ECE 329 – Fall 2022

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

Lecture 20

Lecture 20 – Outline

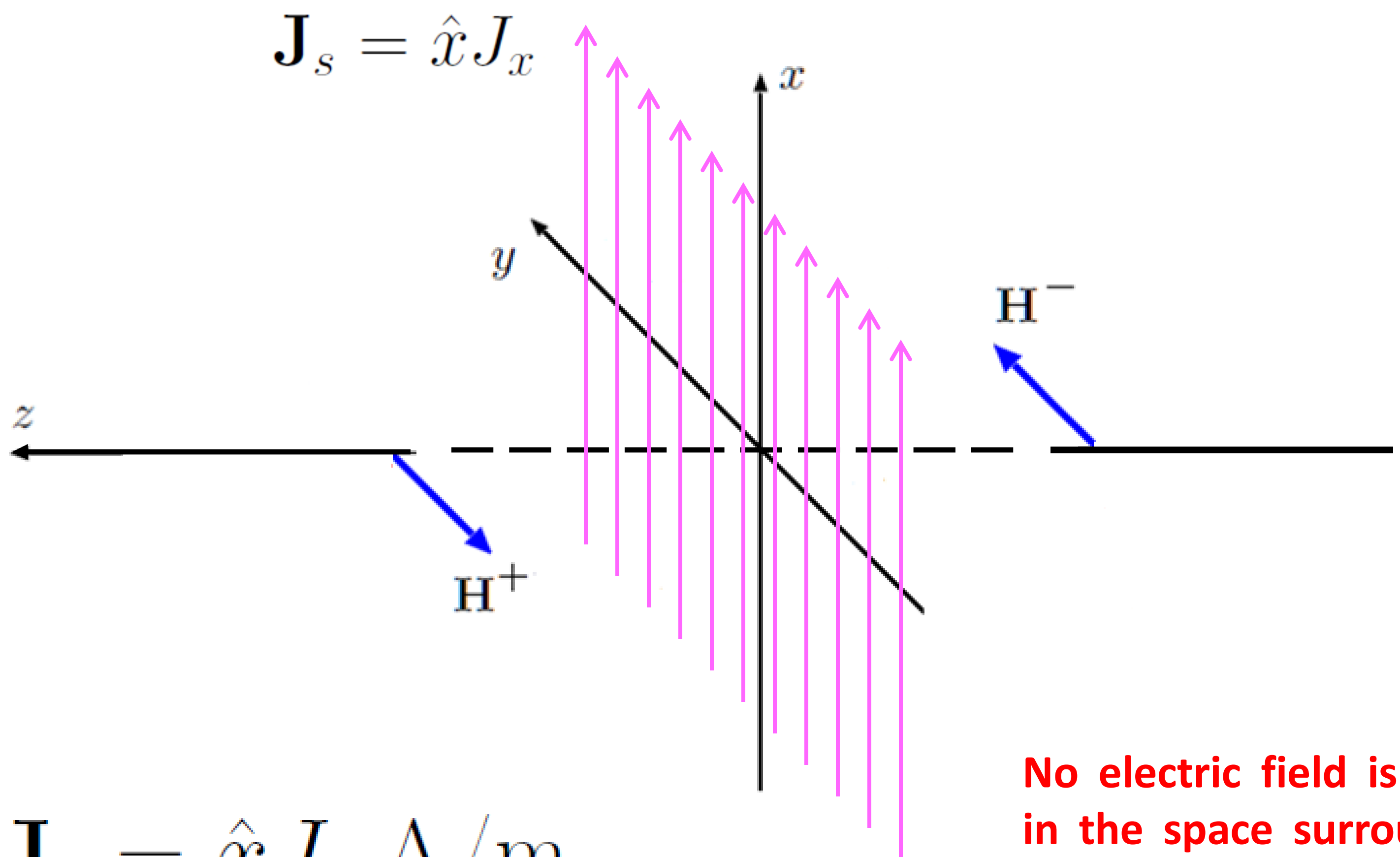
- **Infinite sheet of current**
- **Poynting vector as power flux carried by EM fields**
- **Poynting Theorem**
- **Time-harmonic source**
- **Monochromatic wave**

Reading assignment

Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:

20) Poynting theorem and monochromatic waves

Sheet of constant current produces uniform magnetic field

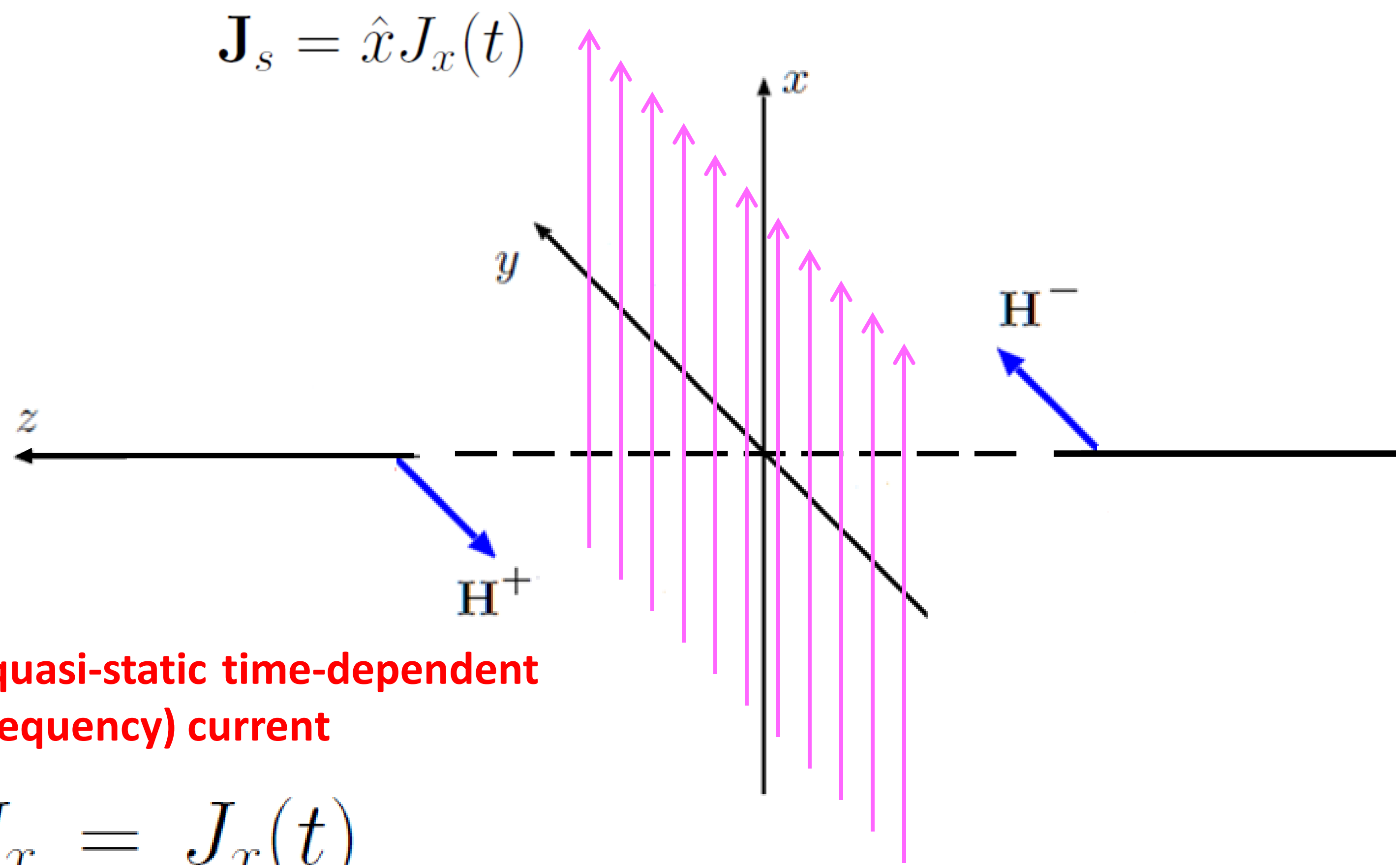


No electric field is generated in the space surrounding the current sheet.

$$\mathbf{J}_s = \hat{x} J_x \ \mathrm{A/m}$$

$$\mathbf{H}(z) = \mp \hat{y} \frac{J_x}{2} \ \mathrm{A/m} \text{ for } z \gtrless 0$$

Sheet of "quasi-static" current



For a quasi-static time-dependent (low frequency) current

$$J_x = J_x(t)$$

$$\mathbf{H}(z,t) \approx \mp \hat{y} \frac{J_x(t)}{2} \text{ A/m for } z \gtrless 0$$

Approximation valid **close to the surface** were the delay term $\frac{z}{v}$ is negligible.

The exact field solution of Maxwell's equations valid for any location z and any frequency is simply obtained by replacing

$$J_x(t) \longrightarrow J_x(t \mp \tfrac{z}{v})$$

which gives

$$\mathbf{H}(z,t) = \mp\hat{y}\frac{J_x(t \mp \frac{z}{v})}{2}\ \mathrm{A/m\ for}\ z \gtrless 0$$

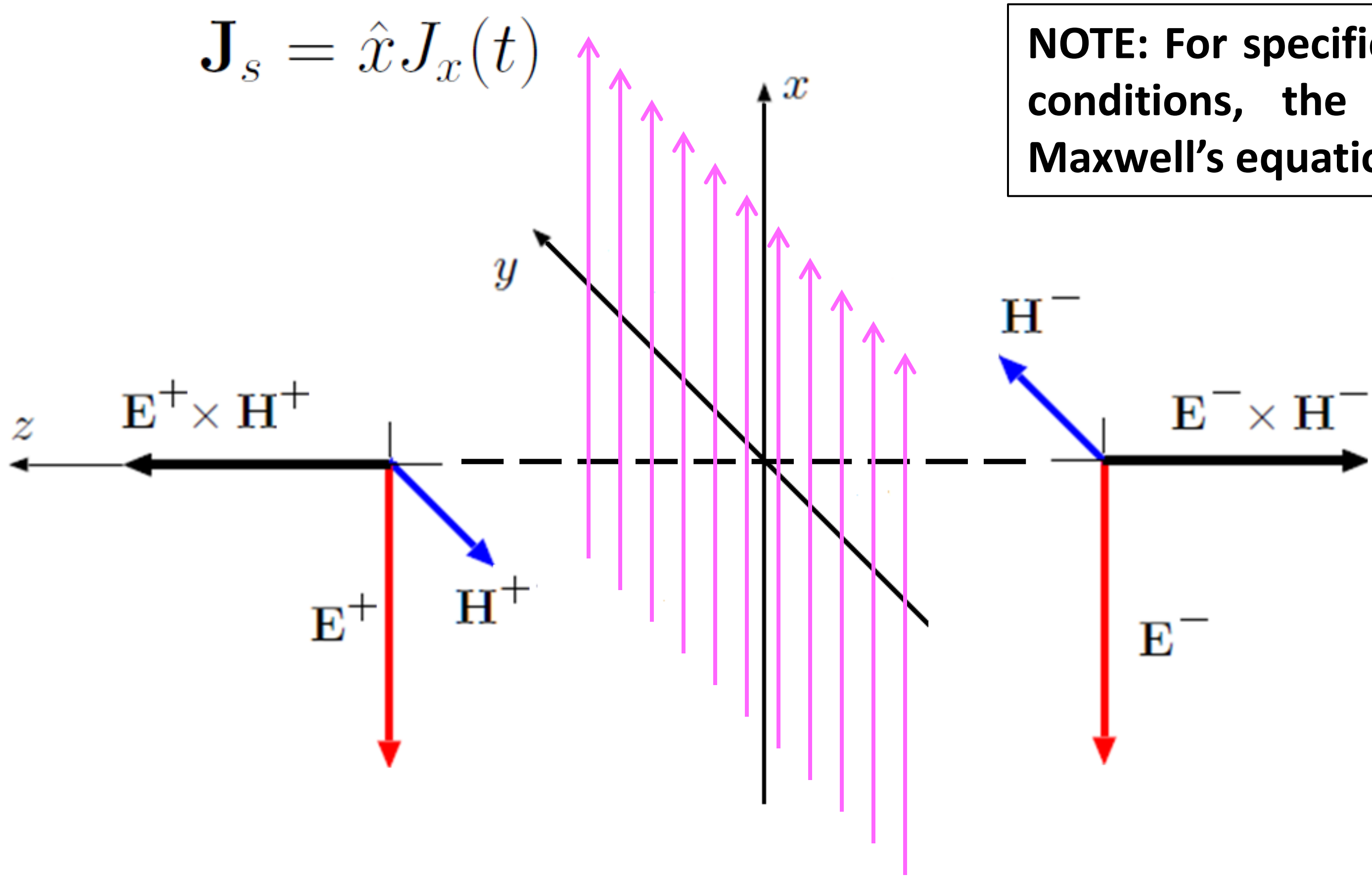
There is also an electric field aligned consistently with

$\mathbf{E}\times\mathbf{H}$ **along** $\mp z$

$$\mathbf{E}(z,t) = -\hat{x}\frac{\eta}{2}J_x(t \mp \frac{z}{v})\ \mathrm{V/m\ for}\ z \gtrless 0$$

Sheet of time-dependent current produces plane waves

NOTE: For specified boundary conditions, the solution of Maxwell's equations is unique



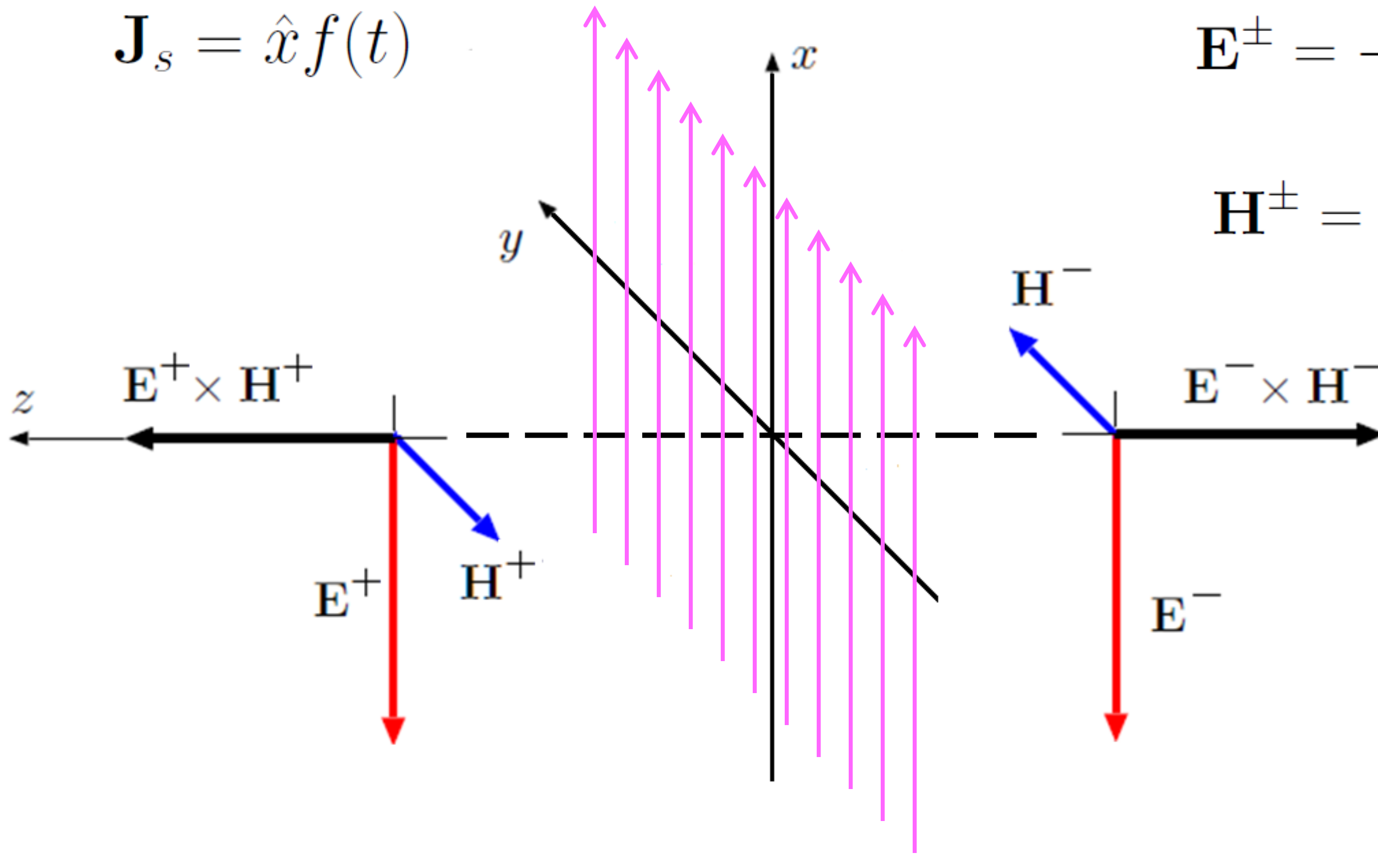
$$\mathbf{H}(z,t) = \mp \hat{y} \frac{J_x(t \mp \frac{z}{v})}{2} \ \mathrm{A/m} \text{ for } z \gtrless 0$$

$$\mathbf{E}(z,t) = -\hat{x} \frac{\eta}{2} J_x(t \mp \frac{z}{v}) \ \mathrm{V/m} \text{ for } z \gtrless 0$$

$$\mathbf{J}_s = \hat{x} f(t)$$

$$\mathbf{E}^{\pm} = -\hat{x}\,\frac{\eta f(t \mp \frac{z}{v})}{2}$$

$$\mathbf{H}^{\pm} = \mp\hat{y}\,\frac{f(t \mp \frac{z}{v})}{2}$$



- For any z the fields are proportional to the "delayed" value of the current

$$\mathbf{E},\, \mathbf{H} \propto f(t \mp \frac{z}{v})$$

- The reference directions of E and J_S oppose one another
- The electric field E is tangent and continuous at the sheet surface (consistent with Boundary Condition)
- The magnetic field H is discontinuous at the sheet surface by an amount equal to the current density (consistent with Boundary Condition)

Example

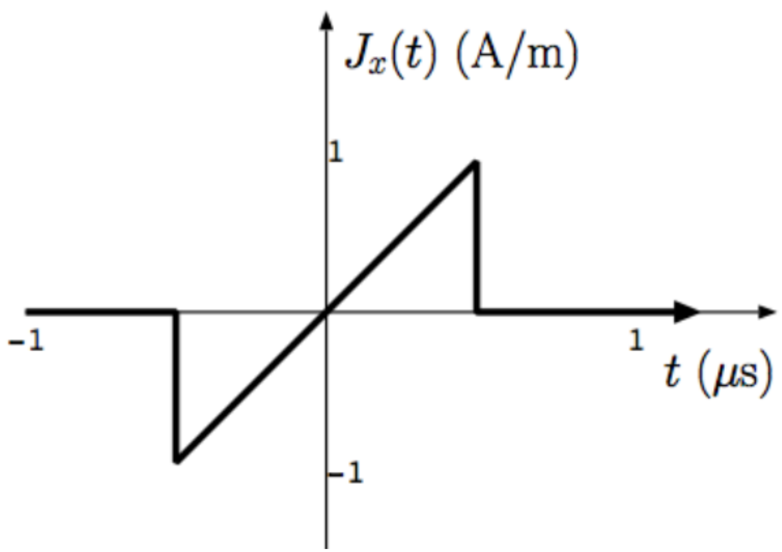
A current sheet on $z = 0$ surface is described by

$$\mathbf{J}_s(t) = \hat{x} f(t), \quad \text{with} \quad f(t) = At\,\mathrm{rect}(\frac{t}{\tau}),$$

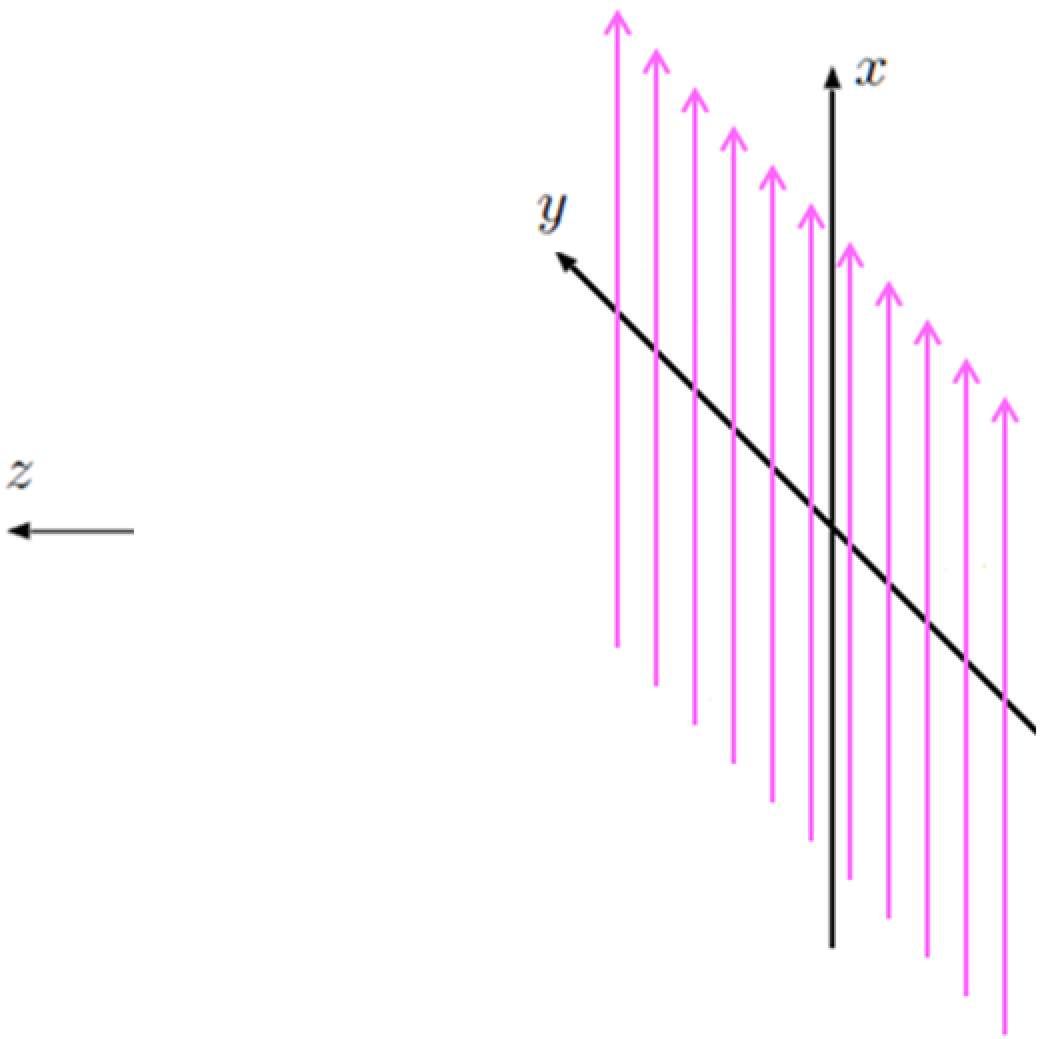
where $\tau = 1\,\mu\text{s}$ and $A = 2\,\frac{\text{A/m}}{\mu\text{S}}$. Assuming that the current sheet is embedded in free space, construct the following plots:

(a) Radiated $H_y(z, t = 2\mu\text{s})$ vs z,

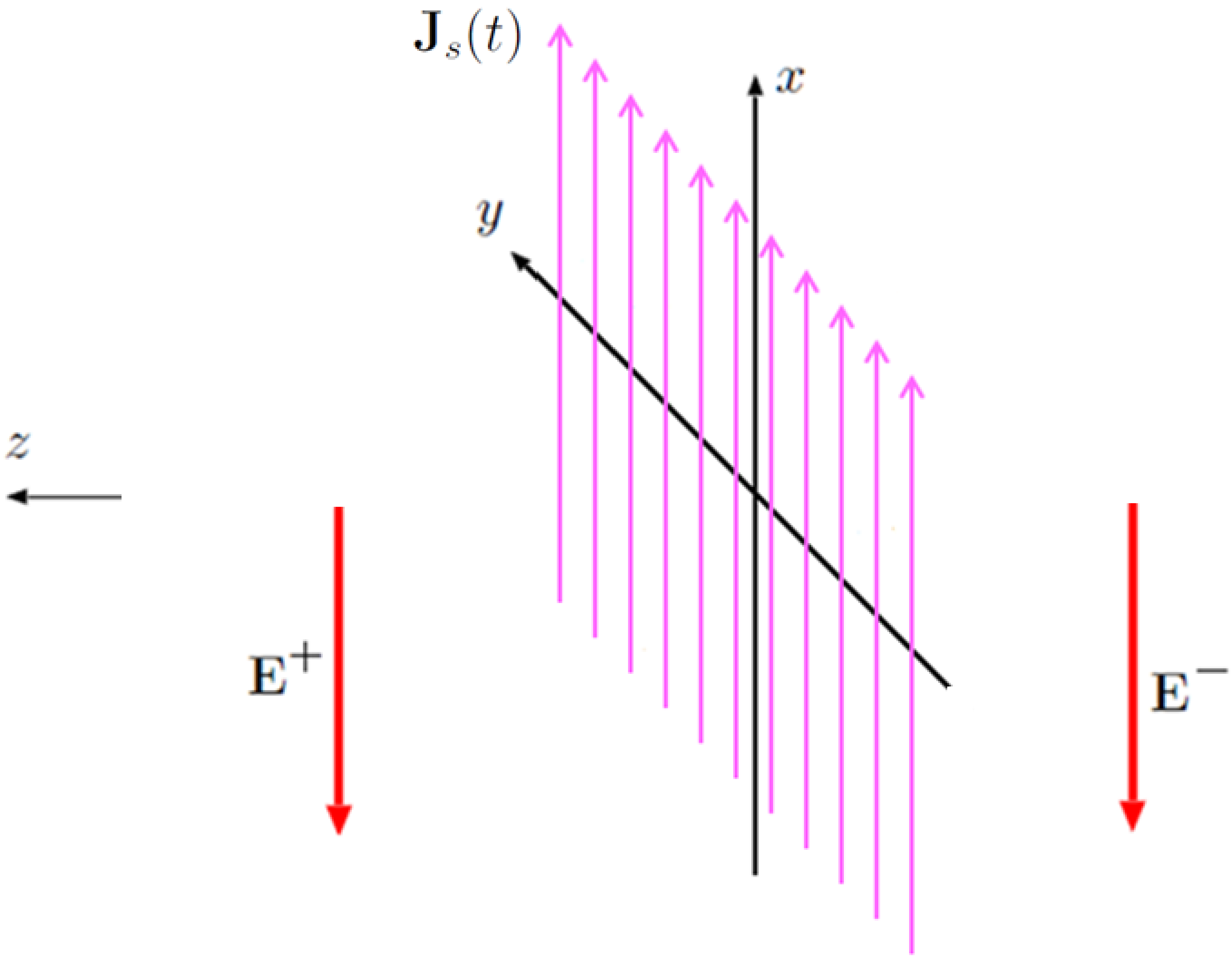
(b) Radiated $E_x(z, t = 2\mu\text{s})$ vs z.



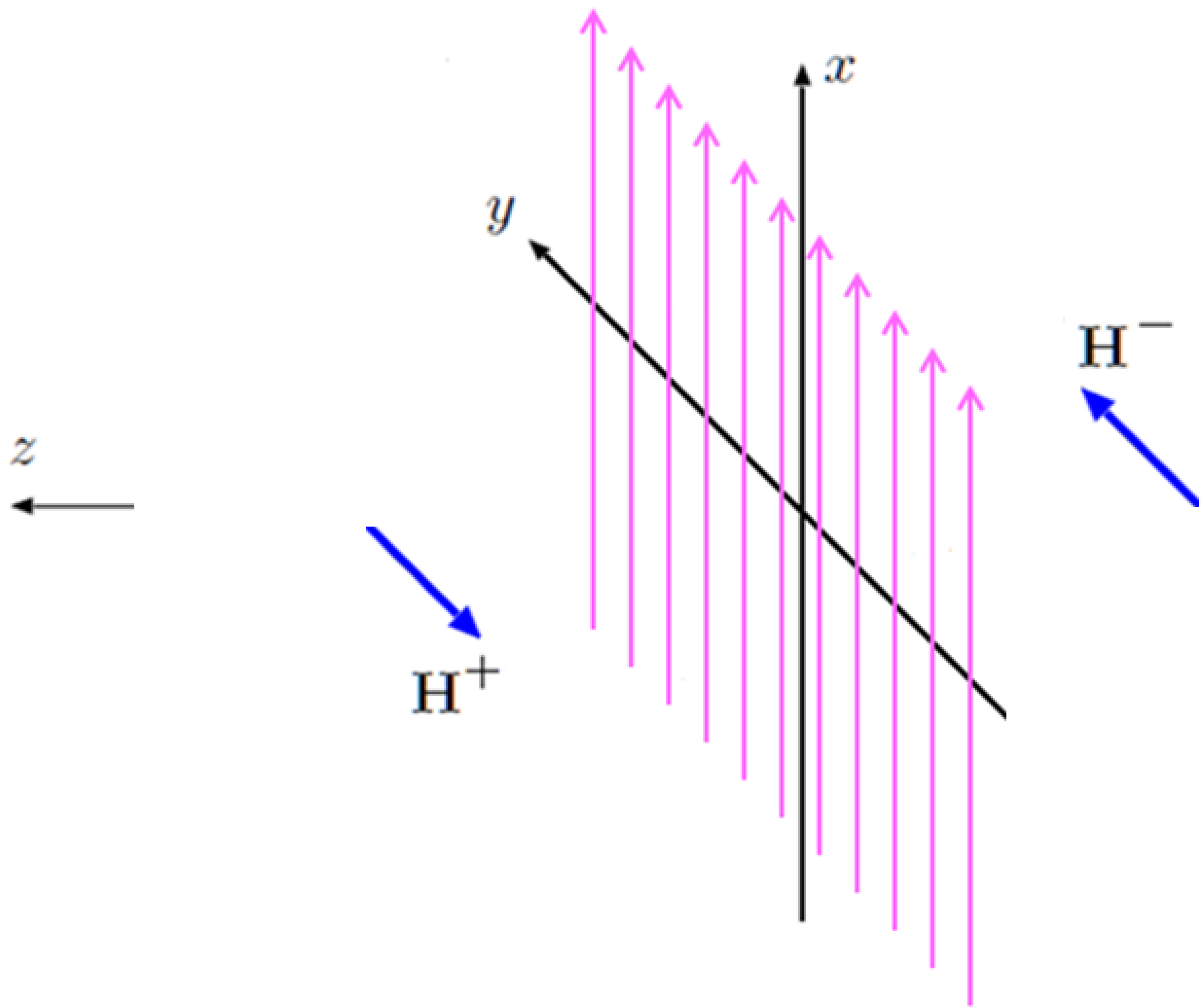
How are the electric fields oriented on the two sides?



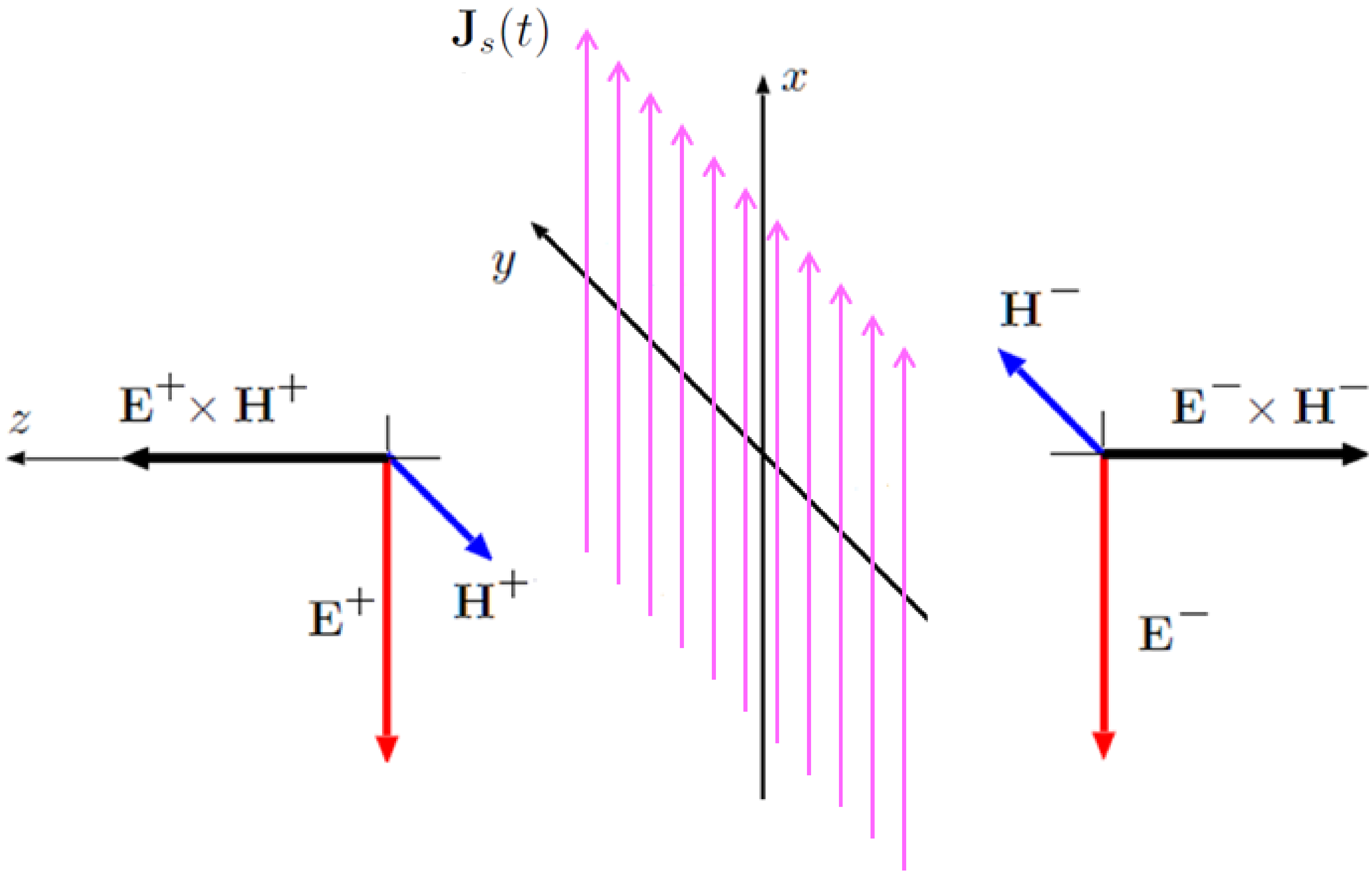
Electric Fields. **How are the magnetic fields oriented on the two sides?**



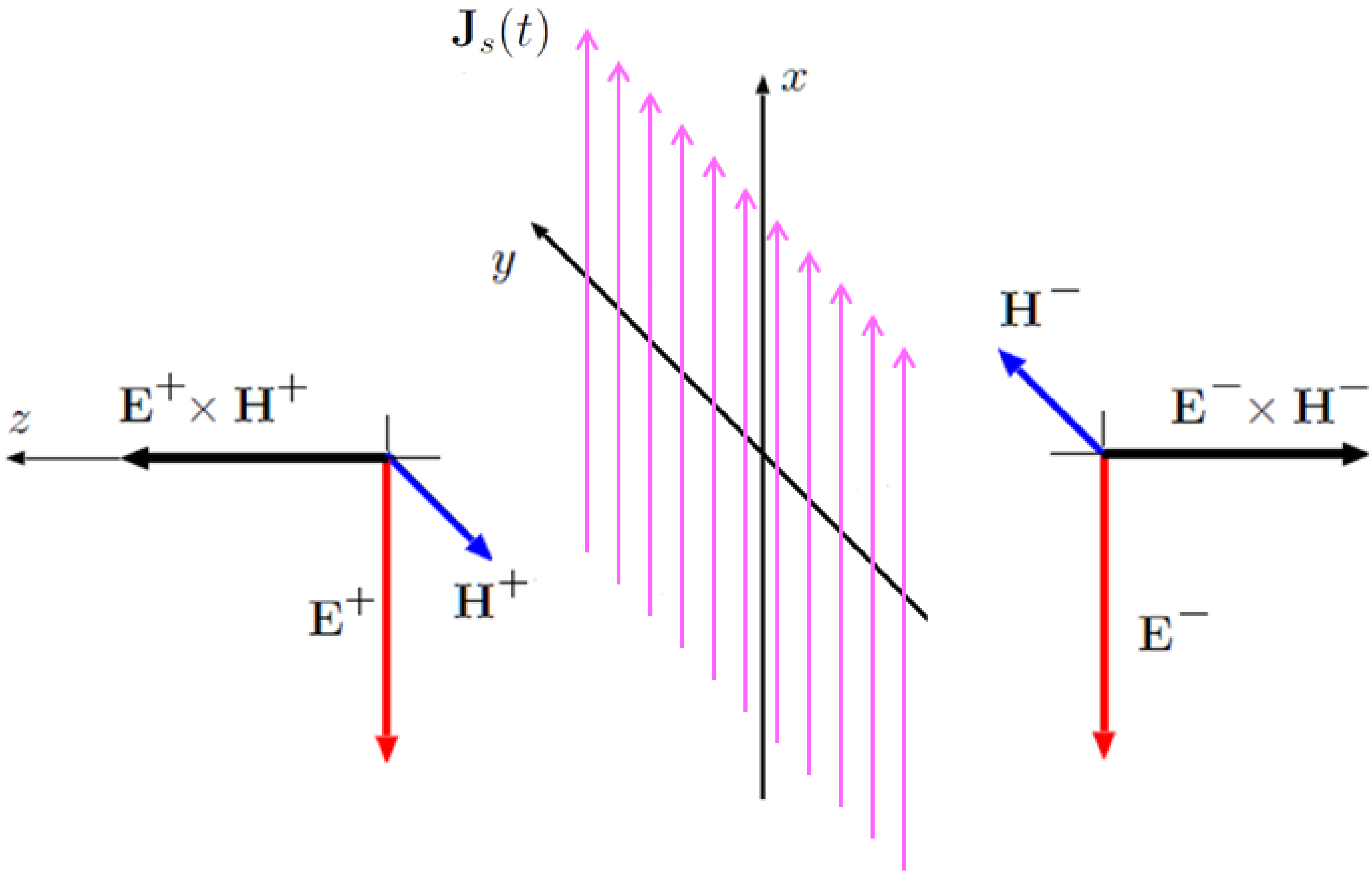
Magnetic fields



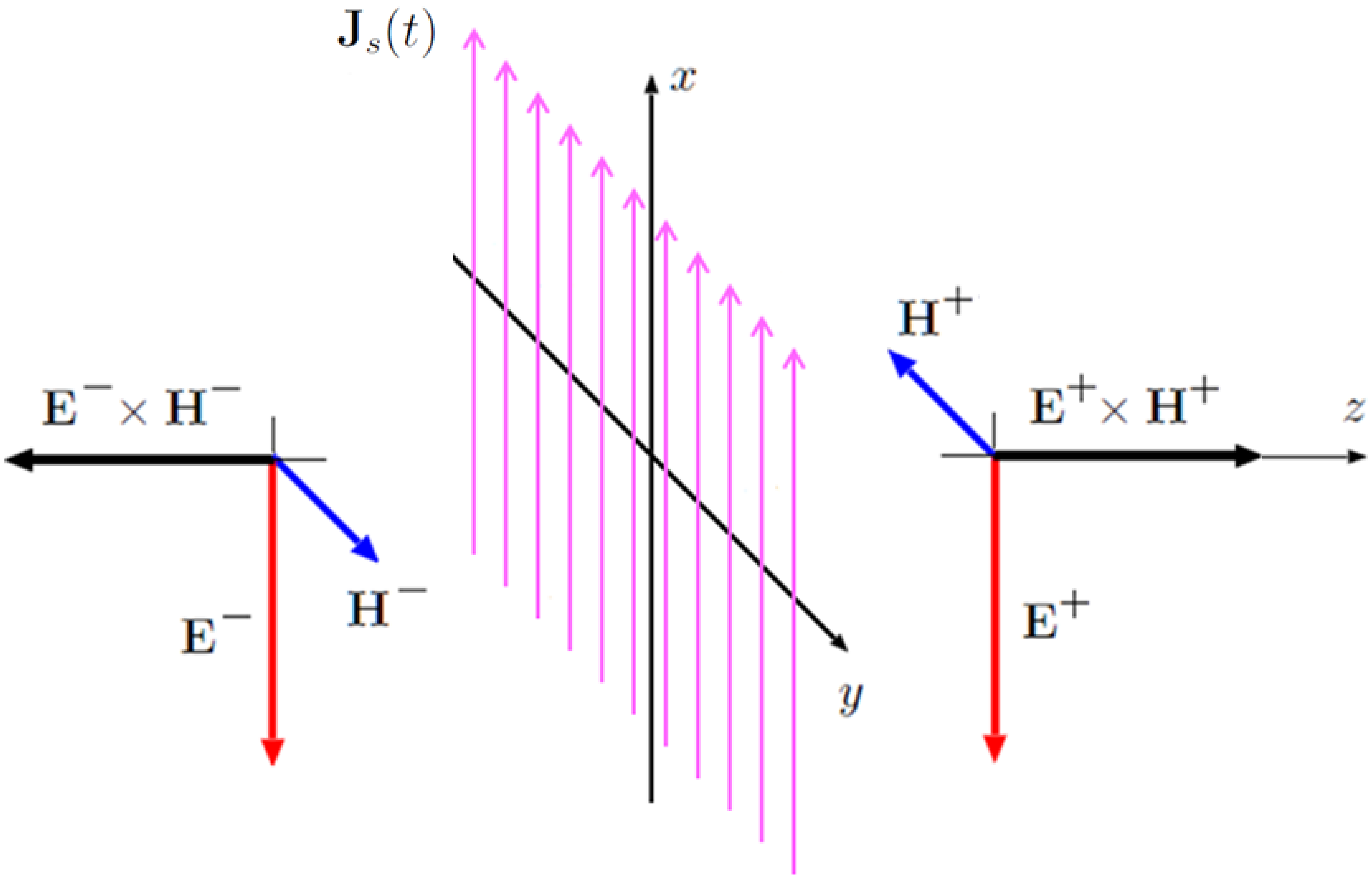
The complete diagram



3D Spatial Visualization Practice: Rotate by 180° about x in your head

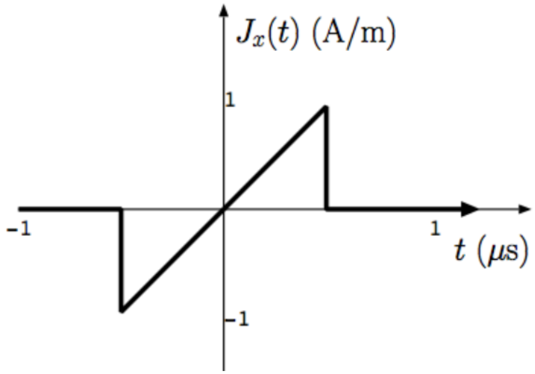


The rotated diagram



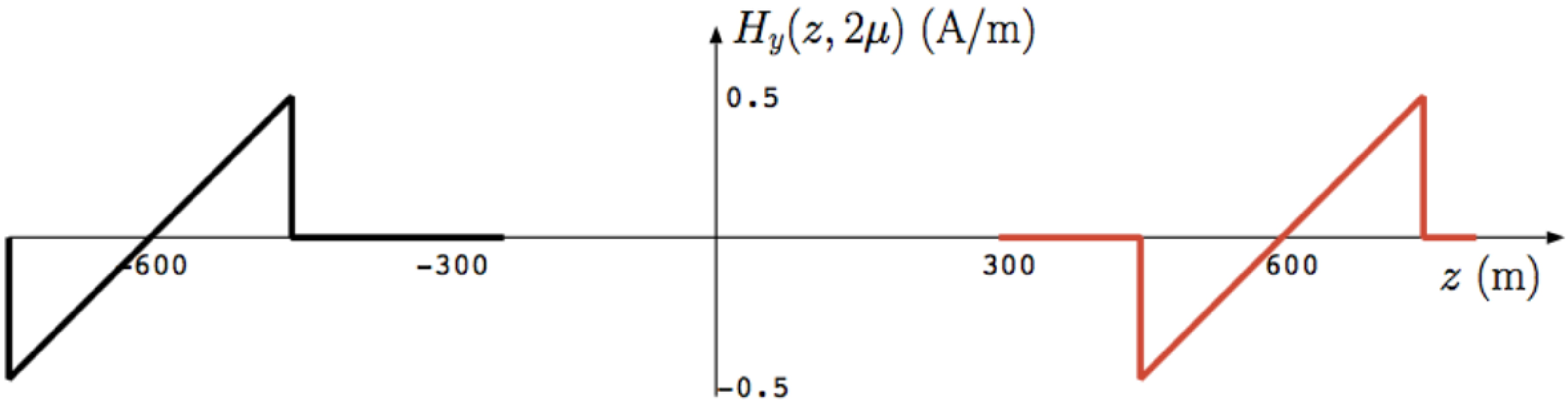
$$\mathbf{J}_s(t) = \hat{x} f(t), \quad \text{with} \quad f(t) = At\,\mathrm{rect}(\frac{t}{\tau})$$

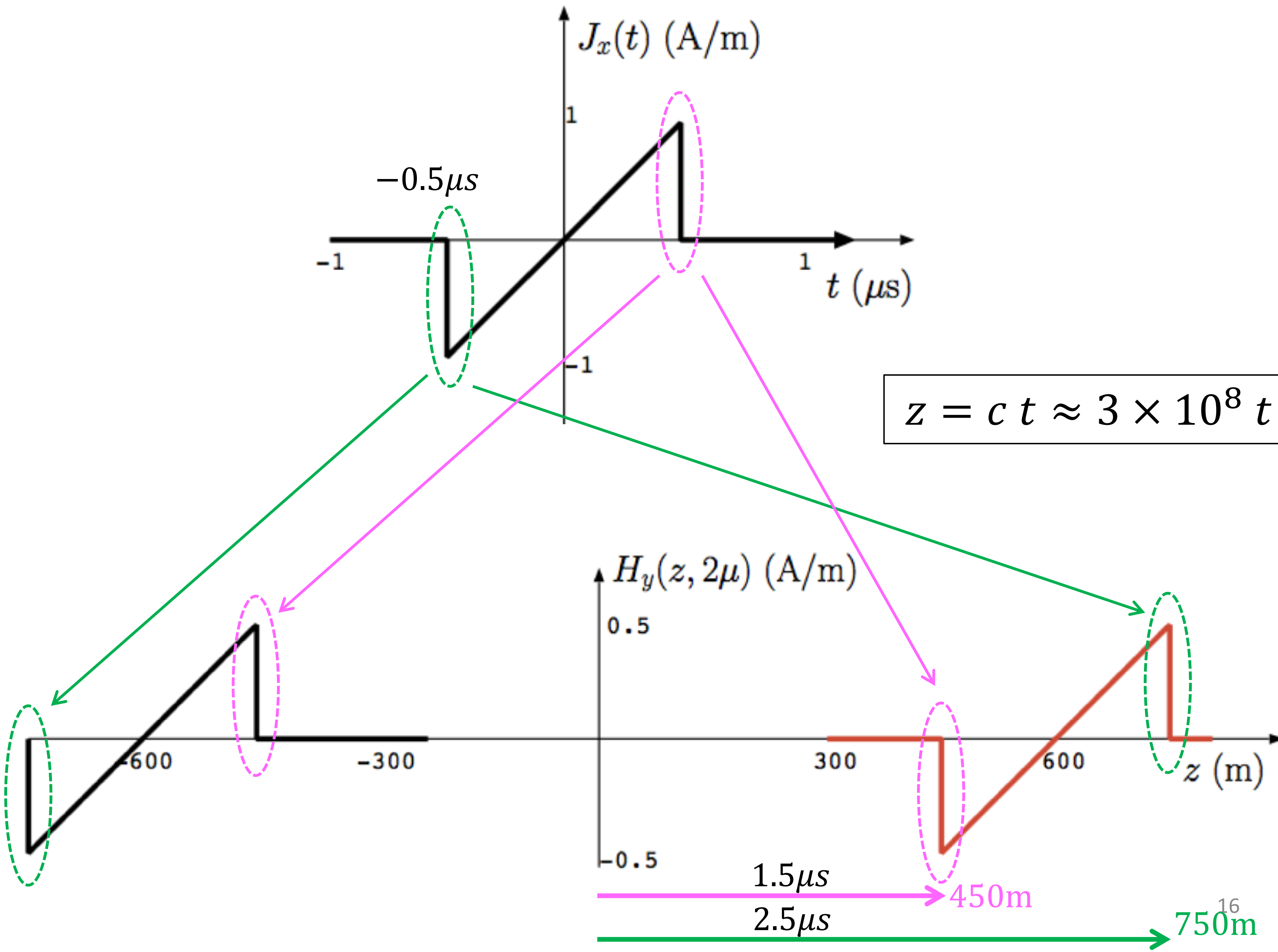
$$\tau = 1\,\mu\text{s} \qquad\qquad A = 2\,\frac{\text{A/m}}{\mu\text{S}}$$



(a) Radiated $H_y(z, t = 2\mu\text{s})$ vs z

$$\boxed{H_y(z, 2\mu\text{s}) = \mp(2\mu \mp \frac{z}{c})\,\mathrm{rect}(\frac{2\mu \mp \frac{z}{c}}{1\mu})\,\frac{\text{A}}{\text{m}} \text{ for } z \gtrless 0.}$$

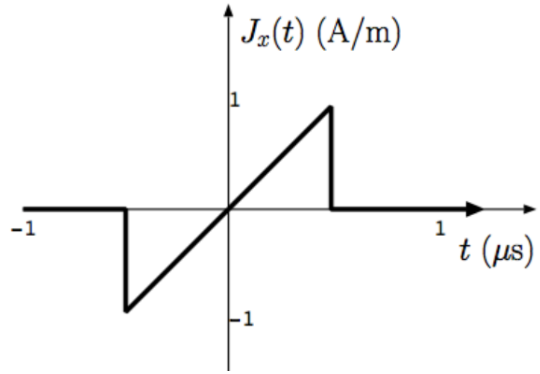




$$z = c\,t \approx 3 \times 10^8\,t$$

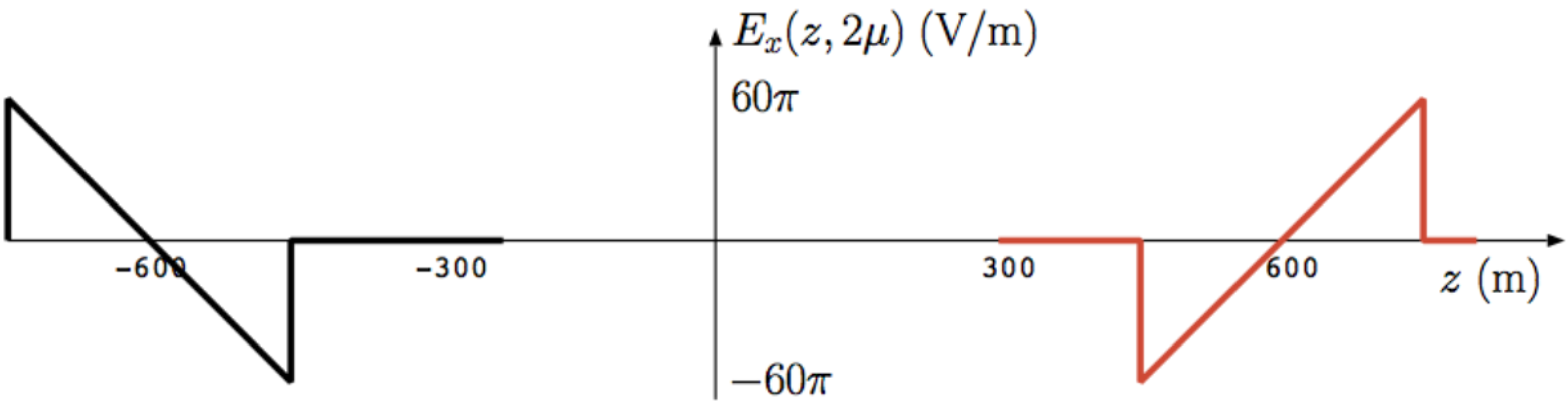
$$\mathbf{J}_s(t) = \hat{x} f(t), \quad \text{with} \quad f(t) = At \,\mathrm{rect}(\frac{t}{\tau})$$

$$\tau = 1\,\mu\mathrm{s} \qquad A = 2\,\frac{\mathrm{A/m}}{\mu\mathrm{S}}$$

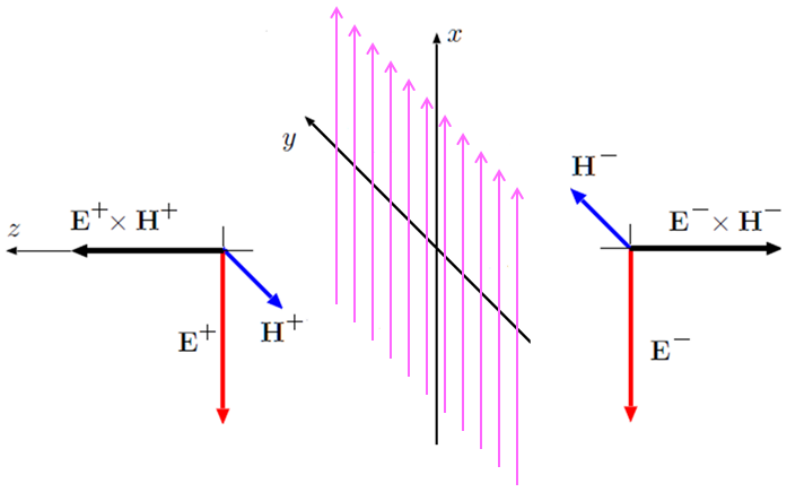


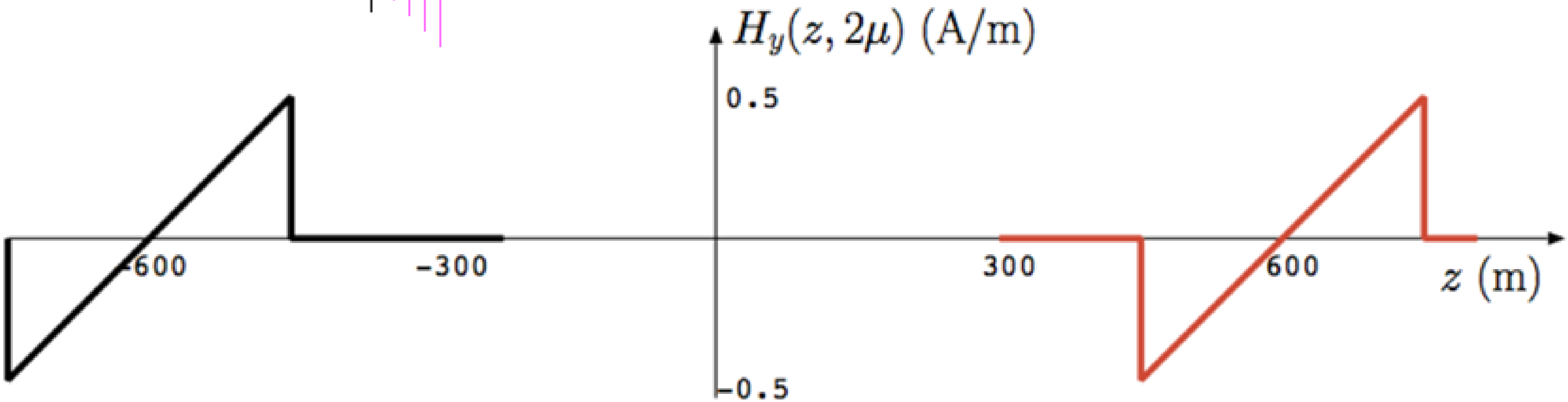
(b) Radiated $E_x(z, t = 2\mu\mathrm{s})$ vs z

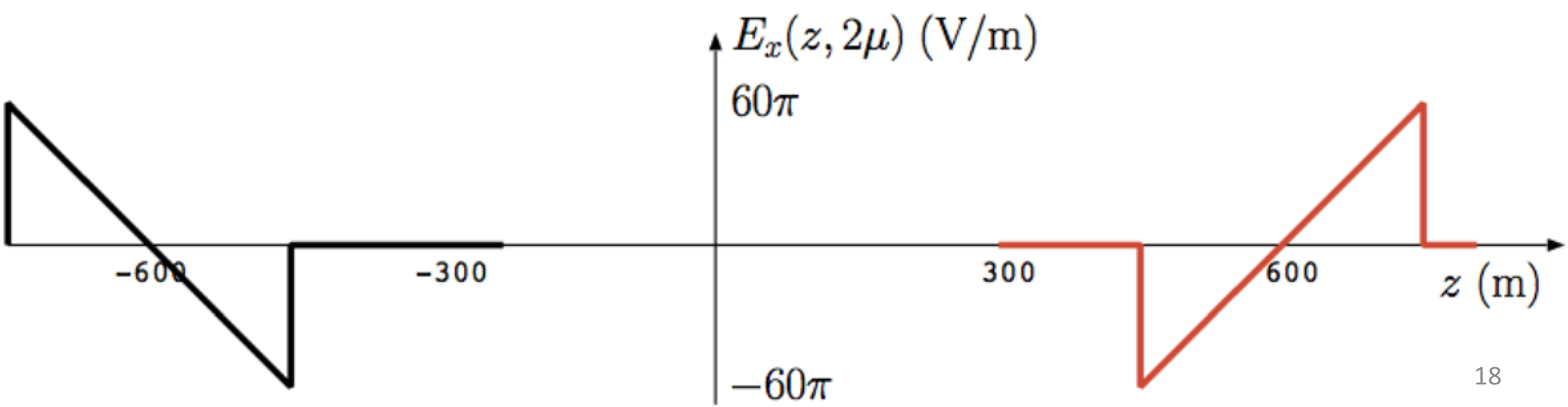
$$\boxed{E_x(z, 2\mu\mathrm{s}) = -120\pi(2\mu \mp \frac{z}{c})\,\mathrm{rect}(\frac{2\mu \mp \frac{z}{c}}{1\mu})\,\frac{\mathrm{V}}{\mathrm{m}} \text{ for } z \gtrless 0}$$



SUMMARY


x
y
z
$E^+ \times H^+$
E^+
H^+
H^-
$E^- \times H^-$
E^-

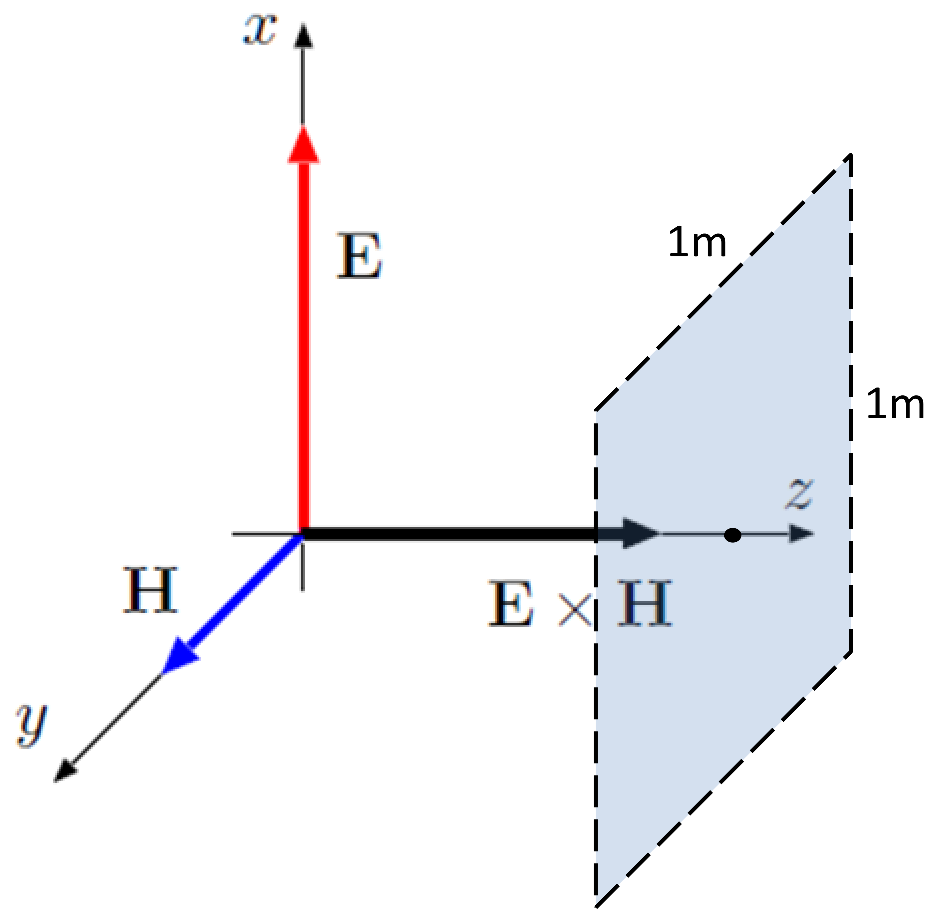

$H_y(z, 2\mu)$ (A/m)
0.5
-0.5
-600
-300
300
600
z (m)


$E_x(z, 2\mu)$ (V/m)
60π
-60π
-600
-300
300
600
z (m)

Poynting Vector and Energy Flux

- **The magnitude of the Poynting Vector represents instantaneous power (energy per second) per unit area carried by an EM wave**

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \qquad (\mathrm{W/m^2})$$



Poynting Theorem – Derivation from Maxwell's equations

Dot Faraday and Ampere law by $\mathbf{H}$ and $\mathbf{E}$, respectively

$$(\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}) \rightarrow (\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}) \cdot \mathbf{H}$$

$$(\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) \qquad (\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) \cdot \mathbf{E}$$

and take the difference

$$\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} = -\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E} - \mathbf{J} \cdot \mathbf{E}$$

We are going to use

$$\boxed{\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})}$$

The various terms can be manipulated as

$$\boxed{\mathbf{H}\cdot\nabla\times\mathbf{E}-\mathbf{E}\cdot\nabla\times\mathbf{H} \;=\nabla\cdot(\mathbf{E}\times\mathbf{H})}$$

$$\boxed{-\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H}\;=-\frac{\partial\mu\mathbf{H}}{\partial t}\cdot\mathbf{H}=-\frac{\partial}{\partial t}(\frac{1}{2}\mu\mathbf{H}\cdot\mathbf{H})}$$

$$\boxed{-\frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{E}\;=-\frac{\partial\,\epsilon\mathbf{E}}{\partial t}\cdot\mathbf{E}\;=-\frac{\partial}{\partial t}(\frac{1}{2}\epsilon\mathbf{E}\cdot\mathbf{E})}$$

Putting it all together

$$\nabla\cdot(\mathbf{E}\times\mathbf{H})=-\frac{\partial}{\partial t}(\frac{1}{2}\epsilon\mathbf{E}\cdot\mathbf{E}+\frac{1}{2}\mu\mathbf{H}\cdot\mathbf{H})\;-\mathbf{J}\cdot\mathbf{E}$$

Energy conservation law

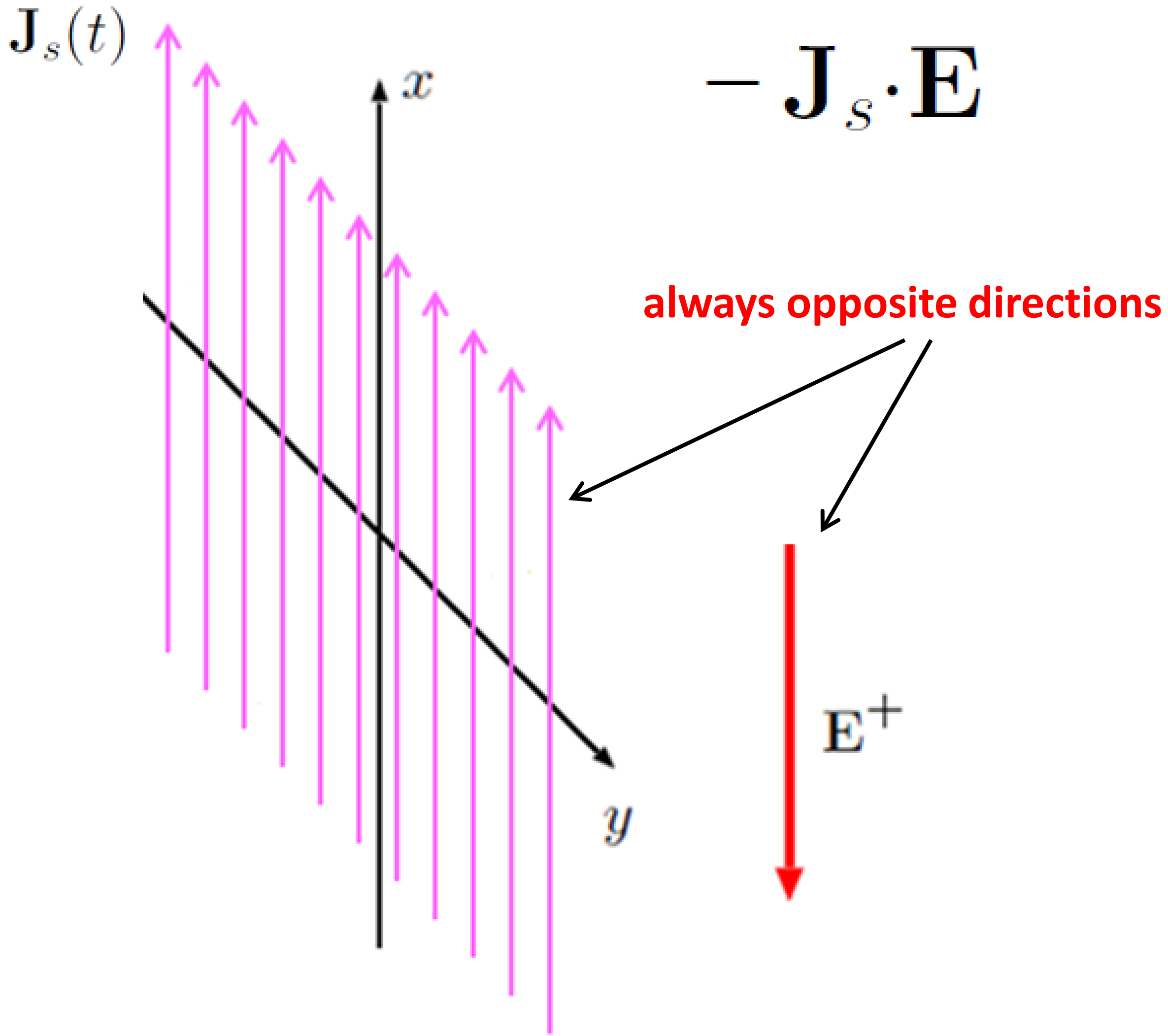
$$\underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon\mathbf{E}\cdot\mathbf{E}+\frac{1}{2}\mu\mathbf{H}\cdot\mathbf{H}\right)}_{\text{rate of change in time of electric energy and magnetic energy}}+\underbrace{\nabla\cdot(\mathbf{E}\times\mathbf{H})}_{\text{flux of power out of elementary volume}}+\underbrace{\mathbf{J}\cdot\mathbf{E}}_{\text{Joule Heating (power absorbed per unit volume)}}=0$$

$$\boxed{\mathbf{J}\cdot\mathbf{E}=\sigma\mathbf{E}\cdot\mathbf{E}=\sigma\mathbf{E}^2}$$

positive value if current density *induced* by the wave causes loss in medium with finite conductivity σ

However, a negative value of $\mathbf{J}\cdot\mathbf{E}$ indicates *generation of power* fed to the wave. For instance

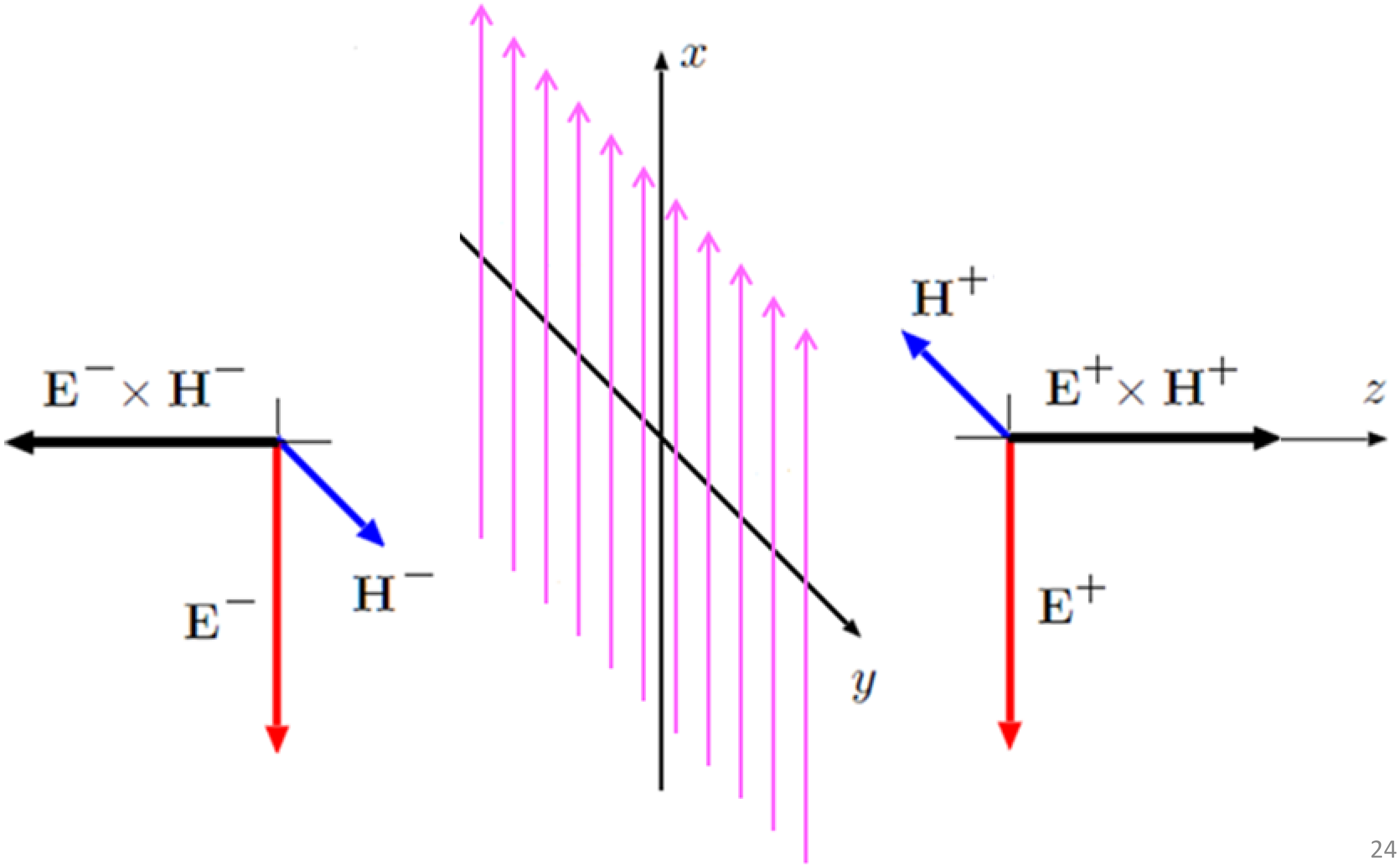
$$-\mathbf{J}_s\cdot\mathbf{E}$$



Example – Time-harmonic surface current

$$\mathbf{J}_s = \hat{x} f(t) = \hat{x} 2\cos(\omega t)\,\frac{\mathrm{A}}{\mathrm{m}}$$

ω is an arbitrary angular frequency of oscillation



Example – Time-harmonic surface current

$$\mathbf{J}_s = \hat{x} f(t) = \hat{x} 2 \cos(\omega t) \, \frac{\text{A}}{\text{m}}$$

ω is an arbitrary angular frequency of oscillation

(a) Determine the radiated TEM wave fields $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ in the regions $z \gtrless 0$,

(b) The associated Poynting vectors $\mathbf{E} \times \mathbf{H}$, and

(c) $\mathbf{J}_s \cdot \mathbf{E}$ on the current sheet.

Consider free space

$$\beta = \frac{\omega}{c} \quad \text{and} \quad \eta = \eta_o \approx 120\pi \, \Omega$$

Example – Time-harmonic surface current

$$\mathbf{J}_s = \hat{x} f(t) = \hat{x} 2\cos(\omega t)\ \frac{\mathrm{A}}{\mathrm{m}}$$

(a) Determine the radiated TEM wave fields $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ in the regions $z \gtrless 0$

$$\boxed{f(t \mp \frac{z}{v}) = 2\cos\left[\omega(t \mp \frac{z}{v})\right] = 2\cos(\omega t \mp \frac{\omega z}{v}) = 2\cos(\omega t \mp \beta z)}$$

Electric Field

$$E_x = -\frac{\eta}{2} f(t \mp \frac{z}{v}) = -\eta\cos(\omega t \mp \beta z)\ \ \frac{\mathrm{V}}{\mathrm{m}}$$

$$\boxed{\mathbf{E}(z,t) = E_x\ \hat{x}\frac{\mathrm{V}}{\mathrm{m}} = -\eta\cos(\omega t \mp \beta z)\hat{x}\frac{\mathrm{V}}{\mathrm{m}}}$$

Example – Time-harmonic surface current

$$\mathbf{J}_s = \hat{x} f(t) = \hat{x} 2 \cos(\omega t) \ \frac{\text{A}}{\text{m}}$$

(a) Determine the radiated TEM wave fields $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ in the regions $z \gtrless 0$

$$\boxed{f(t \mp \frac{z}{v}) = 2\cos\left[\omega(t \mp \frac{z}{v})\right] = 2\cos(\omega t \mp \frac{\omega z}{v}) = 2\cos(\omega t \mp \beta z)}$$

Magnetic Field

$$H_y = \mp \frac{1}{2} f(t \mp \frac{z}{v}) = \mp \cos(\omega t \mp \beta z) \quad \frac{\text{A}}{\text{m}}$$

$$\boxed{\mathbf{H}(z,t) = H_y \, \hat{y} \, \frac{\text{A}}{\text{m}} = \mp \cos(\omega t \mp \beta z) \hat{y} \, \frac{\text{A}}{\text{m}}}$$

Example – Time-harmonic surface current

$$\mathbf{J}_s = \hat{x} f(t) = \hat{x} 2 \cos(\omega t) \ \frac{\mathrm{A}}{\mathrm{m}}$$

(b) The associated Poynting vectors $\mathbf{E} \times \mathbf{H}$

$$\mathbf{E} \times \mathbf{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_x & 0 & 0 \\ 0 & H_y & 0 \end{vmatrix} = \hat{z}\, E_x\, H_y$$

$$E_x = -\frac{\eta}{2} f(t \mp \frac{z}{v}) = -\eta \cos(\omega t \mp \beta z) \quad \frac{\mathrm{V}}{\mathrm{m}}$$

$$H_y = \mp \frac{1}{2} f(t \mp \frac{z}{v}) = \mp \cos(\omega t \mp \beta z) \quad \frac{\mathrm{A}}{\mathrm{m}}$$

$$\boxed{\mathbf{S} = \mathbf{E} \times \mathbf{H} = \pm \eta \cos^2(\omega t \mp \beta z) \hat{z} \ \frac{\mathrm{W}}{\mathrm{m}^2}}$$

Example – Time-harmonic surface current

$$\mathbf{J}_s = \hat{x} f(t) = \hat{x} 2 \cos(\omega t) \, \frac{\text{A}}{\text{m}}$$

(c) $\mathbf{J}_s \cdot \mathbf{E}$ on the current sheet

$\boxed{z = 0}$

$$\mathbf{E}(0, t) = -\eta \cos(\omega t) \hat{x} \, \frac{\text{V}}{\text{m}}$$

$$\mathbf{J}_s(t) \cdot \mathbf{E}(0, t) = (\hat{x} 2 \cos(\omega t) \, \frac{\text{A}}{\text{m}}) \cdot (-\eta \cos(\omega t) \hat{x} \, \frac{\text{V}}{\text{m}})$$

$$\boxed{\mathbf{J}_s(t) \cdot \mathbf{E}(0, t) = -2\eta \cos^2(\omega t) \, \frac{\text{W}}{\text{m}^2}}$$

This term is negative and behaving like a source

The time-harmonic source we have examined has produced *monochromatic waves* characterized by a single frequency (literally, a single color).

For a monochromatic wave, the *instantaneous* Poynting vector is proportional to the square of the cosine term that can be also written as

$$\cos^2(\omega t + \phi) = \frac{1}{2}[1 + \cos(2\omega t + 2\phi)]$$

For a periodic signal, it is more meaningful to evaluate the time-average of the Poynting vector, since it quantifies the overall power flow over time.

Time average of the Poynting vector

$$\langle \mathbf{S}(t) \rangle = \frac{1}{T}\int_0^T \mathbf{S}(t)\,dt = \frac{1}{T}\int_0^T \mathbf{E}(t)\times\mathbf{H}(t)\,dt$$

For our example: $\langle \cos^2(\omega t + \phi) \rangle = \langle \frac{1}{2}[1 + \cos(2\omega t + 2\phi)] \rangle = \frac{1}{2}$

$$\langle \mathbf{S}(t) \rangle = \pm\hat{z}\frac{\eta}{T}\int_0^T \cos^2\left(\omega t \mp \beta z\right) dt$$

$$= \pm\hat{z}\frac{\eta}{T}\int_0^T \frac{1}{2}\left[1 + \cos\left(2\omega t \mp 2\beta z\right)\right] dt$$

$$= \pm\hat{z}\frac{\eta}{2}\,\frac{\mathrm{W}}{\mathrm{m}^2} \approx \pm\hat{z}\,60\pi\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

time-average power per unit area transported by the radiated waves on each side of the sheet of current.

Injected (generated) Power Density

We have calculated earlier the instantaneous power density injected by the sheet of current (including both sides):

$$-\mathbf{J}_s \cdot \mathbf{E} \;=\; 2\eta\cos^2(\omega t)\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

The time average is obtained from the same integration:

$$\langle -\mathbf{J}_s \cdot \mathbf{E} \rangle \;=\; \eta\,\frac{\mathrm{W}}{\mathrm{m}^2} = 120\pi\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

which is indeed equal to the total time-average power injected in the space surrounding the sheet of current, as it should be for conservation of energy.