# ECE 329 – Fall 2022

**Prof. Ravaioli – Office: 2062 ECEB**

Section E – 1:00pm

Lecture 21

# Lecture 21 – Outline

- **Monochromatic waves**
- **Phasors**
- **EM Spectrum**
- **Time-average Poynting vector with phasors**

**Reading assignment**

**Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:**

**21) Monochromatic waves and phasor notation**

## Example – Time-harmonic surface current

$$\mathbf{J}_s = \hat{x} f(t) = \hat{x} 2 \cos(\omega t)\, \frac{\mathrm{A}}{\mathrm{m}}$$

**ω is an arbitrary angular frequency of oscillation**

(a) Determine the radiated TEM wave fields $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ in the regions $z \gtrless 0$,

(b) The associated Poynting vectors $\mathbf{E} \times \mathbf{H}$, and

(c) $\mathbf{J}_s \cdot \mathbf{E}$ on the current sheet.

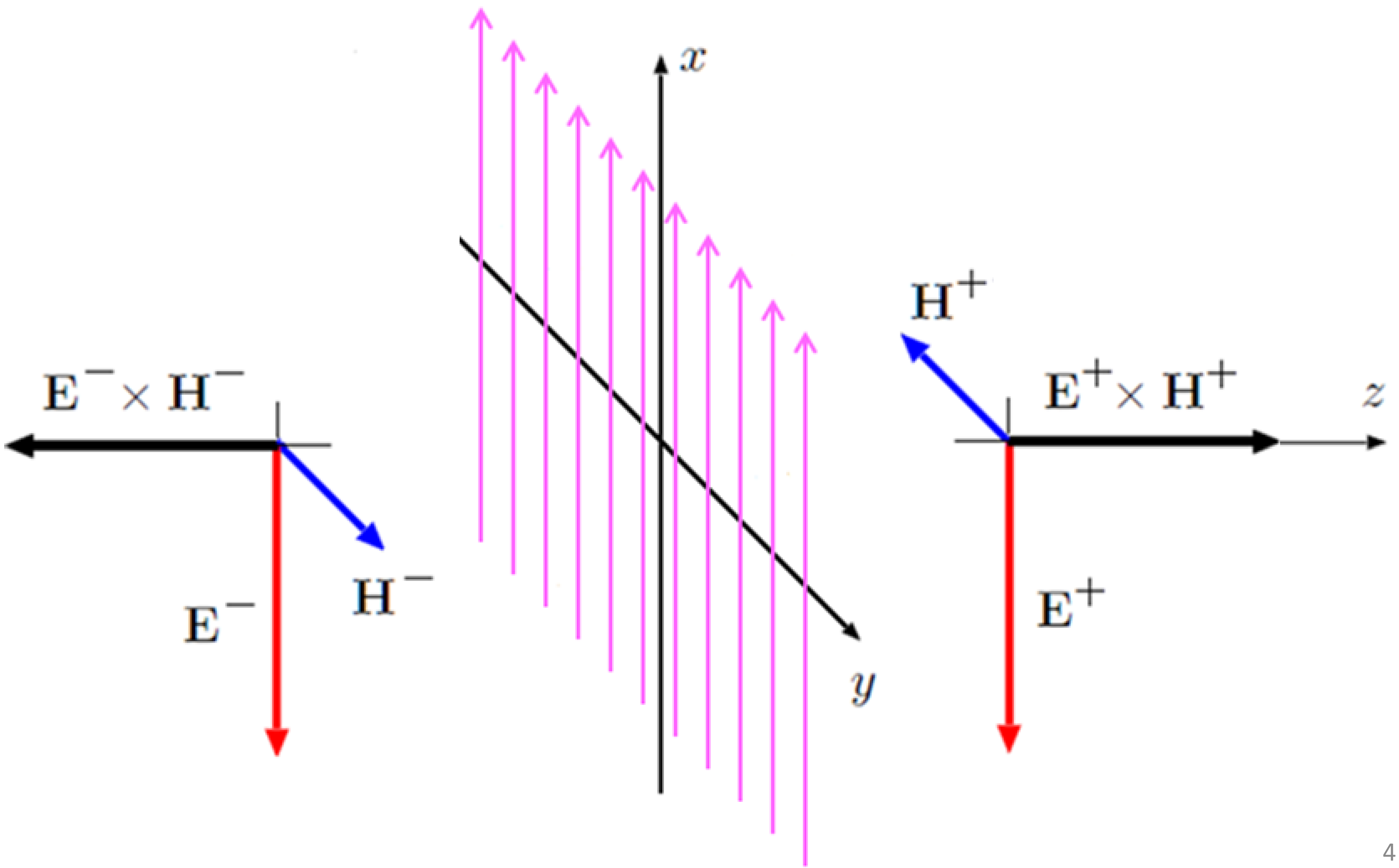
**Consider free space**

$$\beta = \frac{\omega}{c} \quad \text{and} \quad \eta = \eta_o \approx 120\pi\,\Omega$$

# Example – Time-harmonic surface current

$$\mathbf{J}_s = \hat{x} f(t) = \hat{x} 2 \cos(\omega t) \frac{\text{A}}{\text{m}}$$

ω is an arbitrary angular frequency of oscillation



## Example – Time-harmonic surface current

$$\mathbf{J}_s = \hat{x} f(t) = \hat{x} 2 \cos(\omega t) \ \frac{\mathrm{A}}{\mathrm{m}}$$

(a) Determine the radiated TEM wave fields $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ in the regions $z \gtrless 0$

$$\boxed{f(t \mp \frac{z}{v}) = 2\cos\left[\omega\left(t \mp \frac{z}{v}\right)\right] = 2\cos(\omega t \mp \frac{\omega z}{v}) = 2\cos(\omega t \mp \beta z)}$$

## Electric Field

$$E_x = -\frac{\eta}{2} f(t \mp \frac{z}{v}) = -\eta \cos(\omega t \mp \beta z) \ \frac{\mathrm{V}}{\mathrm{m}}$$

$$\boxed{\mathbf{E}(z,t) = E_x \ \hat{x} \frac{\mathrm{V}}{\mathrm{m}} = -\eta \cos(\omega t \mp \beta z) \hat{x} \frac{\mathrm{V}}{\mathrm{m}}}$$

## Example – Time-harmonic surface current

$$\mathbf{J}_s = \hat{x} f(t) = \hat{x} 2 \cos(\omega t) \, \frac{\mathrm{A}}{\mathrm{m}}$$

(a) Determine the radiated TEM wave fields $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ in the regions $z \gtrless 0$

$$\boxed{f(t \mp \frac{z}{v}) = 2\cos\left[\omega(t \mp \frac{z}{v})\right] = 2\cos(\omega t \mp \frac{\omega z}{v}) = 2\cos(\omega t \mp \beta z)}$$

### Magnetic Field

$$H_y = \mp \frac{1}{2} f(t \mp \frac{z}{v}) = \mp \cos(\omega t \mp \beta z) \quad \frac{\mathrm{A}}{\mathrm{m}}$$

$$\boxed{\mathbf{H}(z,t) = H_y \, \hat{y} \, \frac{\mathrm{A}}{\mathrm{m}} = \mp \cos(\omega t \mp \beta z) \hat{y} \, \frac{\mathrm{A}}{\mathrm{m}}}$$

## Example – Time-harmonic surface current

$$\mathbf{J}_s = \hat{x} f(t) = \hat{x} 2 \cos(\omega t) \; \frac{\mathrm{A}}{\mathrm{m}}$$

(b) The associated Poynting vectors $\mathbf{E} \times \mathbf{H}$

$$\mathbf{E} \times \mathbf{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_x & 0 & 0 \\ 0 & H_y & 0 \end{vmatrix} = \hat{z}\, E_x\, H_y$$

$$E_x = -\frac{\eta}{2} f(t \mp \frac{z}{v}) = -\eta \cos(\omega t \mp \beta z) \quad \frac{\mathrm{V}}{\mathrm{m}}$$

$$H_y = \mp \frac{1}{2} f(t \mp \frac{z}{v}) = \mp \cos(\omega t \mp \beta z) \quad \frac{\mathrm{A}}{\mathrm{m}}$$

$$\boxed{\mathbf{S} = \mathbf{E} \times \mathbf{H} = \pm \eta \cos^2(\omega t \mp \beta z)\, \hat{z}\, \frac{\mathrm{W}}{\mathrm{m}^2}}$$

## Example – Time-harmonic surface current

$$\mathbf{J}_s = \hat{x} f(t) = \hat{x} 2 \cos(\omega t) \frac{\mathrm{A}}{\mathrm{m}}$$

(c) $\mathbf{J}_s \cdot \mathbf{E}$ on the current sheet

$\boxed{z = 0}$

$$\mathbf{E}(0,t) = -\eta \cos(\omega t) \hat{x} \frac{\mathrm{V}}{\mathrm{m}}$$

$$\mathbf{J}_s(t) \cdot \mathbf{E}(0,t) = (\hat{x} 2 \cos(\omega t) \frac{\mathrm{A}}{\mathrm{m}}) \cdot (-\eta \cos(\omega t) \hat{x} \frac{\mathrm{V}}{\mathrm{m}})$$

$$\boxed{\mathbf{J}_s(t) \cdot \mathbf{E}(0,t) = -2\eta \cos^2(\omega t) \frac{\mathrm{W}}{\mathrm{m}^2}}$$

**This term is negative and behaving like a source**

**The time-harmonic source we have examined has produced *monochromatic waves* characterized by a single frequency (literally, a single color).**

**For a monochromatic wave, the *instantaneous* Poynting vector is proportional to the square of the cosine term that can be also written as**

$$\cos^2(\omega t + \phi) = \frac{1}{2}[1 + \cos(2\omega t + 2\phi)]$$

**For a periodic signal, it is more meaningful to evaluate the time-average of the Poynting vector, since it quantifies the overall power flow over time.**

# Time average of the Poynting vector

$$\langle \mathbf{S}(t) \rangle = \frac{1}{T}\int_0^T \mathbf{S}(t)\,dt = \frac{1}{T}\int_0^T \mathbf{E}(t)\times\mathbf{H}(t)\,dt$$

**For our example:** $\langle \cos^2(\omega t+\phi)\rangle = \langle \frac{1}{2}[1+\cos(2\omega t+2\phi)]\rangle = \frac{1}{2}$

$$\langle \mathbf{S}(t) \rangle = \pm\hat{z}\frac{\eta}{T}\int_0^T \cos^2(\omega t \mp \beta z)\,dt$$

$$= \pm\hat{z}\frac{\eta}{T}\int_0^T \frac{1}{2}\left[1+\cos(2\omega t \mp 2\beta z)\right]dt$$

$$= \pm\hat{z}\frac{\eta}{2}\,\frac{\mathrm{W}}{\mathrm{m}^2} \approx \pm\hat{z}\,60\pi\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

**time-average power per unit area transported by the radiated waves on each side of the sheet of current.**

# Injected (generated) Power Density

We have calculated earlier the instantaneous power density injected by the sheet of current (including both sides):

$$-\mathbf{J}_s \cdot \mathbf{E} \;=\; 2\eta\cos^2(\omega t)\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

The time average is obtained from the same integration:

$$\langle -\mathbf{J}_s \cdot \mathbf{E} \rangle \;=\; \eta\,\frac{\mathrm{W}}{\mathrm{m}^2} = 120\pi\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

which is indeed equal to the total time-average power injected in the space surrounding the sheet of current, as it should be for conservation of energy.

**Earlier we discussed that general solutions of the wave equation**

$$\mathbf{E},\,\mathbf{H} \propto f(t \mp \frac{z}{v})$$

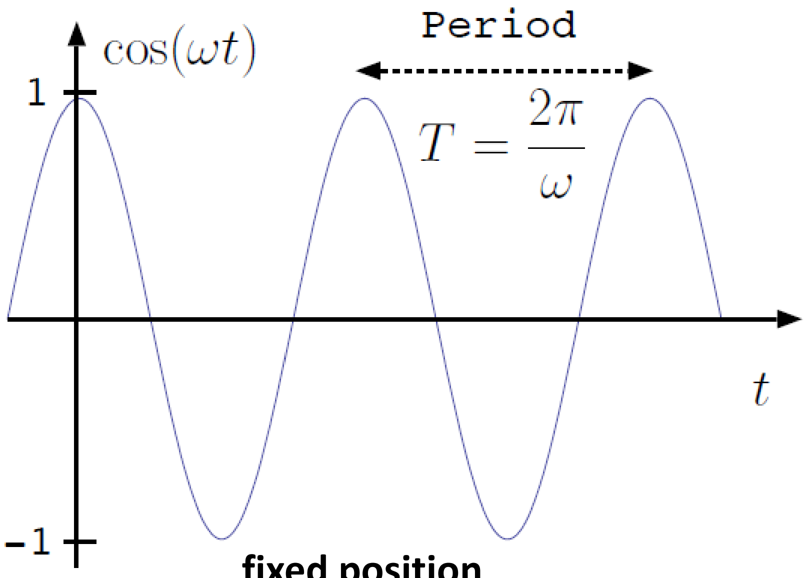
**can be expressed as a superposition of *monochromatic* waves**

$$A\cos[\omega(t \mp \frac{z}{v})] = A\cos(\omega t \mp \beta z)$$

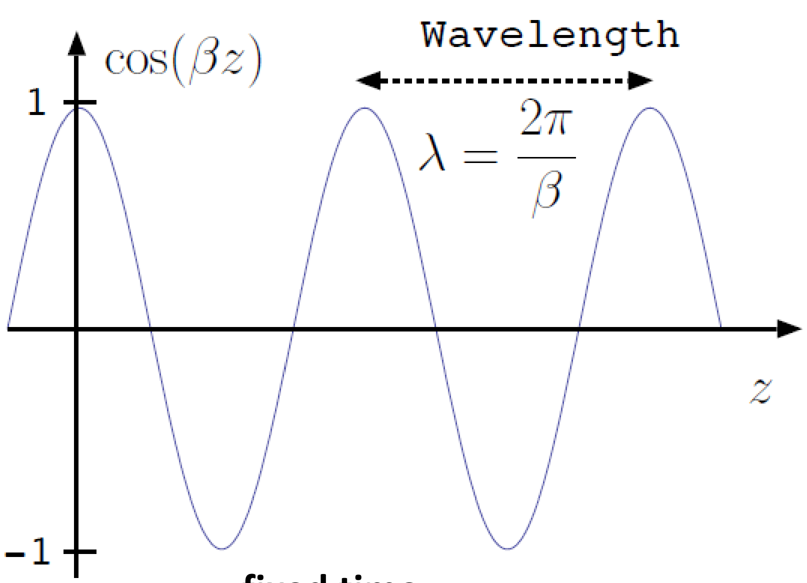
**with wave-number**

$$\beta \equiv \frac{\omega}{v} = \omega\sqrt{\mu\epsilon} \;\; = \frac{2\pi}{\lambda}$$

# Monochromatic waves are co-sinusoidal in time and space




fixed position




fixed time

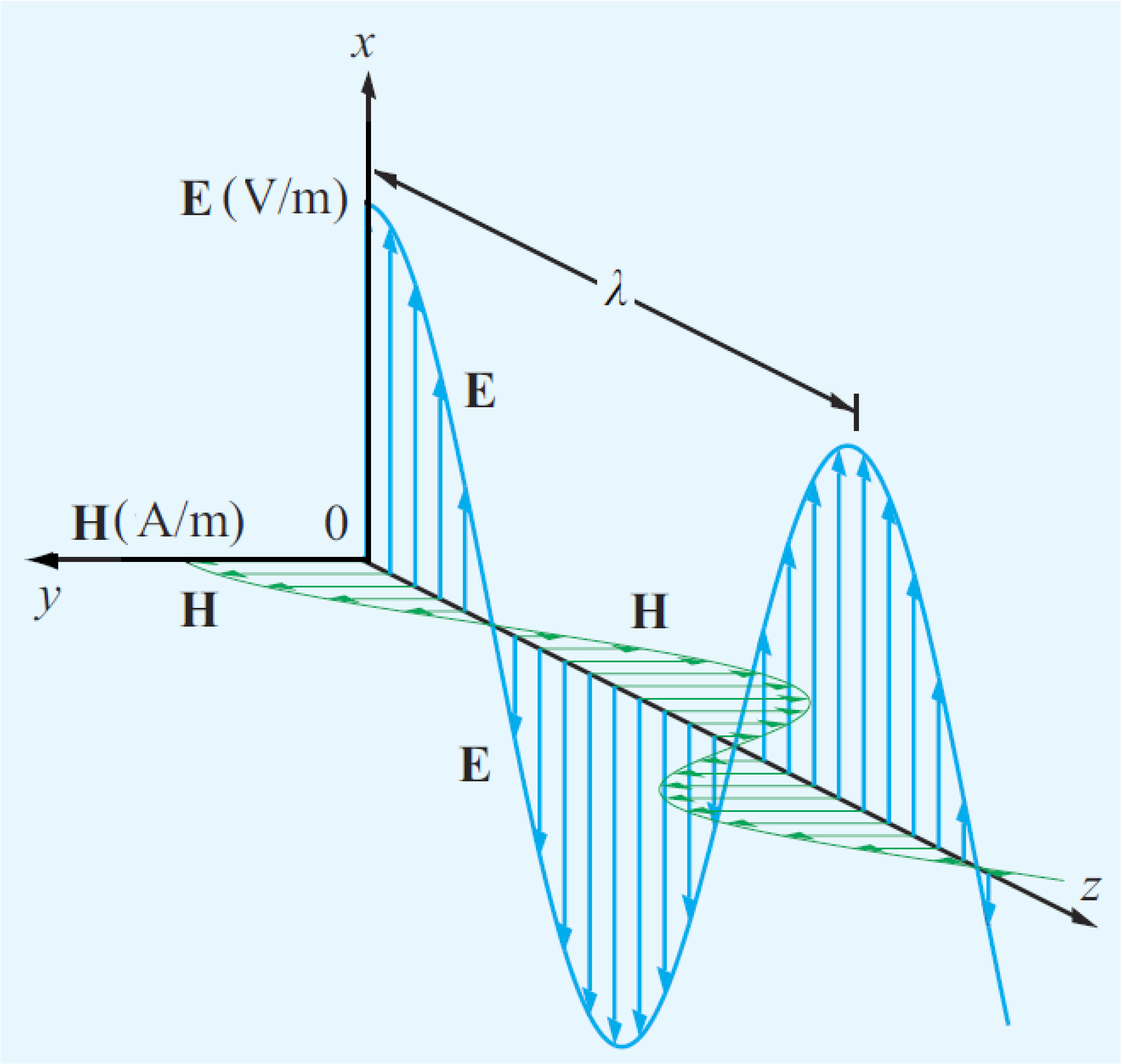
**period** $$T = \frac{2\pi}{\omega} \equiv \frac{1}{f}$$

**wavelength** $$\lambda \equiv \frac{2\pi}{\beta}$$

$$\lambda = \frac{v}{f} = \frac{c}{f}\frac{1}{\sqrt{\mu_r \varepsilon_r}} = \lambda_0 \frac{1}{\sqrt{\mu_r \varepsilon_r}}$$

**wavelength in free space**

# Representation of Electric and Magnetic field in space



# The function linking ω and β is called dispersion relation

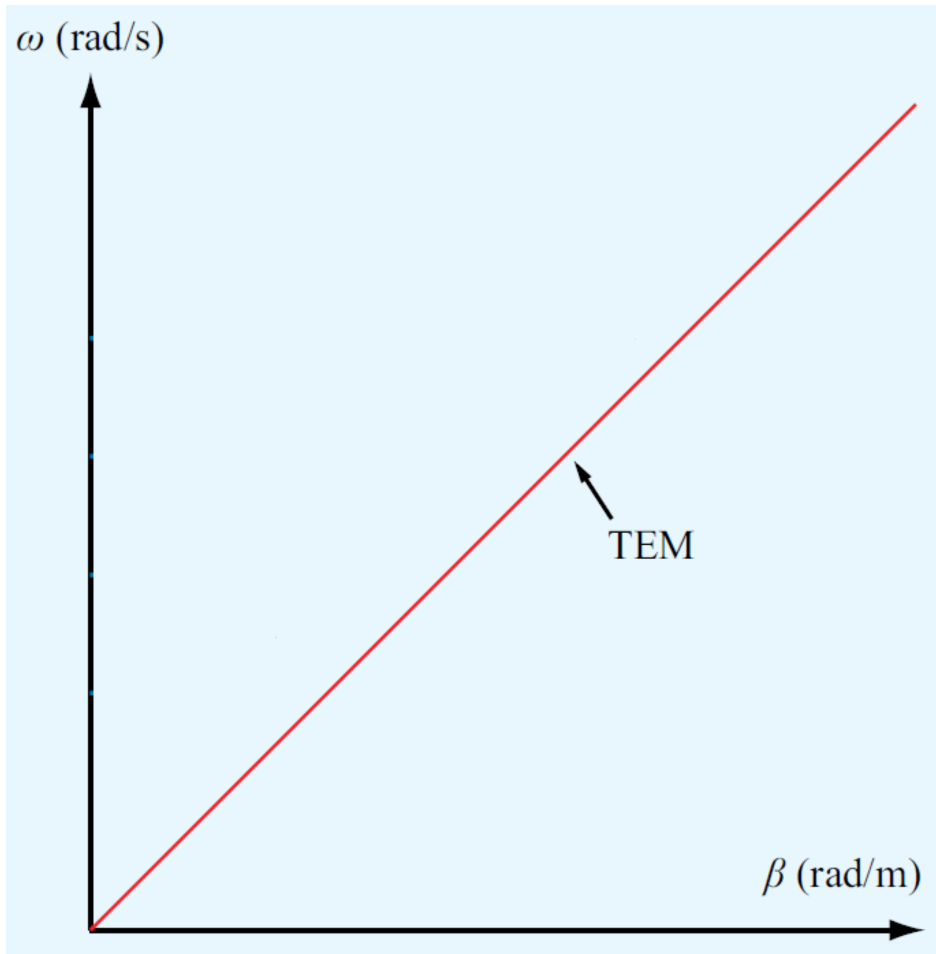
**From the dispersion relation we obtain the velocity**

$$v \equiv \frac{\omega}{\beta}$$

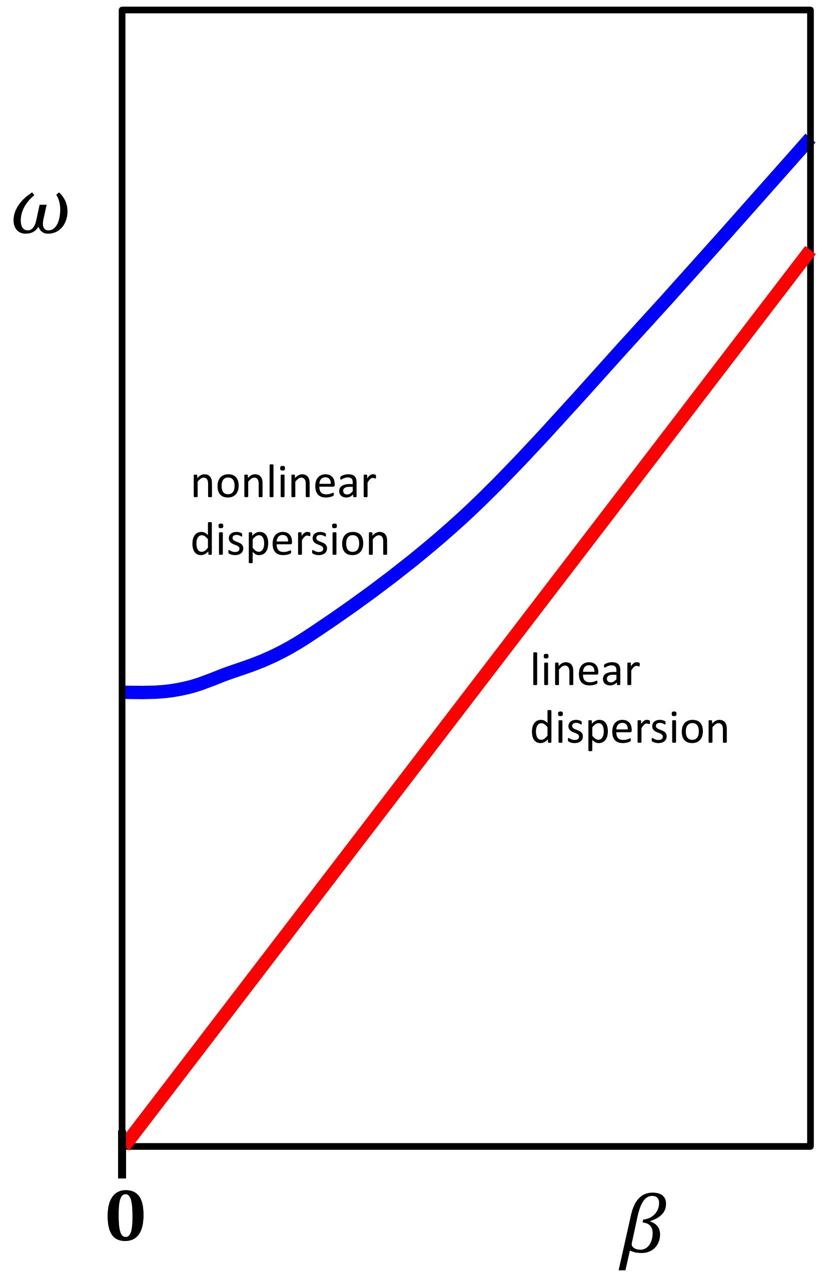
This is also called **phase velocity**

For a TEM wave propagating in a homogeneous perfect dielectric the dispersion relation is linear and it depends only on the properties of the medium.

$$\boxed{\beta = \omega\sqrt{\mu\epsilon}}$$



# Typical behavior of nonlinear dispersion



# Phasor notation

**A monochromatic *x*-polarized wave**

$$\mathbf{E} = E_o \cos(\omega t \mp \beta z)\, \hat{x}\, \frac{\mathrm{V}}{\mathrm{m}}$$

**can be expressed in phasor form**

$$\tilde{\mathbf{E}} = E_o e^{\mp j \beta z}\, \hat{x}\, \frac{\mathrm{V}}{\mathrm{m}}$$

**where**

$$\Re\mathfrak{e}\{\tilde{\mathbf{E}} e^{j\omega t}\} = E_o \cos(\omega t \mp \beta z)\, \hat{x} = \mathbf{E}$$

## Just in case you are rusty on phasors, the derivation is

$$\mathbf{E} = E_0 \cos(\omega t \mp \beta z)\hat{x}$$

$$= \Re e\left\{E_0 \left[\cos(\omega t \mp \beta z) + j\sin(\omega t \mp \beta z)\right]\right\}$$

$$= \Re e\left\{E_0\, e^{j(\omega t \mp \beta z)}\right\}$$

$$= \Re e\left\{E_0\, e^{j\omega t} e^{\mp j\beta z}\right\}$$

$$= \Re e\left\{\tilde{\mathbf{E}}\, e^{j\omega t}\right\}$$

## Also useful:

$$\mathbf{E} = E_0 \sin\left(\omega t \mp \beta z\right)\hat{x}$$

$$= \Re e\left\{-jE_0\left[\cos\left(\omega t \mp \beta z\right) + j\sin\left(\omega t \mp \beta z\right)\right]\right\}$$

$$= \Re e\left\{e^{-j\frac{\pi}{2}} E_0\, e^{j(\omega t \mp \beta z)}\right\}$$

$$= \Re e\left\{E_0\, e^{j\omega t} e^{\mp j(\beta z \pm \pi/2)}\right\}$$

$$= \Re e\left\{\tilde{\mathbf{E}}\, e^{j\omega t}\right\} \qquad \tilde{\mathbf{E}} = E_0\, e^{\mp j(\beta z \pm \pi/2)}$$

**In fact,** $\cos\left(\omega t \mp \beta z - \pi/2\right) = \sin\left(\omega t \mp \beta z\right)$

# Time-average Poynting vector in terms of phasors

$$\boxed{\langle \mathbf{E} \times \mathbf{H} \rangle = \frac{1}{2}\, \Re\mathfrak{e}\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\}}$$

**obtained starting from**

$$\langle \mathbf{S}(t) \rangle = \frac{1}{T}\int_0^T \mathbf{E}(t) \times \mathbf{H}(t)\, dt$$

**EXTRA**

# Proof (part 1)

$$\mathbf{E}(t) = \Re e\left\{\tilde{\mathbf{E}}e^{j\omega t}\right\} = \Re e\left[\left(\Re e\left\{\tilde{\mathbf{E}}\right\} + j\Im \mathrm{m}\left\{\tilde{\mathbf{E}}\right\}\right)\left(\cos\left(\omega t\right) + j\sin\left(\omega t\right)\right)\right]$$
$$= \Re e\left\{\tilde{\mathbf{E}}\right\}\cos\left(\omega t\right) - \Im \mathrm{m}\left\{\tilde{\mathbf{E}}\right\}\sin\left(\omega t\right)$$

$$\mathbf{H}(t) = \Re e\left\{\tilde{\mathbf{H}}e^{j\omega t}\right\} = \Re e\left[\left(\Re e\left\{\tilde{\mathbf{H}}\right\} + j\Im \mathrm{m}\left\{\tilde{\mathbf{H}}\right\}\right)\left(\cos\left(\omega t\right) + j\sin\left(\omega t\right)\right)\right]$$
$$= \Re e\left\{\tilde{\mathbf{H}}\right\}\cos\left(\omega t\right) - \Im \mathrm{m}\left\{\tilde{\mathbf{H}}\right\}\sin\left(\omega t\right)$$

$$\mathbf{E}(t)\times\mathbf{H}(t) = \Re e\left\{\tilde{\mathbf{E}}\right\}\times\Re e\left\{\tilde{\mathbf{H}}\right\}\cos^2\left(\omega t\right)$$
$$+\Im \mathrm{m}\left\{\tilde{\mathbf{E}}\right\}\times\Im \mathrm{m}\left\{\tilde{\mathbf{H}}\right\}\sin^2\left(\omega t\right)$$
$$-\left(\Re e\left\{\tilde{\mathbf{E}}\right\}\times\Im \mathrm{m}\left\{\tilde{\mathbf{H}}\right\} + \Im \mathrm{m}\left\{\tilde{\mathbf{E}}\right\}\times\Re e\left\{\tilde{\mathbf{H}}\right\}\right)\cos\left(\omega t\right)\sin\left(\omega t\right)$$

EXTRA

# Proof (part 2)

**Now, evaluate** $$\langle \mathbf{S}(t)\rangle = \frac{1}{T}\int_0^T \mathbf{E}(t)\times\mathbf{H}(t)\,dt$$

**The time-dependent integration of the three parts gives**

$$\frac{1}{T}\int_0^T \cos^2(\omega t)\,dt = \frac{1}{T}\left[\frac{t}{2}+\frac{\sin(2\omega t)}{4\omega}\right]_0^T = \frac{1}{2}$$

$$\frac{1}{T}\int_0^T \sin^2(\omega t)\,dt = \frac{1}{T}\left[\frac{t}{2}-\frac{\sin(2\omega t)}{4\omega}\right]_0^T = \frac{1}{2}$$

$$\frac{1}{T}\int_0^T \cos(\omega t)\sin(\omega t)\,dt = \frac{1}{T}\left[\frac{\sin^2(\omega t)}{2\omega}\right]_0^T = 0$$

EXTRA

## Proof (part 3)

**Finally we have**

$$\langle \mathbf{S}(t) \rangle = \frac{1}{T}\int_0^T \mathbf{E}(t)\times\mathbf{H}(t)\,dt$$

$$= \frac{1}{2}\Big[\underbrace{\Re e\{\tilde{\mathbf{E}}\}\times\Re e\{\tilde{\mathbf{H}}\} + \Im m\{\tilde{\mathbf{E}}\}\times\Im m\{\tilde{\mathbf{H}}\}}_{\boxed{\Re e\{\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^*\}}}\Big]$$

**Since we observe that**

$$\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^* = \Re e\{\tilde{\mathbf{E}}\}\times\Re e\{\tilde{\mathbf{H}}\} + \Im m\{\tilde{\mathbf{E}}\}\times\Im m\{\tilde{\mathbf{H}}\}$$
$$+ j\left(\Im m\{\tilde{\mathbf{E}}\}\times\Re e\{\tilde{\mathbf{H}}\} - \Re e\{\tilde{\mathbf{E}}\}\times\Im m\{\tilde{\mathbf{H}}\}\right)$$

$$\boxed{\langle \mathbf{S}(t) \rangle = \frac{1}{2}\Re e\left\{\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^*\right\}}$$

# Time-average Poynting vector

$$\boxed{\langle \mathbf{S}(t) \rangle = \frac{1}{2} \Re e \left\{ \tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* \right\}}$$

**Often the following quantity is called "complex Poynting vector"**

$$\vec{\mathbf{S}} = \frac{1}{2} \tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*$$

**but beware, it is NOT a phasor.**

$$\boxed{\langle \mathbf{S}(t) \rangle = \Re e \left\{ \vec{\mathbf{S}} \right\}}$$

**It means this**

$$\xcancel{\mathbf{S}(t) = \Re e \left\{ \vec{\mathbf{S}} e^{j\omega t} \right\}}$$

**NEVER this!**

## In phasor notation Maxwell's equations become

$$\nabla \cdot \widetilde{\mathbf{E}} = \tilde{\rho}_{\mathrm{V}}/\epsilon,$$

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$$

$$\nabla \cdot \widetilde{\mathbf{H}} = 0$$

$$\nabla \times \widetilde{\mathbf{H}} = \widetilde{\mathbf{J}} + j\omega\epsilon\widetilde{\mathbf{E}}$$

# In absence of charges and currents

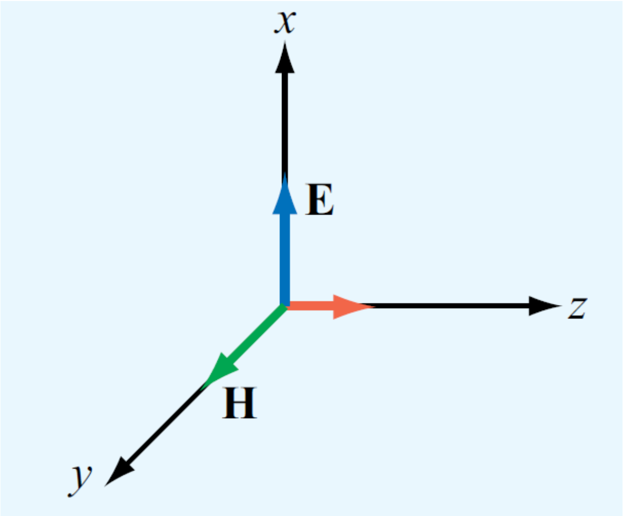
$$\nabla \cdot \widetilde{\mathbf{E}} = 0$$

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$$

$$\nabla \cdot \widetilde{\mathbf{H}} = 0$$

$$\nabla \times \widetilde{\mathbf{H}} = j\omega\epsilon_c\widetilde{\mathbf{E}}$$

# TEM wave in phasor notation




$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\widetilde{E}_x^+(z) = \hat{\mathbf{x}}E_{x0}^+ e^{-j\beta z}$$

$$\widetilde{\mathbf{H}}(z) = \hat{\mathbf{y}}\frac{\widetilde{E}_x^+(z)}{\eta} = \hat{\mathbf{y}}\frac{E_{x0}^+}{\eta}e^{-j\beta z}$$

**The field amplitudes may be <u>complex</u> (there is an additional phase)**

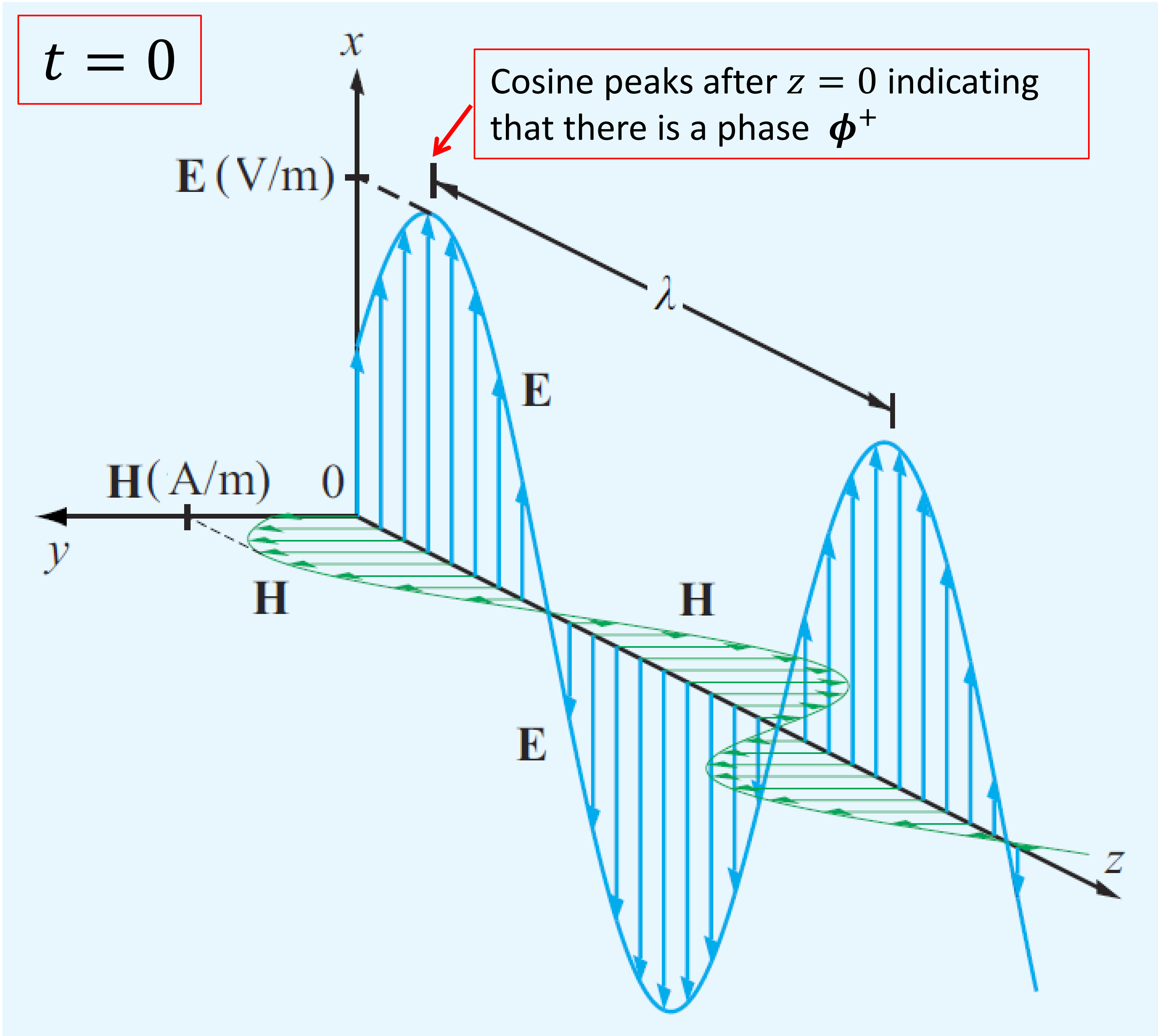
$$E_{x0}^+ = |E_{x0}^+|e^{j\phi^+}$$

**The corresponding time-dependent fields are**

$$\mathbf{E}(z,t) = \Re\mathfrak{e}\left[\widetilde{\mathbf{E}}(z)\, e^{j\omega t}\right] = \hat{\mathbf{x}}|E_{x0}^+|\cos(\omega t - \beta z + \phi^+) \quad \text{(V/m)}$$

$$\mathbf{H}(z,t) = \Re\mathfrak{e}\left[\widetilde{\mathbf{H}}(z)\, e^{j\omega t}\right] = \hat{\mathbf{y}}\frac{|E_{x0}^+|}{\eta}\cos(\omega t - \beta z + \phi^+) \quad \text{(A/m)}$$

# Representation of Electric and Magnetic field in space



**Example –** The electric field of a **1 MHz** plane wave, traveling along $+z$ in air, is $\hat{x}$-polarized. This field reaches a peak value of $1.2\pi$ (mV/m) at $t$ = 0 and $z$ = 50 m. Obtain expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$ and then plot them as a function of $z$ at $t$ = 0.

**At 1 MHz:**

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{1 \times 10^6} = 300 \text{ m}$$

$$\beta = (2\pi/300) \text{ (rad/m)}$$

$$\mathbf{E}(z, t) = \hat{\mathbf{x}}|E_{x0}^+| \cos(\omega t - \beta z + \phi^+)$$

$$= \hat{\mathbf{x}}\, 1.2\pi \cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \phi^+\right) \text{ (mV/m)}$$

**The field is maximum when the argument of the cosine equals 0 or 2π. Peak value was stated to be 1.2*π* (mV/m) at *t* = 0 and *z* = 50 m**

$$\cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \phi^+\right)$$

$$-\frac{2\pi \times 50}{300} + \phi^+ = 0 \quad \Rightarrow \quad \boxed{\phi^+ = \frac{\pi}{3}}$$

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\, 1.2\pi \cos\left(2\pi \times 10^6 t \boxed{- \frac{2\pi z}{300}} + \frac{\pi}{3}\right) \quad (\text{mV/m})$$

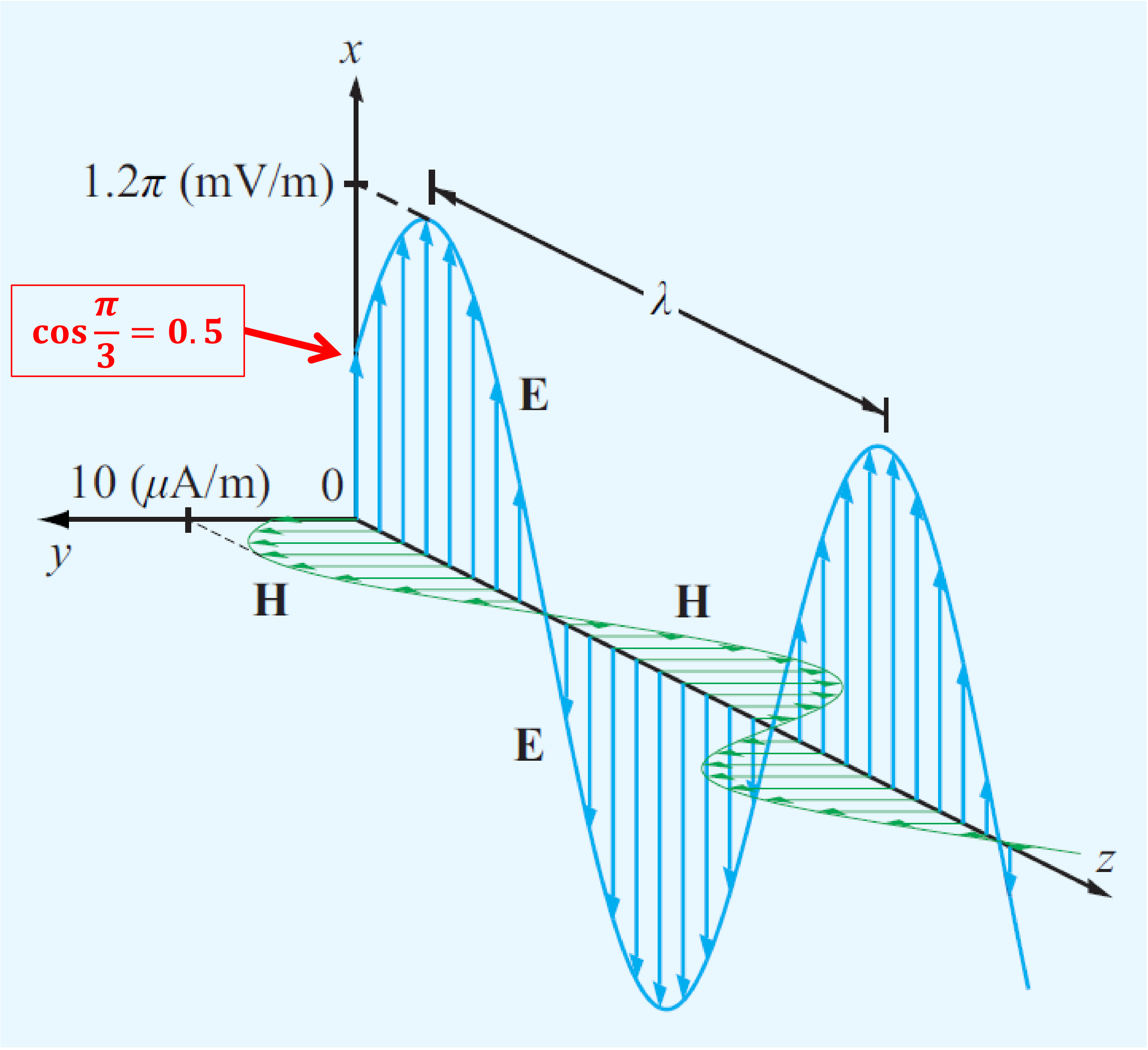
$= -\pi/3$ when $z = 50\text{m}$

$$\mathbf{H}(z,t) = \hat{\mathbf{y}}\, \frac{E(z,t)}{\eta_0}$$

using the approximation $\eta_0 \approx 120\pi \ (\Omega)$

$$= \hat{\mathbf{y}}\, 10 \cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \frac{\pi}{3}\right) \quad (\mu\text{A/m})$$

note choice of units



$x$
1.2π (mV/m)
λ
$\cos\frac{\pi}{3} = 0.5$
E
10 (μA/m)
0
$y$
H
H
E
$z$

$t = 0$