Lecture 21 Outline

• Transmission Matrix Method
• Gratings
• Coupled Mode Theory
Transfer Matrix Method

Sections 3.1-3.5 in Coldren, Corzine and Mašanović
Definition of the Scattering Matrix

**Linear Networks**  \( \mathbf{S} \) allows you to find outputs from inputs

**Inputs:**  \( a_n \)  \( \mathbf{b} = \mathbf{S} \mathbf{a} \)

**Outputs:**  \( b_n \)  \( b_i = \sum_j S_{ij} a_j \)

Can measure \( S_{ij} \) by

setting \( a_k = 0 \) for \( k \neq j \)

and measuring \( b_i \)

\[
\mathbf{E}(x,y,z,t) = \hat{e} E_0 U(x,y) e^{j(\omega t - \beta z)}
\]

\[
a_j = \frac{E_0}{\sqrt{2 \eta_j}} e^{-j \beta z} \quad \text{where} \quad \eta_j = \frac{377 \Omega}{\bar{\eta}_j}
\]

For  \( \int |U|^2 \, dx \, dy = 1 \) we have  \( a_j a_j^* = P_j^+ \)

The net power flowing into the port is:

\[
P_j = a_j a_j^* - b_j b_j^*
\]

Important case (2-port junction):

\[
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  S_{11} & S_{12} \\
  S_{21} & S_{22}
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2
\end{bmatrix}
\]

If network is reciprocal,  \( \mathbf{S}_r = \mathbf{S} \)

If network is lossless,  \( \mathbf{S} \) is unitary:  \( \mathbf{S}_r^* \mathbf{S} = 1 \)
Definition of the Transmission Matrix

T allows you to cascade networks

Left side: \( A_1 = a_1, B_1 = b_1 \)

Right side: \( A_2 = b_2, B_2 = a_2 \)

\[
\begin{bmatrix}
A_1 \\
B_1
\end{bmatrix} = \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix} \begin{bmatrix}
A_2 \\
B_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} = \begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix} \begin{bmatrix}
a_1 \\
a_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
T_{11} = \frac{1}{S_{21}} \\
T_{12} = -\frac{S_{22}}{S_{12}} \\
T_{21} = \frac{S_{11}}{S_{21}} \\
T_{22} = -\frac{S_{11}S_{22} - S_{12}S_{21}}{S_{21}}
\end{bmatrix}
\]

If network is reciprocal, scattering matrix is symmetric and \( \det T = 1 \)

If network is lossless, \( S \) is unitary: \( S^*S = 1 \)
<table>
<thead>
<tr>
<th>Scattering Matrix</th>
<th>Transmission Matrix</th>
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<tr>
<td><strong>Definition</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b_1 = S_{11}a_1 + S_{12}a_2$</td>
</tr>
<tr>
<td></td>
<td>$b_2 = S_{21}a_1 + S_{22}a_2$</td>
</tr>
<tr>
<td></td>
<td>$A_1 = T_{11}A_2 + T_{12}B_2$</td>
</tr>
<tr>
<td></td>
<td>$B_1 = T_{21}A_2 + T_{22}B_2$</td>
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<tr>
<td><strong>Relation to $r$ and $t$</strong></td>
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</tr>
<tr>
<td>$r_{12} = \frac{b_1}{a_1} \bigg</td>
<td><em>{a_2=0} = S</em>{11}$</td>
</tr>
<tr>
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</tr>
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</tr>
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</tr>
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<td>$S = \begin{bmatrix} r_{12} &amp; t_{21} \ t_{12} &amp; r_{21} \end{bmatrix}$</td>
<td>$T = \frac{1}{t_{12}} \begin{bmatrix} 1 &amp; -r_{21} \ r_{12} &amp; t_{12}t_{21} - r_{12}r_{21} \end{bmatrix}$</td>
</tr>
<tr>
<td>$\det S = S_{11}S_{22} - S_{12}S_{21} = r_{12}r_{21} - t_{12}t_{21}$</td>
<td>$\det T = T_{11}T_{22} - T_{12}T_{21} = t_{21}/t_{12}$</td>
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<td><strong>Relation to $T$-Matrix</strong></td>
<td></td>
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<tr>
<td>$S = \frac{1}{T_{11}} \begin{bmatrix} T_{21} &amp; \det T \ 1 &amp; -T_{12} \end{bmatrix}$</td>
<td>$T = \frac{1}{S_{21}} \begin{bmatrix} 1 &amp; -S_{22} \ S_{11} &amp; -\det S \end{bmatrix}$</td>
</tr>
</tbody>
</table>
### Network Properties and Their Consequences on the Matrix Coefficients

#### Reciprocal Network (valid for normalized fields with and without loss)

\[
S_{r} = S \rightarrow S_{12} = S_{21}
\]

\[
S = \begin{bmatrix}
S_{11} & S_{21} \\
S_{21} & S_{22}
\end{bmatrix} = \frac{1}{T_{11}} \begin{bmatrix}
T_{21} & 1 \\
1 & -T_{12}
\end{bmatrix}
\]

\[
T = \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & (T_{12}T_{21} + 1)/T_{11}
\end{bmatrix} = \frac{1}{S_{21}} \begin{bmatrix}
1 & -S_{22} \\
S_{11} & S_{21}^2 - S_{11}S_{22}
\end{bmatrix}
\]

#### Lossless Reciprocal Network

\[
|S_{11}|^2 + |S_{21}|^2 = 1 \quad |T_{21}|^2 + 1 = |T_{11}|^2
\]

\[
S_{r}^*S = 1 \rightarrow |S_{12}|^2 + |S_{22}|^2 = 1 \rightarrow 1 + |T_{12}|^2 = |T_{11}|^2
\]

\[
S_{11}^*S_{12} + S_{21}^*S_{22} = 0 \quad T_{21}^* - T_{12} = 0
\]

\[
S = \begin{bmatrix}
S_{11} & S_{21} \\
S_{21} & -S_{11}^*(S_{21}/S_{21}^*)
\end{bmatrix} = \frac{1}{T_{11}} \begin{bmatrix}
T_{21} & 1 \\
1 & -T_{21}^*
\end{bmatrix}
\]

\[
T = \begin{bmatrix}
T_{11} & T_{21}^* \\
T_{21} & T_{11}^*
\end{bmatrix} = \begin{bmatrix}
1/S_{21} & S_{11}/S_{21}^* \\
S_{11}/S_{21} & 1/S_{21}^*
\end{bmatrix}
\]

#### Lossless Reciprocal Network with \( r \) and \( t \) Phase Shifts of 0 or \( \pi \)

\[
S_{22} = -S_{11}
\]

\[
S_{11} = S_{11}^* \rightarrow \text{det } S = -1
\]

\[
S_{21} = S_{21}^* \rightarrow T_{22} = T_{11}, \quad T_{12} = T_{21}
\]

\[
S = \begin{bmatrix}
S_{11} & S_{21} \\
S_{21} & -S_{11}
\end{bmatrix} = \frac{1}{T_{11}} \begin{bmatrix}
T_{21} & 1 \\
1 & -T_{21}
\end{bmatrix}
\]

\[
T = \begin{bmatrix}
T_{11} & T_{21} \\
T_{21} & T_{11}
\end{bmatrix} = \frac{1}{S_{21}} \begin{bmatrix}
1 & S_{11} \\
S_{11} & 1
\end{bmatrix}
\]
Dielectric Interface

\[ S_{11} = \frac{b_1}{a_1}\bigg|_{a_2=0} = -r_1 = \frac{n_1 - n_2}{n_1 + n_2} \]

Same as Transmission Line Reflection Coefficient

\[ \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \]
Dielectric Interface

\[ S_{11} = \frac{b_1}{a_1} \bigg|_{a_2=0} = -r_1 = \frac{n_1 - n_2}{n_1 + n_2} \]

Same as Transmission Line Reflection Coefficient

\[ \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \]

using EM wave impedances \( \Rightarrow \)

\[ \frac{\eta_L - \eta_0}{\eta_L + \eta_0} = \frac{\sqrt{\frac{\mu_0}{\varepsilon_2}} - \sqrt{\frac{\mu_0}{\varepsilon_1}}}{\sqrt{\frac{\mu_0}{\varepsilon_1}} + \sqrt{\frac{\mu_0}{\varepsilon_2}}} = \frac{1}{\sqrt{\varepsilon_2}} - \frac{1}{\sqrt{\varepsilon_1}} = \frac{1}{\sqrt{\varepsilon_2} + \sqrt{\varepsilon_1}} \]

\[ = \frac{1}{\frac{n_2}{n_1}} - \frac{1}{\frac{n_1}{n_2}} = \frac{n_1 - n_2}{n_1 n_2} = \frac{n_1 - n_2}{n_1 + n_2} \]
Dielectric Interface

\[ S_{11} = \left. \frac{b_1}{a_1} \right|_{a_2=0} = -r_1 = \frac{n_1 - n_2}{n_1 + n_2} \]

Note \( r_1 \) is a positive real number if \( n_2 > n_1 \).

\[ S_{22} = \left. \frac{b_2}{a_2} \right|_{a_1=0} = r_2 = -(-r_1) = r_1 \]

Note \( r_2 \) is a negative real number if \( n_2 > n_1 \) (\( \pi \)-phase shift).

\[ S_{12} = S_{21} = t = \sqrt{1-r_1^2} \]

\[
S = \begin{bmatrix}
  -r_1 & t \\
  t & r_1
\end{bmatrix}
\]

\[
T = \begin{bmatrix}
  T_{11} & T_{21} \\
  T_{21} & T_{11}
\end{bmatrix} = \frac{1}{S_{21}} \begin{bmatrix}
  1 & S_{11} \\
  S_{11} & 1
\end{bmatrix}
\]

\[
T = \frac{1}{t} \begin{bmatrix}
  1 & -r_1 \\
  -r_1 & 1
\end{bmatrix}
\]
no reflection in uniform transmission line, only transmission

$$\begin{align*}
    b_1 &= a_2 e^{-j\tilde{\beta}L} \\
    b_2 &= a_1 e^{-j\tilde{\beta}L} \\
    \tilde{\beta} &= \beta + j\beta_i \\
    \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} &= \begin{bmatrix} 0 & e^{-j\tilde{\beta}L} \\ e^{-j\tilde{\beta}L} & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}
\end{align*}$$

$$S = \begin{bmatrix} 0 & e^{-j\tilde{\beta}L} \\ e^{-j\tilde{\beta}L} & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} e^{j\tilde{\beta}L} & 0 \\ 0 & e^{-j\tilde{\beta}L} \end{bmatrix}$$
no reflection in uniform transmission line, only transmission

\[
b_2 = a_1 e^{-j\tilde{\beta} L}, \quad a_2 = b_1 e^{+j\tilde{\beta} L} \\
\tilde{\beta} = \beta + j\beta_i
\]
### Summary of Building Blocks for S and T

**Table 3.3: Summary of S- and T-matrices for Simple “Building-Block” Components**

<table>
<thead>
<tr>
<th>Scattering Matrix</th>
<th>Structure</th>
<th>Transmission Matrix</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
    r_{12} & t_{12} \\
    t_{12} & -r_{12}
\end{bmatrix}
\] | \[\begin{array}{c}
\begin{array}{c}
1 \\
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2 \\
2
\end{array}
\end{array}
\] \[\begin{array}{c}
\begin{array}{c}
2 \\
L
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2 \\
2
\end{array}
\end{array}
\] | \[\frac{1}{t_{12}} \begin{bmatrix}
    1 & r_{12} \\
    r_{12} & 1
\end{bmatrix}\] |

\[r_{21} = -r_{12} \text{ and } t_{21} = t_{12}\]

\[r_{12}^2 + t_{12}^2 = 1\]

| \[\begin{bmatrix}
    0 & e^{-j\phi} \\
    e^{-j\phi} & 0
\end{bmatrix}\] | \[\leq \begin{array}{c}
\begin{array}{c}
2 \\
2
\end{array}
\end{array}\] | \[\begin{bmatrix}
    e^{j\phi} & 0 \\
    0 & e^{-j\phi}
\end{bmatrix}\] |

\[\phi = \tilde{\beta}_2 L\]

| \[\begin{bmatrix}
    r_{12} & t_{12}e^{-j\phi} \\
    t_{12}e^{-j\phi} & -r_{12}e^{-j2\phi}
\end{bmatrix}\] | \[\begin{array}{c}
\begin{array}{c}
1 \\
2
\end{array}
\begin{array}{c}
\begin{array}{c}
2 \\
2
\end{array}
\end{array}
\] \[\begin{array}{c}
\begin{array}{c}
2 \\
L
\end{array}
\end{array}\] | \[\frac{1}{t_{12}} \begin{bmatrix}
    e^{j\phi} & r_{12}e^{-j\phi} \\
    r_{12}e^{j\phi} & e^{-j\phi}
\end{bmatrix}\] |

\[r_{12}^2 + t_{12}^2 = 1\]
Fabry-Perot Cavity

We can write these relations

\[ b_1 = -a_1 r_1 + a'_1 t_1, \]
\[ b'_1 = a_1 t_1 + a'_1 r_1, \]
\[ b_2 = a'_2 t_2, \]
\[ b'_2 = a'_2 r_2. \]

assume initially \( a_2 = 0 \).

Solve for

\[ S_{11} = b_1/a_1 \quad S_{21} = b_2/a_1 \]
After a few manipulations:

\[ a_2 = 0 \]
\[ a_1 = 0 \]

\[ S_{11} = -r_1 + \frac{t_1^2 r_2 e^{-2j\beta L}}{1 - r_1 r_2 e^{-2j\beta L}}, \]
\[ S_{21} = \frac{t_1 t_2 e^{-j\beta L}}{1 - r_1 r_2 e^{-2j\beta L}}. \]
\[ S_{22} = -r_2 + \frac{t_2^2 r_1 e^{-2j\beta L}}{1 - r_1 r_2 e^{-2j\beta L}}, \]
\[ S_{12} = S_{21}. \]
The Transmission Matrix for the cavity can be obtained from the Scattering Matrix or by multiplication of elementary Transmission Matrices for the interfaces and the transmission line.

\[
T = T_1 \cdot T_2 \cdot T_3 = \frac{1}{t_1} \begin{bmatrix} 1 & -r_1 \\ -r_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{j\beta L} & 0 \\ 0 & e^{-j\beta L} \end{bmatrix} \cdot \frac{1}{t_2} \begin{bmatrix} 1 & r_2 \\ r_2 & 1 \end{bmatrix}
\]
The Transmission Matrix for the cavity can be obtained from the Scattering Matrix or by multiplication of elementary Transmission Matrices for the interfaces and the transmission line.

\[
T_{11} = \frac{1}{t_1t_2} [e^{j\tilde{\beta}L} - r_1r_2e^{-j\tilde{\beta}L}], \\
T_{12} = -\frac{1}{t_1t_2} [r_1e^{-j\tilde{\beta}L} - r_2e^{j\tilde{\beta}L}], \\
T_{21} = -\frac{1}{t_1t_2} [r_1e^{j\tilde{\beta}L} - r_2e^{-j\tilde{\beta}L}], \\
T_{22} = \frac{1}{t_1t_2} [e^{-j\tilde{\beta}L} - r_1r_2e^{j\tilde{\beta}L}],
\]
Lossless Fabry-Perot Cavity Spectra for S

Etalon is highly reflective if $L=\text{odd quarter-integer multiple of } \lambda/n_2$.

Etalon is 100% transmissive if $L=\text{half-integer multiple of } \lambda/n_2$.

$S_{21}^{\text{max}}<1$ with loss.

Cleaved cavity $r=0.565$. 
Transmission Line – Interface – Transmission Line

(Example 3.1 in Coldren, Corzine and Mašanović)

\[
T = T_1 \cdot T_2 \cdot T_3 = \begin{bmatrix} e^{j\phi_1} & 0 \\ 0 & e^{-j\phi_1} \end{bmatrix} \cdot \frac{1}{t_{12}} \begin{bmatrix} 1 & -r_{12} \\ -r_{12} & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{j\phi_2} & 0 \\ 0 & e^{-j\phi_2} \end{bmatrix}
\]

\[
= \frac{1}{t_{12}} \begin{bmatrix} e^{j(\phi_1+\phi_2)} & -r_{12}e^{j(\phi_1-\phi_2)} \\ -r_{12}e^{j(\phi_2-\phi_1)} & e^{-j(\phi_1+\phi_2)} \end{bmatrix}.
\]
Application to Gratings (Distributed Bragg Reflectors)

- At the Bragg frequency, the period of the grating is half of the average optical wavelength in the medium.
- For each period, multiply 4 simple T-Matrices together
Period of a uniform grating structure

Add one matrix to the previous case to account for the additional interface
T-matrix for the complete grating structure

\[ T_g = [T_i]_m = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}^m \]
Application to Gratings (Distributed Bragg Reflectors)

One period $L_1 + L_2$

\[
L_1 = \frac{\lambda_{\text{design}}}{4n_1}
\]

\[
L_2 = \frac{\lambda_{\text{design}}}{4n_2}
\]

design choice
Application to Gratings (Distributed Bragg Reflectors)

At the Bragg condition, elements for the T-matrix of one period are

\[
T_{11} = \frac{1}{t^2} [e^{j\phi^+} - r^2 e^{-j\phi^-}] \rightarrow -\frac{1 + r^2}{t^2},
\]

\[
T_{21} = \frac{r}{t^2} [e^{j\phi^+} - e^{-j\phi^-}] \rightarrow -\frac{2r}{t^2},
\]

\[
T_{12} = \frac{r}{t^2} [e^{-j\phi^+} - e^{j\phi^-}] \rightarrow -\frac{2r}{t^2},
\]

\[
T_{22} = \frac{1}{t^2} [e^{-j\phi^+} - r^2 e^{j\phi^-}] \rightarrow -\frac{1 + r^2}{t^2},
\]

\[
\phi_{\pm} \equiv \tilde{\beta}_1 L_1 \pm \tilde{\beta}_2 L_2 \text{ becomes either 0 or } \pi \text{ at the Bragg condition.}
\]
Reflected Amplitude/Phase for Gratings

Example with $m = 20$ and $r = 0.1, 0.025, 0.01$

\[ \delta = \beta - \beta_0 = \text{detuning parameter} \]

$\beta$ is the average propagation constant of the grating

$L_g$ is the grating length

\[ \beta = \frac{\beta_1/n_1 + \beta_2/n_2}{1/n_1 + 1/n_2} \]

the phase delay of each layer is $\beta_1 L_1 = \beta_2 L_2 = \pi/2$
Reflected Amplitude for Gratings
Reflected Phase for Gratings

\[ \angle r_g \]

\[ \delta L_g / \pi \]

\[ 2mr = 4 \]

\[ 0.4 \]
Distributed Feedback Structures

Coupled-mode theory
Mode coupling

In an ideal **uniform waveguide**, the basis modes are **orthogonal** and propagate independently.

With the introduction of perturbations, modes may couple. Perturbations characterized by periodicity favors coherent coupling. Practical example is the **grating**.

Two waveguides brought very close together may also experience coupling if the modes in one guides overlap with the ones in the other. **Directional couplers** are based on this principle.

In **non-uniform waveguides** with variable cross-sections, transverse modes may experience a great deal of coupling, depending on the overlap integral for the mode envelopes when progressing from one section to another.
Coupled mode theory

We consider first the coupling between two identical modes which share the same volume in a waveguide but propagate in opposite directions.

First we express the field as the sum of a forward and a backward component, which we plug into the Helmholtz equation, where the permittivity varies along the axis direction according to a periodic perturbation.

Under the assumption of weak perturbation, the field amplitudes vary slowly (similar to the premise of the Beam Propagation method). This allows us to neglect terms containing second order derivatives of the amplitudes in the expansion of the wave equation.
Gratings Structure

A grating structure can be considered as a perturbation on a uniform slab waveguide:
\[ \varepsilon(x, z) = \varepsilon^{(0)}(x) + \Delta\varepsilon(x, z) \]

For the unperturbed slab waveguide

**TE**₀ mode

Forward: \[ A_0 e^{i\beta_0 z} \]  
Backward: \[ B_0 e^{-i\beta_0 z} \]

\( \varepsilon(x, z) \) is periodic in \( z \)

Fourier transform:
\[ \Delta\varepsilon(x, z) = \varepsilon_0 \sum_{p=-\infty}^{\infty} \Delta\varepsilon_p(x) e^{ip\frac{2\pi}{\Lambda} z} \]

For a lossless structure:
\[ \Delta\varepsilon^*_p(x) = \Delta\varepsilon_{-p}(x) \]

period \( \Lambda = L_1 + L_2 \)
Example: Grating Structure

Consider the TE0 fundamental mode for the unperturbed waveguide

The forward wave is composed of partial transmissions of itself at the interfaces and partial reflections of the backward wave

NOTE: Here $\beta_0$ indicates the $z$-component of the wave vector for the TE0 mode in the material (same as $k_z$ used earlier) for the unperturbed waveguide with the average index.
Grating Coupled Mode Equations 1

TE Polarization: $E = \hat{y}E_y(x, z)$

Wave Equation for a time-harmonic field ($e^{-i\omega t}$):

$$
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu \varepsilon^{(0)}(x) \right] E_y = -\omega^2 \mu \Delta \varepsilon(x, z) E_y
$$

Consider the sum of unperturbed $+z$ and $-z$ propagating $TE_0$ modes:

$$
E_y(x, z) = \left[ A_0(z)e^{i\beta_0 z} + B_0(z)e^{-i\beta_0 z} \right] E_{y}^{(0)}(x)
$$

Assume $A_0(z)$ and $B_0(z)$ are slowly varying functions due to the distributed feedback effect.

The transverse modal distribution $E_{y}^{(0)}(x)$ satisfies the equation:

$$
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu \varepsilon^{(0)}(x) \right] E_{y}^{(0)}(x) e^{\pm i\beta_0 z} = 0
$$
Grating Coupled Mode Equations 1

TE Polarization: \( \mathbf{E} = \hat{y}E_y(x, z) \)

Wave Equation for a time-harmonic field \( \left( e^{-i\omega t} \right) \):

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu \varepsilon^{(0)}(x) \right] E_y = -\omega^2 \mu \Delta \varepsilon(x, z) E_y
\]

Consider the sum of unperturbed +z and -z propagating \( TE_0 \) modes:

\[
E_y(x, z) = \left[ A_0(z)e^{i\beta_0 z} + B_0(z)e^{-i\beta_0 z} \right] E_y^{(0)}(x)
\]

Assume \( A_0(z) \) and \( B_0(z) \) are slowly varying functions due to the distributed feedback effect.

The transverse modal distribution \( E_y^{(0)}(x) \) satisfies the equation:

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu \varepsilon^{(0)}(x) \right] E_y^{(0)}(x) e^{\pm i\beta_0 z} = 0
\]
Grating Coupled Mode Equations 2

Substitute \( E = \left[ A_0(z)e^{i\beta_0z} + B_0(z)e^{-i\beta_0z} \right] E_y^{(0)}(x) \hat{y} \) into the wave equation:

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu \varepsilon^{(0)}(x) \right] \left[ A_0(z)e^{i\beta_0z} + B_0(z)e^{-i\beta_0z} \right] E_y^{(0)}(x) = -\omega^2 \mu \Delta \varepsilon(x, z) \left[ A_0(z)e^{i\beta_0z} + B_0(z)e^{-i\beta_0z} \right] E_y^{(0)}(x)
\]

Use the product rule for \( z \)-derivative. The unperturbed solution gives zero when differentiating \( e^{\pm i\beta_0z} \) twice. Use the assumption that \( A, B \) are slowly varying functions to drop the 2\(^{nd} \) order derivatives of \( A_0(z), B_0(z) \). This leaves the cross-term with 1 derivative of \( e^{\pm i\beta_0z} \) times 1 derivative of \( A_0(z), B_0(z) \):

**Product Rule:**

\[
\frac{\partial^2}{\partial z^2} fg = \frac{\partial^2 f}{\partial z^2} g + 2 \frac{\partial f}{\partial z} \frac{\partial g}{\partial z} + f \frac{\partial^2 g}{\partial z^2}
\]
We obtain:

\[
\left( 2i \beta_0 \frac{\partial A_0}{\partial z} e^{i \beta_0 z} - 2i \beta_0 \frac{\partial B_0}{\partial z} e^{-i \beta_0 z} \right) E_y^{(0)}(x) = 
\]

\[-\omega^2 \mu \varepsilon_0 \sum_p \Delta \varepsilon_p (x) e^{i p \frac{2\pi}{\Lambda} z} \left[ E_y^{(0)}(x) \right] \left[ A_0(z) e^{i \beta_0 z} + B_0(z) e^{-i \beta_0 z} \right] \]

Multiply both sides by \(- H_x^{(0)*}(x) = (\beta_0 / 2 \omega \mu) E_y^{(0)*}(x)\)

-x component of magnetic field for TE\(_0\) mode

Use normalization of guided power \( \frac{\beta_0}{2 \omega \mu} \int_{-\infty}^{\infty} |E_y^{(0)}(x)|^2 \, dx = 1 \)
Grating Coupled Mode Equations 4a

The coupled modes equation may have a solution that coherently adds to the forward wave the components of the backward wave which are reflected in the forward direction

\[
\left(2i\beta_0 \frac{\partial A_0}{\partial z} e^{i\beta_0 z} - 2i\beta_0 \frac{\partial B_0}{\partial z} e^{-i\beta_0 z}\right) E_y^{(0)}(x) = -\omega^2 \mu_0 \varepsilon_0 \sum_p \Delta \varepsilon_p(x) e^{i p \frac{2\pi}{\Lambda} z} E_y^{(0)}(x) \left[A_0(z) e^{i\beta_0 z} + B_0(z) e^{-i\beta_0 z}\right]
\]

if there are terms in the expansion of the permittivity which put the reflected backward wave components in phase with the forward wave.
Similarly, there is a solution that coherently adds to the backward wave the components of the forward wave which are reflected in the backward direction

\[
\left( 2i \beta_0 \frac{\partial A_0}{\partial z} e^{i \beta_0 z} - 2i \beta_0 \frac{\partial B_0}{\partial z} e^{-i \beta_0 z} \right) E_y^{(0)}(x) = -\omega^2 \mu \varepsilon_0 \sum_p \Delta \varepsilon_p(x) e^{i p \frac{2\pi}{\Lambda} z} E_y^{(0)}(x) \left[ A_0(z) e^{i \beta_0 z} + B_0(z) e^{-i \beta_0 z} \right]
\]

if there are terms in the expansion of the permittivity which put the reflected forward wave components in phase with the backward wave.
Grating Coupled Mode Equations 5

\[ e^{i\beta_0 z} \text{ matches up approximately with } e^{-i\beta_0 z + i\left(\frac{2\pi}{\Lambda}\right)z} \text{ when:} \]

\[ -\beta_0 + p\left(\frac{2\pi}{\Lambda}\right) \text{ is close to } \beta_0 \text{ for a particular } p = \ell \]

1st order grating when \((\ell = \pm 1)\)
With negligible coupling to other modes:

\[
\frac{dA_0(z)}{dz} = iK_{ab}B_0(z)e^{-i2\left(\beta_0 - \frac{\ell\pi}{\Lambda}\right)z}
\]

\[\iff\text{ Coupling Relationship}\]

where

\[K_{ab} = \frac{\omega}{4}\varepsilon_0\int_{-\infty}^{\infty} \Delta\varepsilon_\ell(x)|E_y^{(0)}(x)|^2 \, dx\]

coupling coefficient

A second coupled mode equation can then be determined:

\[
\frac{dB_0(z)}{dz} = iK_{ba}A_0(z)e^{i2\left(\beta_0 - \frac{\ell\pi}{\Lambda}\right)z}
\]

where:

\[K_{ba} = -\frac{\omega}{4}\varepsilon_0\int_{-\infty}^{\infty} \Delta\varepsilon_{-\ell}(x)|E_{-y}^{(0)}(x)|^2 \, dx\]

coupling coefficient

Condition for contra-directional coupling (lossless coupler):

\[K_{ba} = -K_{ab}^* \quad \text{since} \quad \Delta\varepsilon_\ell^* = \Delta\varepsilon_{-\ell}\]

Phase matching condition requires:

\[-\beta_0 \approx \beta_0 - (2\pi / \Lambda) \ell\]
Grating Design – The period of the grating $\Lambda$ should be chosen to satisfy the relation:

$$\beta_0 = \beta_B \equiv \ell \frac{\pi}{\Lambda} = \frac{2\pi}{\lambda_B} \bar{n}$$

from which we obtain the Bragg wavelength (defined in free space) at which the grating maximizes reflection

$$\lambda_B = \frac{2\Lambda \bar{n}}{\ell} \quad \rightarrow \quad \Lambda = \ell \frac{\lambda_B}{2\bar{n}}$$

$\Rightarrow$ The length of the grating period should be an integer multiple of Bragg half-wavelengths (Bragg condition).

For first-order grating, we choose $\ell = 1$. Higher orders of $\ell$ allow the use of longer gratings for the same $\lambda_0$. 

This is the “design” wavelength
Coupled Mode Equations for DFBs

\[
\frac{dA_0(z)}{dz} = iK_{ab}B_0(z)e^{-i\left(\beta_0 - \frac{l\pi}{\Lambda}\right)z}
\]
\[
\frac{dB_0(z)}{dz} = iK_{ba}A_0(z)e^{i\left(\beta_0 - \frac{l\pi}{\Lambda}\right)z}
\]

This set of coupled differential equations is not yet amenable to easy analytical treatment.

We introduce:

\[\Delta\beta = \beta_0 - \beta_B \quad \text{where} \quad \beta_B = \frac{\pi}{\Lambda} = \frac{2\pi}{\lambda_{design}}\]

\[A(z) = A_0(z)e^{i\Delta\beta z}\]
\[B(z) = B_0(z)e^{-i\Delta\beta z}\]

Rework the coupled differential equations into this form:

\[
\frac{dA(z)}{dz} = A_0(z)\frac{d}{dz}\left(e^{i\Delta\beta z}\right) + \frac{dA_0(z)}{dz}e^{i\Delta\beta z}
\]
Solution for index grating DFB structure

\[
\frac{dA(z)}{dz} = i\Delta\beta A(z) + iK_{ab} B(z) \\
\frac{dB(z)}{dz} = iK_{ba} A(z) - i\Delta\beta B(z)
\]

or equivalently in matrix form:

Coupled Mode Equations:

\[
\frac{d}{dz} \begin{bmatrix} A(z) \\ B(z) \end{bmatrix} = i \begin{bmatrix} \Delta\beta & K_{ab} \\ K_{ba} & -\Delta\beta \end{bmatrix} \begin{bmatrix} A(z) \\ B(z) \end{bmatrix}
\]

incident \hspace{1cm} A(0) \hspace{1cm} \rightarrow \hspace{1cm} \text{transmitted} \hspace{1cm} A(L) \hspace{1cm} \rightarrow \hspace{1cm} \text{reflected} \hspace{1cm} B(L) = 0

z = 0 \hspace{1cm} \hspace{1cm} \hspace{1cm} z = L
Solution steps

Assume lossless gratings \(( K \equiv K_{ab} \quad \text{and} \quad K_{ba} = -K^* )\) and eigensolution with form

\[
\begin{bmatrix}
A(z) \\
B(z)
\end{bmatrix} = \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} e^{iqz}
\]

The corresponding eigenequation is

\[
\begin{bmatrix}
\Delta \beta - q & K \\
-K^* & \Delta \beta - q
\end{bmatrix}
\begin{bmatrix}
A_0 \\
B_0
\end{bmatrix} = 0
\]

with eigenvalues

\[
q_{\pm} = \pm \sqrt{(\Delta \beta)^2 - |K|^2} = \pm iS
\]

\[
S = \sqrt{|K|^2 - (\Delta \beta)^2}
\]
Solution steps

We keep only two counterpropagating modes for forward and backward TE\(_0\) propagation with wave vectors

\[
\beta_\pm = \frac{\pi}{\Lambda} \pm \sqrt{(\Delta \beta)^2 - |K|^2}
\]

For

\[
|\Delta \beta| = \left| \beta(\omega) - \frac{\pi}{\Lambda} \right| < K
\]

with eigenvalues

\[
\beta_\pm = \frac{\pi}{\Lambda} \pm i \sqrt{|K|^2 - (\Delta \beta)^2} = \frac{\pi}{\Lambda} \pm i S
\]

where \( S = \sqrt{|K|^2 - (\Delta \beta)^2} \) defines a circle of radius \(|K|\)

In momentum space this circle defines a “stopband” for solutions which do not correspond to propagation. This is analogous to the forbidden energy gap in the band structure of semiconductors (the grating behaves like a 1D crystal).
\[
\delta \to \Delta \beta \\
\beta_0 \to \beta_B
\]

\[
\beta_0 \to \beta_B
\]

in Chuang

**FIGURE 6.4:** An \( \omega - \beta \) diagram for the coupled \( \beta_g \)-solutions to contradirectional coupling in a grating, with the uncoupled solutions denoted by thinner straight lines. Each of the grating-generated replica solutions (dashed) and ordinary forward and backward wave solutions (solid) correspond to one of the \( A \) or \( B \) coefficients indicated. The extent of the stopband in both \( \omega \) and \( \beta \) directions is also shown (where \( v_g \) is the group velocity of the unperturbed mode). However, the complex \( \tilde{\beta}_g \)-solutions which exist throughout the stopband are not shown. Finally, the scale of the stopband has been exaggerated somewhat, considering that \( \kappa \ll \pi/\Lambda \) to satisfy the weak coupling criterion.
Analytical solution for reflection and transmission coefficients

\[ \Gamma(0) = \frac{B(0)}{A(0)} \]

\[ t(L) = \frac{A(L)}{A(0)} \]

reflection

transmitted

\[ z = 0 \]

\[ z = L \]

incident

transmitted

\[ A(0) \]

\[ B(0) \]

\[ A(L) \]

\[ B(L) = 0 \]

\[ \Gamma(0) = \frac{-iK_{ba} \sin qL}{q \cos qL - i\Delta\beta \sin qL} \]

\[ t(L) = \frac{q}{a \cos qL - i\Delta\beta \sin qL} \]

within the stop band:

\[ \Gamma(0) = \frac{-iK^* \sinh SL}{S \cosh SL - i\Delta\beta \sinh SL} \]

\[ t(L) = \frac{iS}{\Delta\beta \sinh SL + iS \cosh SL} \]

at \( \Delta\beta = 0, \ S = |K| \)

\[ |\Gamma(0)| = \tanh |KL| \]
Applications of gratings in lasers – Distributed Feedback (DFB) Laser

\[ L_a = \frac{\lambda}{4\bar{n}} \]

from Coldren, Corzine and Mašanović
Applications of gratings in lasers – Distributed Bragg Reflector (DBR) laser

Edge emitting lasers

from Coldren, Corzine and Mašanović
Applications of gratings in lasers – Distributed Bragg Reflector (DBR) laser

Vertical Cavity Surface Emitting (VCSEL) lasers

from Coldren, Corzine and Mašanović
Reading Assignments:

• Section 8.6 of Chuang’s book
• Section 10.6 of Chuang’s book

• Sections 3.1-3.5 in Coldren, Corzine and Mašanović
• Chapter 6 in Coldren, Corzine and Mašanović (supplemental)
• Section 8.2.5 in Coldren, Corzine and Mašanović