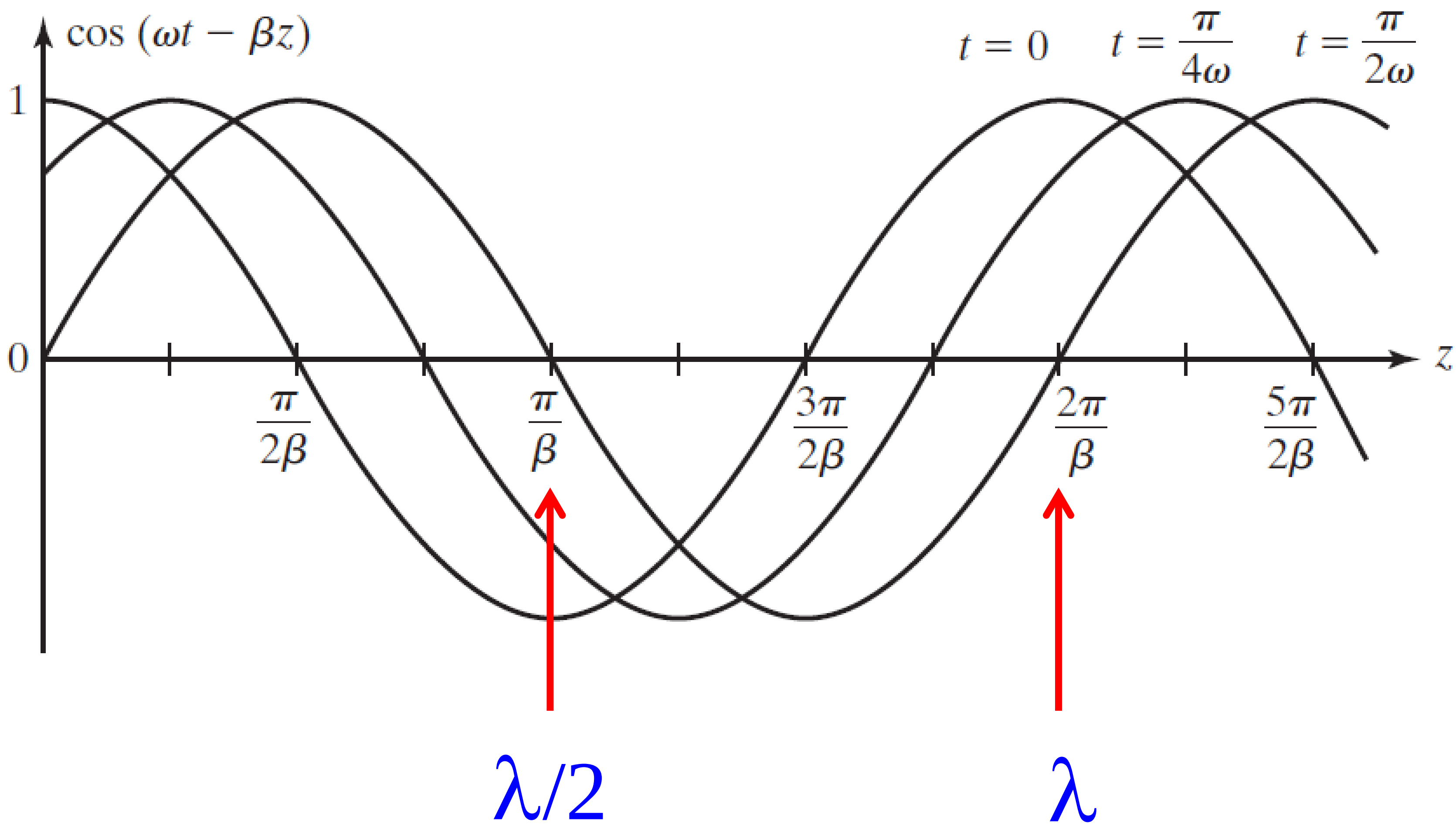
wavelength $\boxed{\lambda \equiv \frac{v}{f}}$

wave number $\boxed{\beta \equiv \frac{2\pi}{\lambda}}$

propagation in space at different times steps

$$\cos(\omega t - \beta z)$$

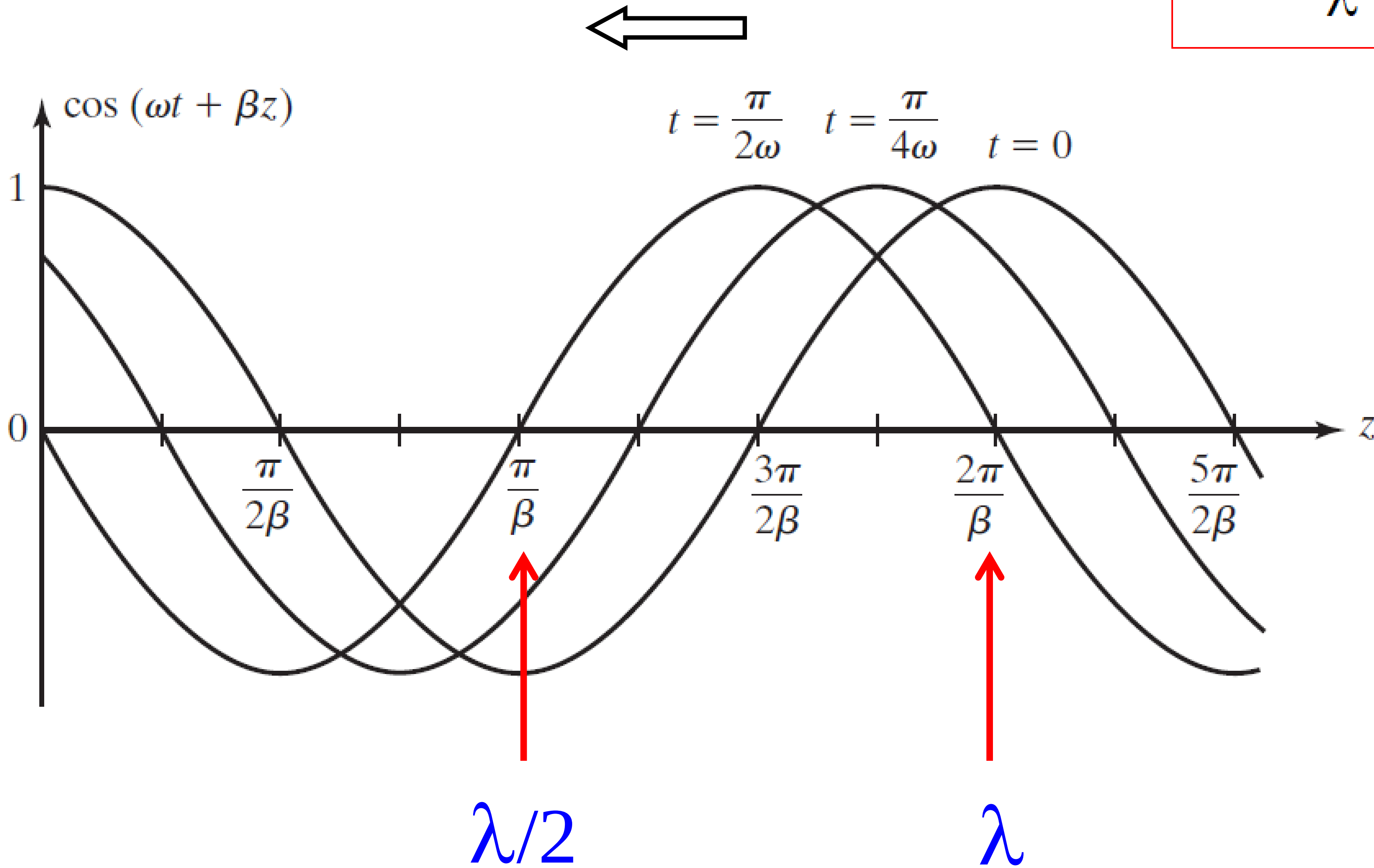
$$\boxed{\beta \equiv \frac{2\pi}{\lambda}}$$



propagation in space at different times steps

$$\cos(\omega t + \beta z)$$

$$\boxed{\beta \equiv \frac{2\pi}{\lambda}}$$



The magnetic field H corresponding to

$$\boxed{E_x = \cos(\omega(t \mp \frac{z}{v}))}$$

is obtained by taking the curl

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \hat{y}\frac{\partial E_x}{\partial z} = \pm \hat{y} \sin(\omega(t \mp \frac{z}{v}))\frac{\omega}{v}$$

From Faraday's law $$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$-\mu \frac{\partial \mathbf{H}}{\partial t} = \pm \hat{y} \sin(\omega(t \mp \frac{z}{v}))\frac{\omega}{v}$$

From

$$-\mu\frac{\partial \mathbf{H}}{\partial t} = \pm\hat{y}\sin(\omega(t \mp \frac{z}{v}))\frac{\omega}{v}$$

we can easily obtain, using also

$$v \equiv \frac{1}{\sqrt{\mu\epsilon}}$$

$$\boxed{\mathbf{H} = \pm\hat{y}\sqrt{\frac{\epsilon}{\mu}}\cos(\omega(t \mp \frac{z}{v}))}$$

In compact form

$$\mathbf{E} = \hat{x} f(t \mp \frac{z}{v})$$

$$\mathbf{H} = \pm\hat{y}\frac{f(t \mp \frac{z}{v})}{\eta}$$

where the field waveform is

$$f(t) \equiv \cos(\omega t) = \mathrm{Re}\{e^{j\omega t}\} = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

and we have the intrinsic impedance

$$\eta \equiv \sqrt{\frac{\mu}{\epsilon}} \quad [\,\Omega\,]$$

The field solution can be generalized by taking a superposition of possible solutions, following Fourier analysis (see ECE 210)

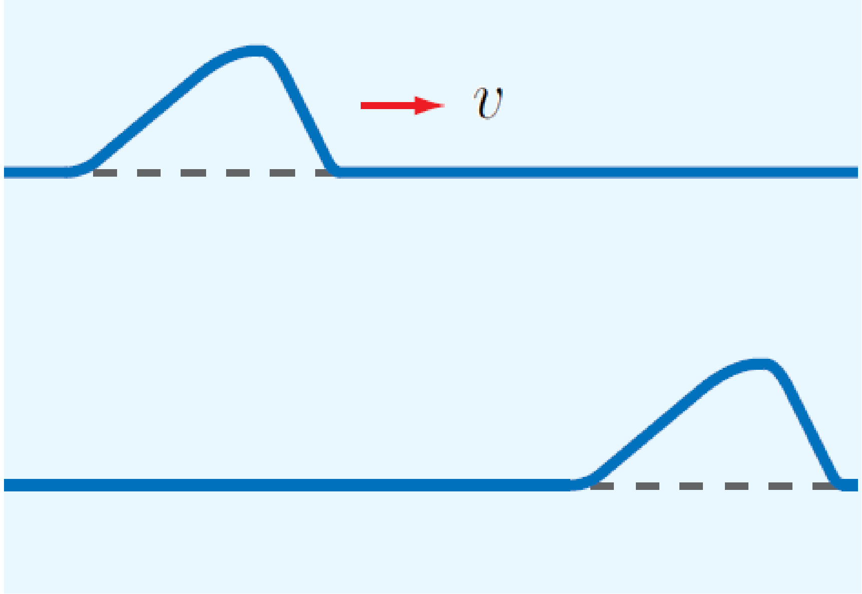
$$f(t) = \sum_n A_n \cos(\omega_n t + \theta_n)$$

where

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Essentially, all practical signal can be represented in this way, meaning that the field solution obtained are valid for any arbitrary waveform $f(t)$

For unbounded propagation without attenuation (no energy loss), wave components at different frequencies travel with the same velocity (the speed of light in that medium). This means that waveforms propagate undistorted.



This is not necessarily true when waveforms propagate in confined structures or in lossy media, experiencing *dispersion* with components travelling at different speeds.

Solutions of the 1D scalar wave equation with arbitrary $f(t)$

$$\mathbf{E},\ \mathbf{H} \propto f(t \mp \frac{z}{v})$$

are known as *D'Alembert* wave solutions

$$\mathbf{E},\ \mathbf{H} \propto f(t - \frac{z}{v}) \qquad +z \text{ direction} \longrightarrow$$

$$\mathbf{E},\ \mathbf{H} \propto f(t + \frac{z}{v}) \qquad -z \text{ direction} \longleftarrow$$

The travel speed in both cases is the speed of light

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

in free space: $\dfrac{1}{\sqrt{\mu_o \epsilon_o}} \equiv c \approx 3 \times 10^8 \,\mathrm{m/s}$

The cross product

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$$

is a vector in the direction of propagation called *Poynting vector*.

The intrinsic impedance for free space is

$$\sqrt{\frac{\mu_o}{\epsilon_o}} \equiv \eta_o \approx 120\pi \quad [\,\Omega\,]$$

The currently accepted value is **376.730313668(57) Ω** (determined experimentally since now the free space speed of light has an exact value in the SI system)

For a wave travelling along z, the electric field can be polarized in any direction of the (x, y) plane. For instance, we can have the following y-polarized wave solutions

$$\mathbf{E} = \hat{y} f(t \mp \frac{z}{v})$$

$$\mathbf{H} = \mp \hat{x} \frac{f(t \mp \frac{z}{v})}{\eta}$$

However, there cannot be any z-polarized wave solutions in this case, otherwise the divergence-free condition would be violated. Uniform plane wave solutions can only be Transverse Electromagnetic (TEM) waves, with E and H <u>transverse</u> to the direction of propagation.

Plane Wave propagation Java App.