Lecture 19 – Outline

• Wave Equation solutions
• TEM Waves
• Current sheets and plane wave generation

Reading assignment
Prof. Kudeki’s ECE 329 Lecture Notes on Fields and Waves:
19) d’Alembert wave solutions, radiation from current sheets
1-D Wave Equation

\[
\frac{\partial^2 E_x}{\partial z^2} = \mu \epsilon \frac{\partial^2 E_x}{\partial t^2}
\]

This equation is satisfied by elementary solutions:

\[
E_x = \cos(\omega(t - \sqrt{\mu \epsilon}z))
\]

\[
E_x = \cos(\omega(t + \sqrt{\mu \epsilon}z))
\]

which represent \( x \)-polarized periodic field solutions with oscillation angular frequency \( \omega \). We can write them as

\[
E_x = \cos(\omega(t \mp \frac{z}{v}))
\]

where

\[
v \equiv \frac{1}{\sqrt{\mu \epsilon}}
\]

has dimensions of velocity (m/s).
propagation in space at different times steps

\[ \cos(\omega t - \beta z) \]

\[ \beta \equiv \frac{2\pi}{\lambda} \]

\[ t = 0 \quad t = \frac{\pi}{4\omega} \quad t = \frac{\pi}{2\omega} \]
propagation in space at different times steps

\[ \cos(\omega t + \beta z) \]

\[ \beta \equiv \frac{2\pi}{\lambda} \]

- \( t = \frac{\pi}{2\omega} \)
- \( t = \frac{\pi}{4\omega} \)
- \( t = 0 \)

\( \lambda/2 \)
\( \lambda \)
The magnetic field $H$ corresponding to

$$E_x = \cos(\omega(t \mp \frac{z}{v}))$$

is obtained by taking the curl

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \hat{y} \frac{\partial E_x}{\partial z} = \pm \hat{y} \sin(\omega(t \mp \frac{z}{v})) \frac{\omega}{v}$$

From Faraday’s law

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$-\mu \frac{\partial \mathbf{H}}{\partial t} = \pm \hat{y} \sin(\omega(t \mp \frac{z}{v})) \frac{\omega}{v}$$
From

\[-\mu \frac{\partial H}{\partial t} = \pm \hat{y} \sin(\omega(t \mp \frac{z}{v})) \frac{\omega}{v}\]

we can easily obtain, using also

\[v \equiv \frac{1}{\sqrt{\mu \varepsilon}}\]

\[\frac{1}{\mu} \frac{1}{v} = \frac{\sqrt{\mu \varepsilon}}{\mu} = \sqrt{\frac{\varepsilon}{\mu}}\]

\[H = \pm \hat{y} \sqrt{\frac{\varepsilon}{\mu}} \cos(\omega(t \mp \frac{z}{v}))\]
In compact form

\[ E = \hat{x} f(t \mp \frac{z}{v}) \]

\[ H = \pm \hat{y} \frac{f(t \mp \frac{z}{v})}{\eta} \]

where the field waveform is

\[ f(t) \equiv \cos(\omega t) = \text{Re}\{e^{j\omega t}\} = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \]

and we have the intrinsic impedance

\[ \eta \equiv \sqrt{\frac{\mu}{\epsilon}} \quad [\Omega] \]
The field solution can be generalized by taking a superposition of possible solutions, following Fourier analysis (see ECE 210)

\[ f(t) = \sum_n A_n \cos(\omega_n t + \theta_n) \]

where

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \]

Essentially, all practical signal can be represented in this way, meaning that the field solution obtained are valid for any arbitrary waveform

\[ f(t) \]
For unbounded propagation without attenuation (no energy loss), wave components at different frequencies travel with the same velocity (the speed of light in that medium). This means that waveforms propagate undistorted.

This is not necessarily true when waveforms propagate in confined structures or in lossy media, experiencing dispersion with components travelling at different speeds.
Solutions of the 1D scalar wave equation with arbitrary $f(t)$

\[ E, \ H \propto f(t \mp \frac{\vec{z}}{v}) \]

are known as \textit{D’Alembert wave solutions}

\[ E, \ H \propto f(t - \frac{\vec{z}}{v}) \quad \text{+z direction} \]

\[ E, \ H \propto f(t + \frac{\vec{z}}{v}) \quad \text{−z direction} \]

The travel speed in both cases is the speed of light in free space:

\[ v = \frac{1}{\sqrt{\mu \varepsilon}} \quad \text{in free space:} \quad \frac{1}{\sqrt{\mu_o \varepsilon_o}} \equiv c \approx 3 \times 10^8 \text{ m/s} \]
The cross product

\[ \mathbf{S} \equiv \mathbf{E} \times \mathbf{H} \]

is a vector in the direction of propagation called \textit{Poynting vector}.

The intrinsic impedance for free space is

\[ \sqrt{\frac{\mu_0}{\varepsilon_0}} \equiv \eta_0 \approx 120\pi \quad [\Omega] \]

The currently accepted value is \(376.730313668(57)\ \Omega\) (determined experimentally since now the free space speed of light has an exact value in the SI system)
For a wave travelling along $z$, the electric field can be polarized in any direction of the $(x, y)$ plane. For instance, with 90° rotation we can have the following $y$-polarized wave solutions

$$E = \hat{y} f(t \mp \frac{z}{v})$$

$$H = \mp \hat{x} \frac{f(t \mp \frac{z}{v})}{\eta}$$

All other possible orientations can be obtained by arbitrary rotation on the $(x, y)$ plane.
However, there cannot be any $z$-polarized wave solutions in this case, otherwise the divergence-free condition would be violated. Uniform plane wave solutions can only be **Transverse Electromagnetic (TEM)** waves, with $E$ and $H$ transverse to the direction of propagation.
Plane Wave propagation Java App.

Module 7.2 Plane Wave

E-field Amplitude (z=0)

Input:
- Frequency: $f = 1.0 \times 10^9$ Hz
- Conductivity: $\sigma = 0.0$ S/m
- Relative Permittivity: $\varepsilon_r = 1.0$
- Relative Permeability: $\mu_r = 1.0$
- E-field Amplitude (z=0): $E_0 = 1.0$ V/m
- E-field Phase (z=0): $\phi = 0.0$ rad
- Length: $l = 10.0 \lambda$

Output:
- Wave Length: $\lambda = 30.0$ cm
- Phase Velocity: $v_p = 3.0 \times 10^8$ m/s
- Period: $T = 1.0 \times 10^{-9}$ s
- Impedance of the Medium: $Z = 376.991118 + j0.0$
- Penetration (Skin) Depth: $\delta_s = \infty$
- Phase and Attenuation Constants: $\beta = 20.94395$ m$^{-1}$, $\alpha = 0.0$ Ne/m

The material is vacuum (perfect dielectric)
TEM Waves Summary

• We found that from Maxwell’s equations we can derive a wave equation. This means that electromagnetic fields in dynamic conditions exist as EM waves.

• For a 1D situation, with the electric field polarized along a specific direction, we write solutions in the form of a uniform plane wave

\[
\begin{align*}
E &= \hat{x} f\left(t - \frac{z}{v}\right) \\
H &= \hat{y} \frac{f\left(t - \frac{z}{v}\right)}{\eta} \\
E &= \hat{x} f\left(t + \frac{z}{v}\right) \\
H &= -\hat{y} \frac{f\left(t + \frac{z}{v}\right)}{\eta}
\end{align*}
\]

Forward wave  Backward wave
TEM Waves Summary

- The waves have constant field amplitude and phase on planes normal to the direction of propagation. These planes are called wave fronts and correspond to

\[ \frac{t - z}{v} = \text{const.} \]

- The cross product between \( E \) and \( H \) is called Poynting vector, aligned with the direction of propagation

\[ S \equiv E \times H \]

- \( E, H, \) and \( S \) are normal to each other with right-hand ordering and the waves are called Transverse ElectroMagnetic (TEM).

- The ratio between electric field and magnetic field is a constant in a given medium an it is called intrinsic impedance

\[ \eta \equiv \sqrt{\frac{\mu}{\varepsilon}} \quad [\Omega] \]
Right-Hand orientation of Forward and Backward Waves

\[
E = \hat{x}f(t + \frac{z}{v}) \\
E \times H
\]

\[
E = \hat{x}f(t - \frac{z}{v}) \\
E \times H
\]
Review Questions – Assume a generic waveform for the Electric Field representing propagation of an EM wave in free space.

\[ E(t) = \hat{x} f \left( t + \frac{y}{c} \right) \]

What is the polarization direction for the electric field?

What is the direction of propagation?

What is the speed of propagation?

What is the corresponding magnetic field \( \mathbf{H} \)?

What is the Poynting vector

What is the corresponding magnetic flux density vector \( \mathbf{B} \)?
Review Questions – Assume a generic waveform for the Electric Field representing propagation of an EM wave in free space.

\[ E(t) = \hat{x} f\left(t + \frac{y}{c}\right) \quad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \]

What is the polarization direction for the electric field?
The electric field is polarized along the \( x \)-axis

What is the direction of propagation?
The wave propagates along the negative \( y \)-direction

What is the speed of propagation?
The speed of light \( c \)
Review Questions – Assume a generic waveform for the Electric Field representing propagation of an EM wave in free space.

\[
E(t) = \hat{x} f(t + \frac{y}{c})
\]

\[
c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}
\]

What is the corresponding magnetic field \( \mathbf{H} \)?

Divide the electric field by \( \eta_0 \) and rotate the polarization axis so that \( \mathbf{E} \times \mathbf{H} \) is aligned with \(-\mathbf{y}\).

\[
H(t) = \hat{z} \frac{f(t + \frac{y}{c})}{\eta_0} = \hat{z} \frac{f(t + \frac{y}{c})}{\sqrt{\mu_0 / \varepsilon_0}}
\]
Review Questions – Assume a generic waveform for the Electric Field representing propagation of an EM wave in free space.

\[ E(t) = \hat{x} f(t + \frac{y}{c}) \]

\[ c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \]

What is the Poynting vector?

\[ E \times H = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_x & 0 & 0 \\ 0 & 0 & H_z \end{vmatrix} = -\hat{y} E_x H_z = -\hat{y} \frac{f \left( t + \frac{y}{c} \right)^2}{\eta_0} \]

What is the corresponding magnetic flux density vector \( \mathbf{B} \)?

\[ \mathbf{B} = \mu_0 \mathbf{H} = \hat{z} \sqrt{\mu_0 \varepsilon_0} f \left( t + \frac{y}{c} \right) = \hat{z} \left( t + \frac{y}{c} \right) \]
Sheet of constant current produces uniform magnetic field

\[ J_s = \hat{x} J_x \ A/m \]

\[ H(z) = \mp \hat{y} \frac{J_x}{2} \ A/m \] for \( z \geq 0 \)

No electric field is generated in the space surrounding the current sheet.
For a quasi-static time-dependent (low frequency) current

\[ J_x = J_x(t) \]

\[ \mathbf{H}(z, t) \approx \mp \hat{y} \frac{J_x(t)}{2} \text{ A/m for } z \geq 0 \]

*Approximation* valid close to the surface were the delay term \( \frac{z}{U} \) is negligible.
The exact field solution of Maxwell's equations valid for any location $z$ and any frequency is simply obtained by replacing

$$J_x(t) \rightarrow J_x(t \mp \frac{z}{v})$$

which gives

$$H(z, t) = \mp \hat{y} \frac{J_x(t \mp \frac{z}{v})}{2} \text{ A/m for } z \geq 0$$

There is also an electric field aligned consistently with

$$E \times H \quad \mp z$$

$$E(z, t) = -\hat{x} \frac{\eta}{2} J_x(t \mp \frac{z}{v}) \text{ V/m for } z \geq 0$$
Sheet of time-dependent current produces plane waves

\[ J_s = \hat{x} J_x(t) \]

NOTE: For specified boundary conditions, the solution of Maxwell’s equations is unique

\[ H(z, t) = \mp \hat{y} \frac{J_x(t \mp \frac{z}{v})}{2} \text{ A/m for } z \geq 0 \]

\[ E(z, t) = -\hat{x} \frac{\eta}{2} J_x(t \mp \frac{z}{v}) \text{ V/m for } z \geq 0 \]
\[ \mathbf{J}_s = \hat{x} f(t) \]

\[ \mathbf{E}^\pm = -\hat{x} \frac{\eta f(t \mp \frac{z}{v})}{2} \]

\[ \mathbf{H}^\pm = \mp \hat{y} \frac{f(t \mp \frac{z}{v})}{2} \]

- For any \( z \) the fields are proportional to the “delayed” value of the current
  \[ \mathbf{E}, \mathbf{H} \propto f(t \mp \frac{z}{v}) \]

- The reference directions of \( \mathbf{E} \) and \( \mathbf{J}_s \) oppose one another
- The electric field \( \mathbf{E} \) is tangent and continuous at the sheet surface (consistent with Boundary Condition)
- The magnetic field \( \mathbf{H} \) is discontinuous at the sheet surface by an amount equal to the current density (consistent with Boundary Condition)
Example

A current sheet on $z = 0$ surface is described by

$$
\mathbf{J}_s(t) = \hat{x} f(t), \quad \text{with} \quad f(t) = A t \, \text{rect}\left(\frac{t}{\tau}\right),
$$

where $\tau = 1 \, \mu s$ and $A = 2 \frac{A}{\mu\text{m}}$. Assuming that the current sheet is embedded in free space, construct the following plots:

(a) Radiated $H_y(z, t = 2\mu s)$ vs $z$,

(b) Radiated $E_x(z, t = 2\mu s)$ vs $z$. 

![Graph showing current density $J_x(t)$ vs time $t$]
How are the electric fields oriented on the two sides?
Electric Fields. How are the magnetic fields oriented on the two sides?
Magnetic fields
The complete diagram
3D Spatial Visualization Practice: Rotate by 180° about \(x\) in your head
The rotated diagram
\( \mathbf{J}_s(t) = \hat{x} f(t) \), with \( f(t) = A t \text{rect}(t/\tau) \)

\( \tau = 1 \mu s \quad A = 2 \frac{\text{A/m}}{\mu \text{s}} \)

(a) Radiated \( H_y(z, t = 2 \mu s) \) vs \( z \)

\[
H_y(z, 2\mu s) = \mp (2\mu \mp \frac{z}{c}) \text{rect}\left(\frac{2\mu \mp \frac{z}{c}}{1\mu} \right) \frac{A}{\text{m}} \quad \text{for} \quad z \geq 0
\]
$z = c t \approx 3 \times 10^8 t$
\( \mathbf{J}_s(t) = \hat{x} f(t), \) with \( f(t) = A t \text{rect}\left(\frac{t}{\tau}\right) \)

\( \tau = 1 \mu s \quad A = 2 \frac{A}{\mu m} \)

(b) Radiated \( E_x(z, t = 2 \mu s) \) vs \( z \)

\[
E_x(z, 2 \mu s) = -\left\{120\pi\right\} \left(2\mu \mp \frac{z}{c}\right) \text{rect}\left(\frac{2\mu \pm \frac{z}{c}}{1\mu}\right) \frac{V}{m} \quad \text{for} \quad z \geq 0
\]