ECE 329 – Fall 2021

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

Lecture 22 & 23

Lecture 22&23 – Outline

- **Phasor wave equation**
- **Conductivity in realistic media**
- **Wave attenuation**
- **Poynting vector (power) attenuation**
- **Examples**

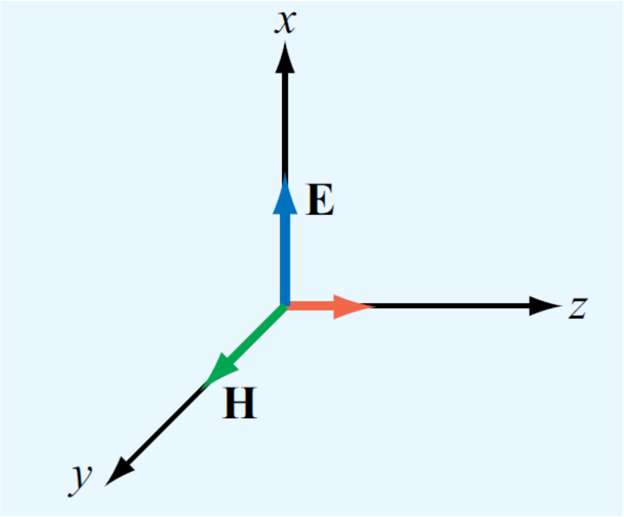
Reading assignment

Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:

22) Phasor form of Maxwell's equations and damped waves in conducting media

23) Imperfect dielectrics, good conductors

TEM wave in phasor notation



$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\widetilde{E}_x^+(z) = \hat{\mathbf{x}}E_{x0}^+ e^{-j\beta z}$$

$$\widetilde{\mathbf{H}}(z) = \hat{\mathbf{y}}\frac{\widetilde{E}_x^+(z)}{\eta} = \hat{\mathbf{y}}\frac{E_{x0}^+}{\eta} e^{-j\beta z}$$

The field amplitudes may be complex (there is an additional phase)

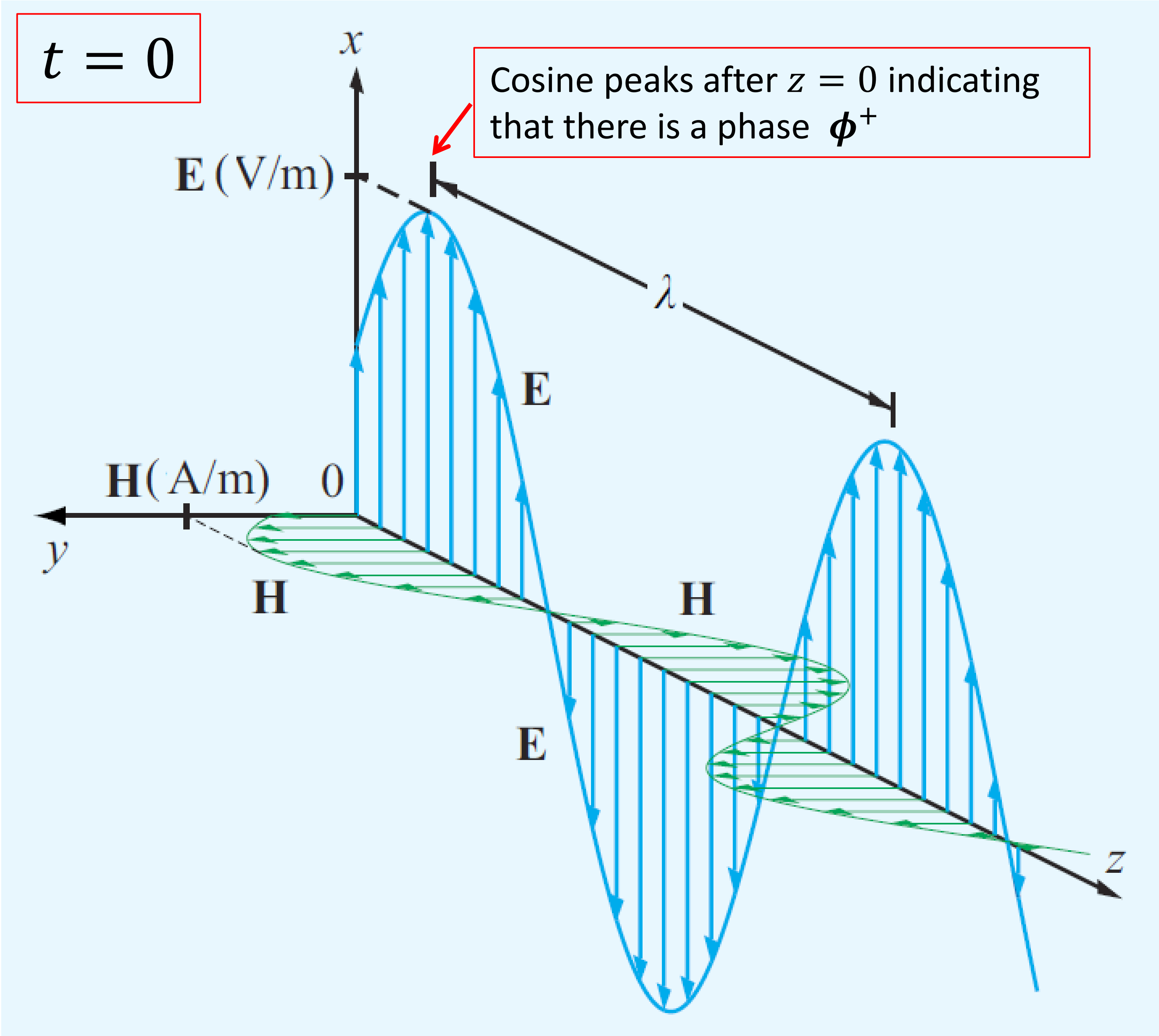
$$E_{x0}^+ = |E_{x0}^+| e^{j\phi^+}$$

The corresponding time-dependent fields are

$$\mathbf{E}(z,t) = \Re\mathfrak{e}\left[\widetilde{\mathbf{E}}(z)\, e^{j\omega t}\right] = \hat{\mathbf{x}}|E_{x0}^+|\cos(\omega t - \beta z + \phi^+) \quad \text{(V/m)}$$

$$\mathbf{H}(z,t) = \Re\mathfrak{e}\left[\widetilde{\mathbf{H}}(z)\, e^{j\omega t}\right] = \hat{\mathbf{y}}\frac{|E_{x0}^+|}{\eta}\cos(\omega t - \beta z + \phi^+) \quad \text{(A/m)}$$

Representation of Electric and Magnetic field in space



Example – The electric field of a **1 MHz** plane wave, traveling along +z in air, is $\hat{x}$-polarized. This field reaches a peak value of 1.2π (mV/m) at t = 0 and z = 50 m. Obtain expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$ and then plot them as a function of z at t = 0.

At 1 MHz:

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{1 \times 10^6} = 300 \text{ m}$$

$$\beta = (2\pi/300) \text{ (rad/m)}$$

$$\mathbf{E}(z, t) = \hat{\mathbf{x}}|E_{x0}^+| \cos(\omega t - \beta z + \phi^+)$$

$$= \hat{\mathbf{x}}\, 1.2\pi \cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \phi^+\right) \text{ (mV/m)}$$

The field is maximum when the argument of the cosine equals 0 or 2π. Peak value was stated to be 1.2*π* (mV/m) at *t* = 0 and *z* = 50 m

$$\cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \phi^+\right)$$

$$-\frac{2\pi \times 50}{300} + \phi^+ = 0 \quad \Rightarrow \quad \boxed{\phi^+ = \frac{\pi}{3}}$$

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\, 1.2\pi \cos\left(2\pi \times 10^6 t \boxed{- \frac{2\pi z}{300}} + \frac{\pi}{3}\right) \quad (\text{mV/m})$$

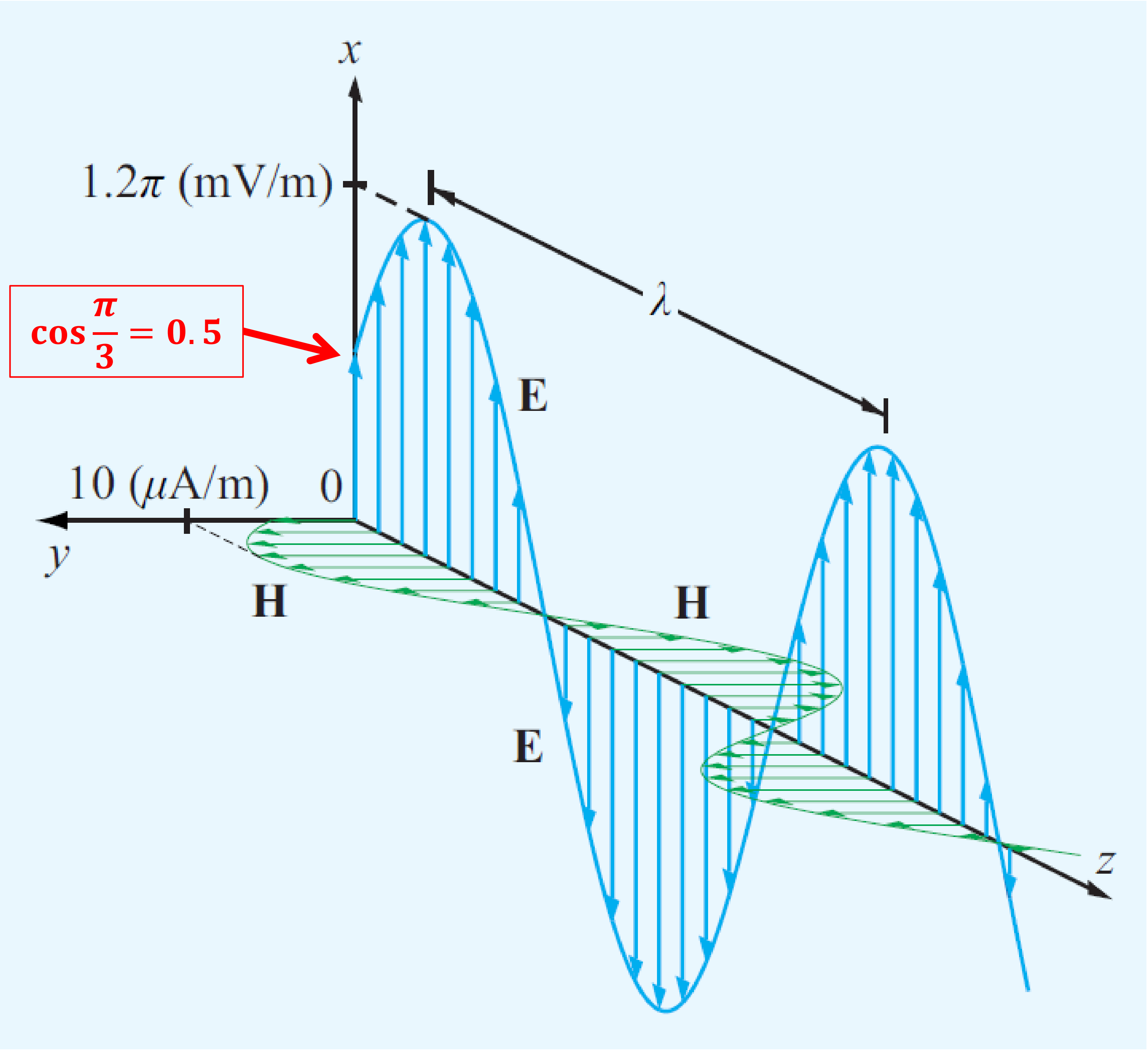
$= -\pi/3$ when $z = 50\text{m}$

$$\mathbf{H}(z,t) = \hat{\mathbf{y}}\, \frac{E(z,t)}{\eta_0}$$

using the approximation $\eta_0 \approx 120\pi \ (\Omega)$

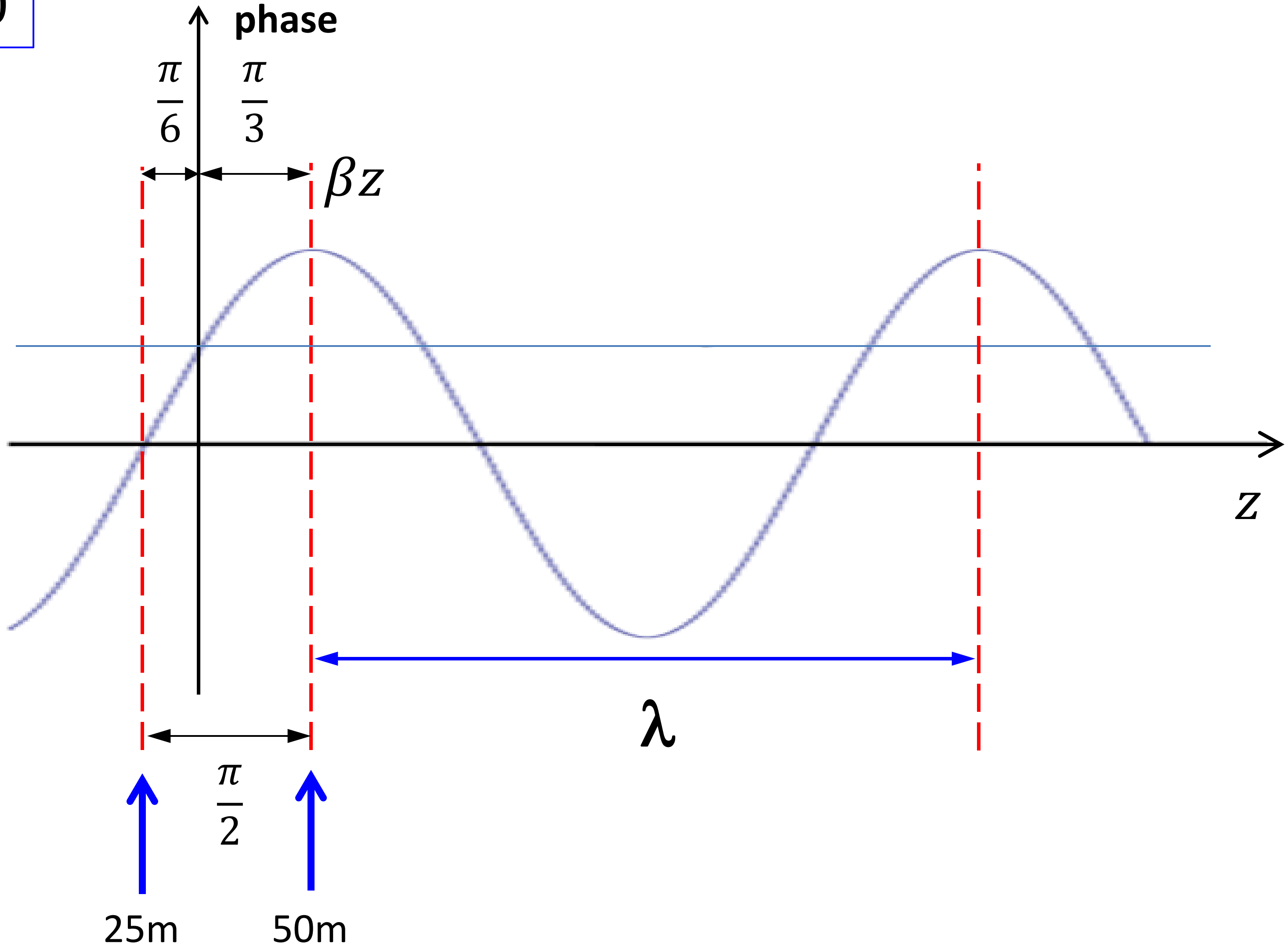
$$= \hat{\mathbf{y}}\, 10 \cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \frac{\pi}{3}\right) \quad (\underline{\mu\text{A/m}})$$

note choice of units



x
1.2π (mV/m)
λ
$\cos\frac{\pi}{3} = 0.5$
E
10 (μA/m)
0
y
H
H
E
z

$t = 0$



In the phasor domain, we can substitute

$$\frac{\partial}{\partial t} \rightarrow j\omega$$

so that Maxwell's equations are rewritten as

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}
\end{aligned}$$

$$\Longrightarrow$$

$$\begin{aligned}
\nabla \cdot \tilde{\mathbf{D}} &= \tilde{\rho} \\
\nabla \cdot \tilde{\mathbf{B}} &= 0 \\
\nabla \times \tilde{\mathbf{E}} &= -j\omega \tilde{\mathbf{B}} \\
\nabla \times \tilde{\mathbf{H}} &= \tilde{\mathbf{J}} + j\omega \tilde{\mathbf{D}}
\end{aligned}$$

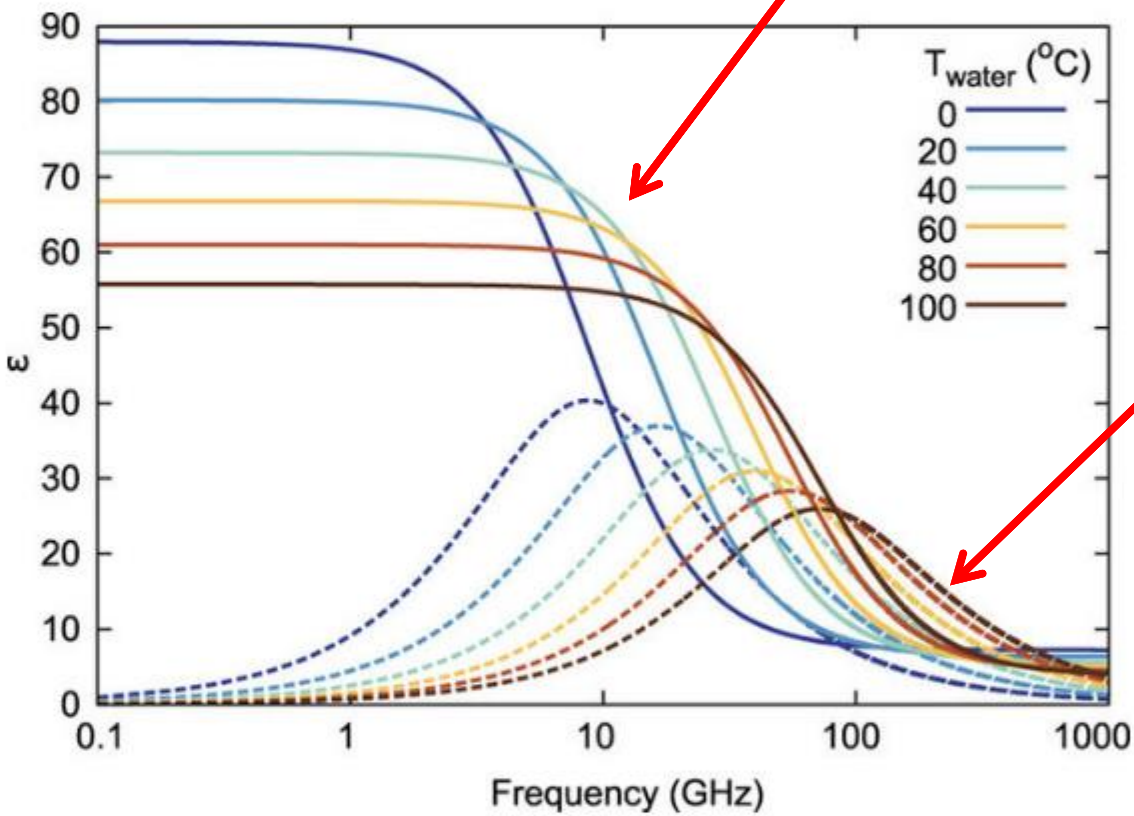
The auxiliary constitutive relations simply become

$$\tilde{\mathbf{D}} = \epsilon \tilde{\mathbf{E}}$$
$$\tilde{\mathbf{B}} = \mu \tilde{\mathbf{H}}$$
$$\tilde{\mathbf{J}} = \sigma \tilde{\mathbf{E}}$$

In general, the physical parameters ϵ, μ, σ are function of frequency.

this is the usual **real** relative permittivity

Example: relative permittivity of water



but there is also an **imaginary** part due to the conductivity losses

Nature – Scientific Reports 5(1):13535, August 2015

Wave equation in phasor form

Let's assume at first that we are away from charges and currents and that the medium has zero conductivity. We can transform directly equation in time-dependent form

d' Alembert equation

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \tilde{\mathbf{E}} = (-j\omega)(-j\omega)\,\mu\epsilon\tilde{\mathbf{E}} = -\omega^2\mu\epsilon\tilde{\mathbf{E}}$$

Helmholtz equation

$$\nabla^2 \tilde{\mathbf{E}} + \omega^2\mu\epsilon\tilde{\mathbf{E}} = 0$$

See in the notes equivalent derivation starting from $\nabla \times \tilde{\mathbf{E}}$

1-D model with x-polarized phasor electric field

$$\tilde{\mathbf{E}} = \hat{x}\tilde{E}_x(z)$$

1-D Helmholtz equation

$$\frac{\partial^2}{\partial z^2}\tilde{E}_x + \omega^2\mu\epsilon\tilde{E}_x = 0$$

Elementary solutions

$$\beta \equiv \omega\sqrt{\mu\epsilon}$$

$$\tilde{E}_x(z) = e^{\mp j\beta z}$$

General solution

$$\tilde{E}_x(z) = \mathrm{A}\,e^{-j\beta z} + \mathrm{B}\,e^{j\beta z}$$

forward wave backward wave

The coefficients A and B are the amplitudes of the wave solutions and can be complex due to a relative phase:

$$\mathrm{A} = \tilde{\mathrm{E}}_0 = E_0 e^{j\varphi} \quad \textbf{where} \quad E_0 = \left|\tilde{\mathrm{E}}_0\right|$$

The complete solution for the forward wave can be written as

$$\tilde{\mathrm{E}}_x(z) = \tilde{\mathrm{E}}_0 e^{-j\beta z}\hat{x} = E_0 e^{j\varphi} e^{-j\beta z}\hat{x}$$

and we recover the time-dependent solution

$$E_x(z,t) = \Re e\left\{\tilde{\mathrm{E}}_x(z) e^{j\omega t}\hat{x}\right\} = \Re e\left\{E_0 e^{j\varphi} e^{-j\beta z} e^{j\omega t}\hat{x}\right\}$$

$$= E_0 \cos\left(\omega t - \beta z + \varphi\right)\hat{x}$$

The corresponding phasor of the magnetic field is obtained from the phasor form of Faraday's law

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}$$

$$\tilde{\mathbf{H}} = -\frac{\nabla \times \tilde{\mathbf{E}}}{j\omega\mu}$$

For our 1-D solution with x-polarized electric field

$$\nabla \times \left(E_0 e^{j\varphi} e^{-j\beta z}\hat{x}\right) = \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_0 e^{j\varphi} e^{-j\beta z} & 0 & 0 \end{bmatrix} =$$

$$\frac{\partial}{\partial z}\left(E_0 e^{j\varphi} e^{-j\beta z}\right)\hat{y} - \cancel{\frac{\partial}{\partial y}\left(E_0 e^{j\varphi} e^{-j\beta z}\right)}\hat{z} = -j\beta E_0 e^{j\varphi} e^{-j\beta z}\hat{y}$$

no y-dependence

Finally we obtain

$$\tilde{\mathrm{H}}_y(z) = -\frac{-j\beta E_0 e^{j\varphi} e^{-j\beta z}}{j\omega\mu}\hat{y}$$

$$= \frac{\omega\sqrt{\mu\varepsilon}}{\omega\mu} E_0 e^{j\varphi} e^{-j\beta z}\hat{y}$$

$$= \sqrt{\frac{\varepsilon}{\mu}} E_0 e^{j\varphi} e^{-j\beta z}\hat{y} = \frac{1}{\eta}\tilde{\mathrm{E}}_0 e^{-j\beta z}\hat{y}$$

Now we consider general media with conductivity due to free charges, but assumed to be neutral with zero net charge. Material properties are described by

$$\boxed{\tilde{\mathbf{J}} = \sigma \tilde{\mathbf{E}}} \qquad \boxed{\varepsilon = \varepsilon_r \varepsilon_0} \qquad \boxed{\mu = \mu_r \mu_0}$$

The curl Maxwell equations become

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}$$

$$\nabla \times \tilde{\mathbf{H}} = \sigma\tilde{\mathbf{E}} + j\omega\varepsilon\tilde{\mathbf{E}} = j\omega\left(\varepsilon - j\frac{\sigma}{\omega}\right)\tilde{\mathbf{E}}$$

It is as if the material conductivity introduces an imaginary part for the permittivity

We can now obtain the Helmholtz wave equation

$$\nabla\times\nabla\times\tilde{\mathbf{E}}=\cancel{\nabla\nabla}\cdot\tilde{\mathbf{E}}-\nabla^2\tilde{\mathbf{E}}=-j\omega\mu\nabla\times\tilde{\mathbf{H}}$$

$$=-j\omega\mu\left(\tilde{\mathbf{J}}+j\omega\varepsilon\tilde{\mathbf{E}}\right)$$

$$\boxed{\nabla^2\tilde{\mathbf{E}}=j\omega\mu\left(\sigma+j\omega\varepsilon\right)\tilde{\mathbf{E}}}$$

Note: we have set the divergence of the electric field to zero, even if there are free charges, because we assume the material to be charge neutral overall.

In 1D the wave equation is

$$\frac{\partial^2 \tilde{\mathrm{E}}_x}{\partial z^2} = j\omega\mu\left(\sigma + j\omega\varepsilon\right)\tilde{\mathrm{E}}_x = \gamma^2 \tilde{\mathrm{E}}_x$$

with general solution

$$\tilde{E}_x(z) = \mathbf{A}\, e^{-\gamma z} + \mathbf{B}\, e^{\gamma z}$$

The magnetic field becomes

$$\tilde{H}_y(z) = -\frac{1}{j\omega\mu}\frac{\partial \tilde{E}_x}{\partial z} = \sqrt{\frac{\sigma + j\omega\varepsilon}{j\omega\mu}}\left(\mathbf{A}\, e^{-\gamma z} - \mathbf{B}\, e^{\gamma z}\right)$$

$$= \frac{1}{\eta}\left(\mathbf{A}\, e^{-\gamma z} - \mathbf{B}\, e^{\gamma z}\right)$$

The propagation factor is complex

$$\gamma = \sqrt{(j\omega\mu)(\sigma + j\omega\varepsilon)} = \alpha + j\beta$$

and also the medium impedance is complex

$$\bar{\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\bar{\eta}|e^{j\tau}$$

These parameters describe the behavior of general media with conductivity which causes attenuation (damping) of electromagnetic waves. They are functions of frequency.

Materials can be classified based on their conductivity, starting from ideal limit cases

			σ ↑
ideal limit case	**Perfect Conductor**	$\sigma \to \infty$	
realistic media	**Good Conductor**	$\sigma \gg$	
	"General Medium"		
	Imperfect Dielectric	$\sigma \ll$	
ideal limit case	**Perfect Dielectric**	$\sigma \to 0$	

NOTE that the classification for a specific material may also vary with frequency

Perfect Dielectric

$\sigma \to 0$

Propagation Constant

$$\beta = \omega\sqrt{\mu_r\mu_0\,\varepsilon_r\varepsilon_0}$$

$$\alpha = 0$$

Phase Velocity

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_r\mu_0\,\varepsilon_r\varepsilon_0}}$$

Medium Impedance

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}}$$

$$= \sqrt{\frac{\mu_r\mu_0}{\varepsilon_r\varepsilon_0}} = \sqrt{\frac{\mu}{\varepsilon}}$$

Wavelength

$$\lambda = \frac{2\pi}{\beta} = \frac{v_p}{f}$$

$$= \frac{1}{f\sqrt{\mu_r\mu_0\,\varepsilon_r\varepsilon_0}}$$

Imperfect Dielectric

$\boldsymbol{\sigma} \ll$

$$\boxed{\frac{\sigma}{\omega\varepsilon} \ll 1}$$

$$\gamma = \sqrt{(j\omega\mu)(\sigma + j\omega\varepsilon)} = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$$

$$\approx j\omega\sqrt{\mu\varepsilon}\left[1 - j\frac{\sigma}{2\omega\varepsilon}\right] = \underbrace{\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}}_{\alpha} + j\underbrace{\omega\sqrt{\mu\varepsilon}}_{\beta}$$

expand in series and truncate

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \qquad \beta \approx j\omega\sqrt{\mu\varepsilon}$$

Imperfect Dielectric

$\boldsymbol{\sigma} \ll$

$$\frac{\sigma}{\omega\varepsilon} \ll 1$$

Phase Velocity

$$v_p = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{\mu\varepsilon}}$$

Wavelength

$$\lambda = \frac{2\pi}{\beta} \approx \frac{1}{f\sqrt{\mu\varepsilon}}$$

Medium Impedance

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}}\left[1 - j\frac{\sigma}{2\omega\varepsilon}\right]^{-\frac{1}{2}} \approx \sqrt{\frac{\mu}{\varepsilon}}$$

Imperfect Dielectric

$\sigma \ll$

$$\frac{\sigma}{\omega\varepsilon} \ll 1$$

When this assumption is verified, with good approximation the material behaves like a perfect dielectric, except that there is a small attenuation.

There is no precise rule to quantify exactly when the approximation is valid. A strict rule of thumb for imperfect dielectric could be:

$$\frac{\sigma}{\omega\varepsilon} \le 0.01$$

This quantity is usually called "loss tangent"

Imperfect Dielectric

$\boldsymbol{\sigma} \ll$

There imperfect dielectric approximation may still be reasonable up to $\sigma/\omega\varepsilon \leq 0.1$ but the dephasing due to the imaginary part of the impedance becomes more noticeable. A more stringent approximation is

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon\left(1 - j\dfrac{\sigma}{\omega\varepsilon}\right)}}$$

$$\approx \sqrt{\frac{\mu}{\varepsilon}}\left(1 + j\frac{\sigma}{2\omega\varepsilon}\right) = \sqrt{\frac{\mu}{\varepsilon}}e^{j\tan^{-1}\frac{\sigma}{2\omega\varepsilon}} \approx \sqrt{\frac{\mu}{\varepsilon}}e^{j\frac{\sigma}{2\omega\varepsilon}}$$

$$\boxed{|\eta| \approx \sqrt{\frac{\mu}{\varepsilon}}}$$

$$\boxed{\tau = \angle\eta \approx \frac{\sigma}{2\omega\varepsilon}}$$

Good Conductor

$\sigma \gg$

$$\boxed{\frac{\sigma}{\omega\varepsilon} \gg 1}$$

$$\gamma = \sqrt{(j\omega\mu)(\sigma + \cancel{j\omega\varepsilon})} \approx \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma}\sqrt{j}$$

$$= \sqrt{\omega\mu\sigma}\, e^{j\frac{\pi}{4}} = \sqrt{\omega\mu\sigma}\left[\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right]$$

$$= \sqrt{\omega\mu\sigma}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = \sqrt{\pi f \mu\sigma}\,(1+j)$$

$$\boxed{\alpha \approx \sqrt{\pi f \mu\sigma} \qquad \beta \approx \sqrt{\pi f \mu\sigma}}$$

Good Conductor

$$\sigma \gg$$

$$\frac{\sigma}{\omega\varepsilon} \gg 1$$

Phase Velocity

$$v_p = \frac{\omega}{\beta} \approx \frac{2\pi f}{\sqrt{\pi f \mu\sigma}} = \sqrt{\frac{4\pi f}{\mu\sigma}}$$

Wavelength

$$\lambda = \frac{2\pi}{\beta} \approx \frac{2\pi}{\sqrt{\pi f \mu\sigma}} = \sqrt{\frac{4\pi}{f\mu\sigma}}$$

Medium Impedance

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}e^{j\frac{\pi}{4}}$$

$$= \sqrt{\frac{\omega\mu}{\sigma}}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{\pi f\mu}{\sigma}}(1+j)$$

Good Conductor

$\boldsymbol{\sigma} \gg$

$$\frac{\sigma}{\omega\varepsilon} \gg 1$$

When this assumption is verified, the attenuation constant $\boldsymbol{\alpha}$ and the propagation constant $\boldsymbol{\beta}$ are approximately equal.

The medium impedance η has nearly equal real and imaginary parts, with phase angle $\boldsymbol{\tau \approx 45° = \pi/4}$.

Therefore, in a good conductor electric and magnetic field have always a phase difference $\boldsymbol{\tau \approx 45° = \pi/4}$.

A strict rule of thumb is that approximations for good conductor can be applied when

$$\frac{\sigma}{\omega\varepsilon} \geq 100$$

Good Conductor

$\sigma \gg$

In a good conductor the fields attenuate rapidly. The distance over which fields are attenuated by a factor $\mathbf{exp(-1)}$ is

$$\frac{1}{\alpha} = \delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

For a good conductor, this is called "skin depth". In general, $1/\alpha$ can be called "penetration depth"

Perfect Conductor

$\sigma \to \infty$

For this ideal material, the attenuation is also infinite and the skin depth goes to zero. This means that the electromagnetic field must go to zero below the perfect conductor surface.

"General Medium"

In this case, we need to use the complete formulation, since approximations would be too inaccurate.

Following the previously introduced rules, we can define the general medium to be in the range:

$$0.01 \leq \frac{\sigma}{\omega\varepsilon} \leq 100$$

Of course, this is arbitrary. The full model is valid for all cases.

"General Medium"

Full result for the propagation constant

$$\gamma = \sqrt{(j\omega\mu)(\sigma + j\omega\varepsilon)} = \alpha + j\beta$$

It can be shown that

$$\alpha = \frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]^{\frac{1}{2}}$$

$$\beta = \frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]^{\frac{1}{2}}$$

"General Medium"

Phase Velocity

$$v_p = \frac{\omega}{\beta} = \frac{\sqrt{2}}{\sqrt{\mu\varepsilon}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{-\frac{1}{2}}$$

Wavelength

$$\lambda = \frac{2\pi}{\beta} = \frac{\sqrt{2}}{f\sqrt{\mu\varepsilon}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{-\frac{1}{2}}$$

Medium Impedance

$$\bar{\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\bar{\eta}|e^{j\tau}$$

Example

The intrinsic impedance of the medium is complex as long as the conductivity is not zero. The phase angle of the intrinsic impedance indicates that electric field and magnetic field are out of phase.

Considering the forward wave solutions

$$\tilde{E}_x(z) = E_0 e^{j\varphi}\, e^{-\gamma z} = E_0 e^{j\varphi}\, e^{-\alpha z} e^{-j\beta z} = E_0\, e^{-\alpha z} e^{-j(\beta z - \varphi)}$$

$$\tilde{H}_y(z) = \frac{1}{\eta} E_0 e^{j\varphi}\, e^{-\gamma z} = \frac{1}{|\eta|} E_0 e^{j\varphi}\, e^{-\alpha z} e^{-j\beta z} e^{-j\tau}$$

$$= \frac{1}{|\eta|} E_0\, e^{-\alpha z} e^{-j(\beta z - \varphi + \tau)}$$

The time-dependent fields are recovered as:

$$E_x(z,t) = \Re e\left\{E_0\, e^{-\alpha z} e^{-j(\beta z-\varphi)}\, e^{j\omega t}\right\}$$

$$= E_0\, e^{-\alpha z} \cos\left(\omega t - \beta z + \varphi\right)$$

$$H_y(z,t) = \frac{1}{|\eta|}\Re e\left\{E_0\, e^{-\alpha z} e^{-j(\beta z-\varphi+\tau)}\, e^{j\omega t}\right\}$$

$$= \frac{1}{|\eta|} E_0\, e^{-\alpha z} \cos\left(\omega t - \beta z + \varphi - \tau\right)$$

Example: A typical good conductor is copper (Cu), which has the following parameters

$$\sigma = 5.80 \times 10^{7}\,[\mathrm{S/m}]$$

$$\varepsilon \approx \varepsilon_o$$

$$\mu \approx \mu_o$$

What is the skin depth of copper at 30 GHz?

$$\frac{1}{\alpha} = \delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

$$= \frac{1}{\sqrt{\pi \times 30 \times 10^{9} \times 4\pi \times 10^{-7} \times 5.8 \times 10^{7}}} = 381\ \mathrm{nm}$$

Example: Although copper is a good conductor, at what frequency it stops being that? At what frequency does it become an imperfect dielectric? Assume that the conductivity is always 5.8×10^7 S/m.

Conventionally, copper ceases to be a good conductor when

$$\frac{\sigma}{\omega\varepsilon} = 100 \quad \rightarrow \quad f = \frac{\sigma}{100 \times 2\pi \times \varepsilon_r \varepsilon_0} = \frac{5.8\times10^{7}}{100\times2\pi\times8.854\times10^{-12}} \approx 10^{17}\,\text{Hz}$$

X-rays

Copper starts to be an imperfect dielectric when

$$\frac{\sigma}{\omega\varepsilon} = 0.01 \quad \rightarrow \quad f = \frac{\sigma}{0.01 \times 2\pi \times \varepsilon_r \varepsilon_0} = \frac{5.8\times10^{7}}{0.01\times2\pi\times8.854\times10^{-12}} \approx 10^{20}\,\text{Hz}$$

Gamma rays

Example: Consider sea water, with parameters

$$\sigma \approx 4.0\ [\mathrm{S/m}]$$

$$\varepsilon \approx 80\varepsilon_o$$

$$\mu \approx \mu_o$$

What type of material is sea water at 25 kHz?

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi \times 25 \times 10^{3} \times 80 \times 8.854 \times 10^{-12}} = 35,958$$

$$\boxed{\frac{\sigma}{\omega\varepsilon} \gg 1}$$

good conductor

Summary of the special cases

	Condition	β	α	$\lvert\eta\rvert$	τ	$\lambda = \frac{2\pi}{\beta}$	$\delta = \frac{1}{\alpha}$
Perfect dielectric	$\sigma = 0$	$\omega\sqrt{\epsilon\mu}$	0	$\sqrt{\frac{\mu}{\epsilon}}$	0	$\frac{2\pi}{\omega\sqrt{\epsilon\mu}}$	∞
Imperfect dielectric	$\frac{\sigma}{\omega\epsilon} \ll 1$	$\sim \omega\sqrt{\epsilon\mu}$	$\beta\frac{1}{2}\frac{\sigma}{\omega\epsilon} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$	$\sim \sqrt{\frac{\mu}{\epsilon}}$	$\sim \frac{\sigma}{2\omega\epsilon}$	$\sim \frac{2\pi}{\omega\sqrt{\epsilon\mu}}$	$\frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$
Good conductor	$\frac{\sigma}{\omega\epsilon} \gg 1$	$\sim \sqrt{\pi f\mu\sigma}$	$\sim \sqrt{\pi f\mu\sigma}$	$\sqrt{\frac{\omega\mu}{\sigma}}$	$45°$	$\sim \frac{2\pi}{\sqrt{\pi f\mu\sigma}}$	$\sim \frac{1}{\sqrt{\pi f\mu\sigma}}$
Perfect conductor	$\sigma = \infty$	∞	∞	0	-	0	0

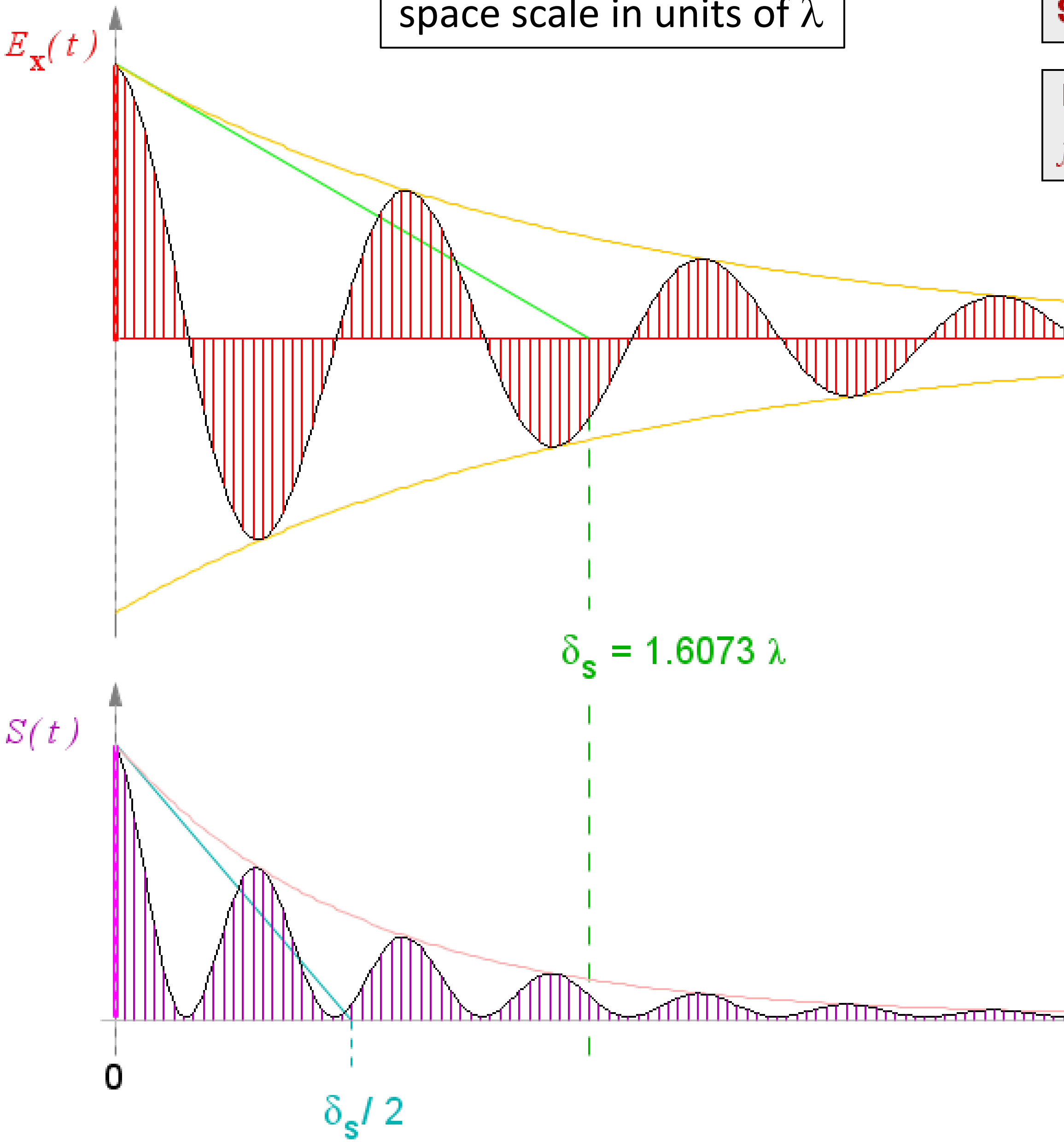
Significance of penetration (skin) depth

space scale in units of λ

Slightly Lossy

E (z = 0) = 10.0 [V/m] σ = 0.001 [S/m]

f = 10.0 MHz ε_r = 9.0



WaveLength $\boxed{\lambda = 9.95085 \ [\mathrm{m}]}$

Phase Velocity $u_p = 9.95085 \times 10^7$ [m / s]

Impedance of the Medium [Ω]

η = 123.831975 + j 12.261782

= 124.437572 $\angle$ 0.0987 rad

= 124.437572 $\angle$ 5.655 °

Penetration (Skin) Depth

δ_s = 1.6073 λ = 15.9941 [m]

Phase and Attenuation Constants

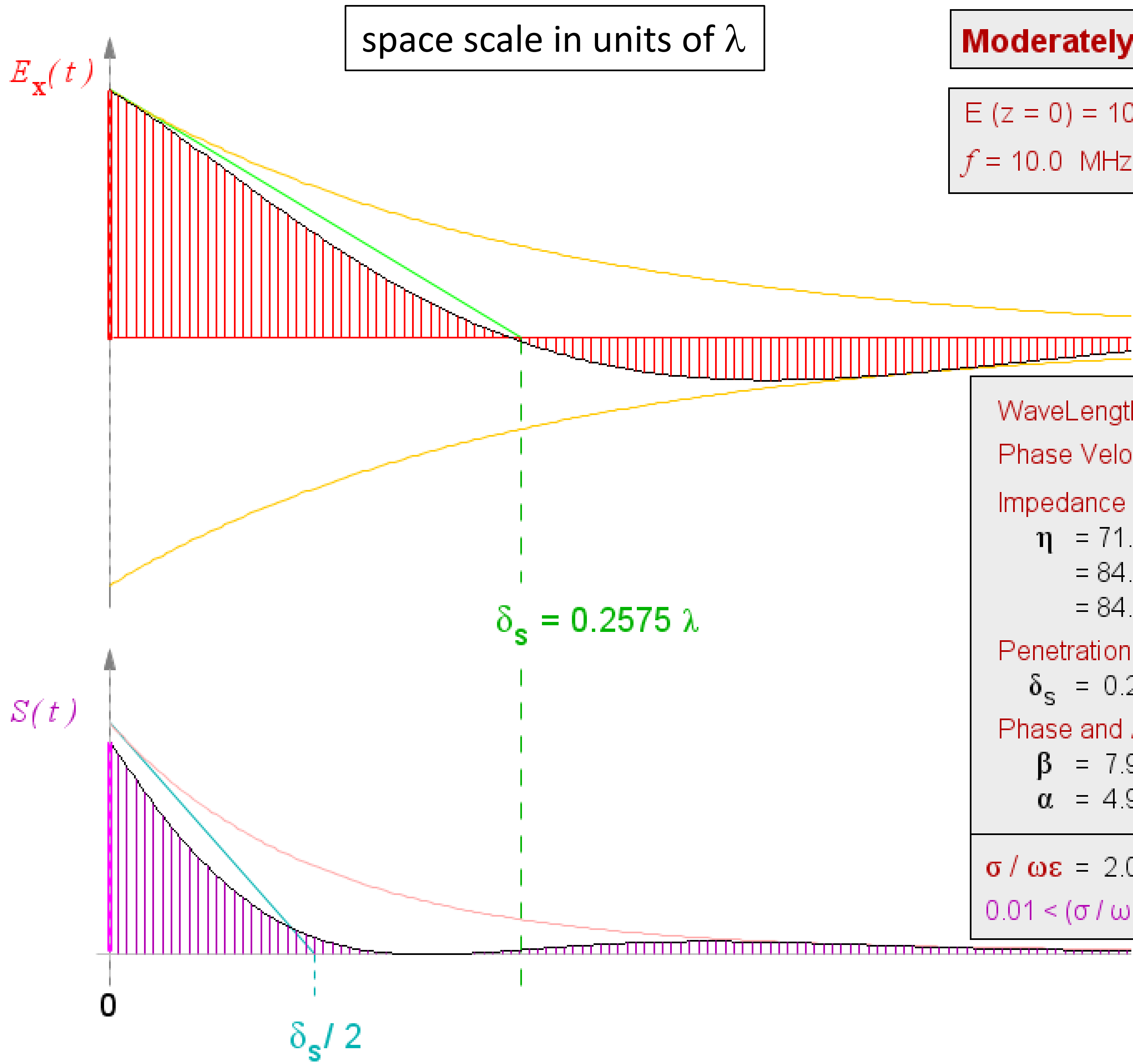
$\beta = 6.31422 \times 10^{-1}$ [m^{-1}]

$\alpha = 6.25231 \times 10^{-2}$ [Ne / m]

$\sigma / \omega\varepsilon$ = 0.2

$0.01 < (\sigma / \omega\varepsilon) < 100.0$ General medium

Significance of penetration (skin) depth



Moderately Lossy

E (z = 0) = 10.0 [V/m] σ = 0.01 [S/m]

f = 10.0 MHz ε_r = 9.0

WaveLength λ = 7.86151 [m]

Phase Velocity $u_p = 7.86151 \times 10^7$ [m / s]

Impedance of the Medium [Ω]

η = 71.485619 + j 44.180542

= 84.036385 ∠ 0.5536 rad

= 84.036385 ∠ 31.7175 °

Penetration (Skin) Depth

δ_s = 0.2575 λ = 2.02448 [m]

Phase and Attenuation Constants

$\beta = 7.99234 \times 10^{-1}$ [m^{-1}]

$\alpha = 4.93953 \times 10^{-1}$ [Ne / m]

$\sigma / \omega\varepsilon$ = 2.0

0.01 < ($\sigma / \omega\varepsilon$) < 100.0 General medium

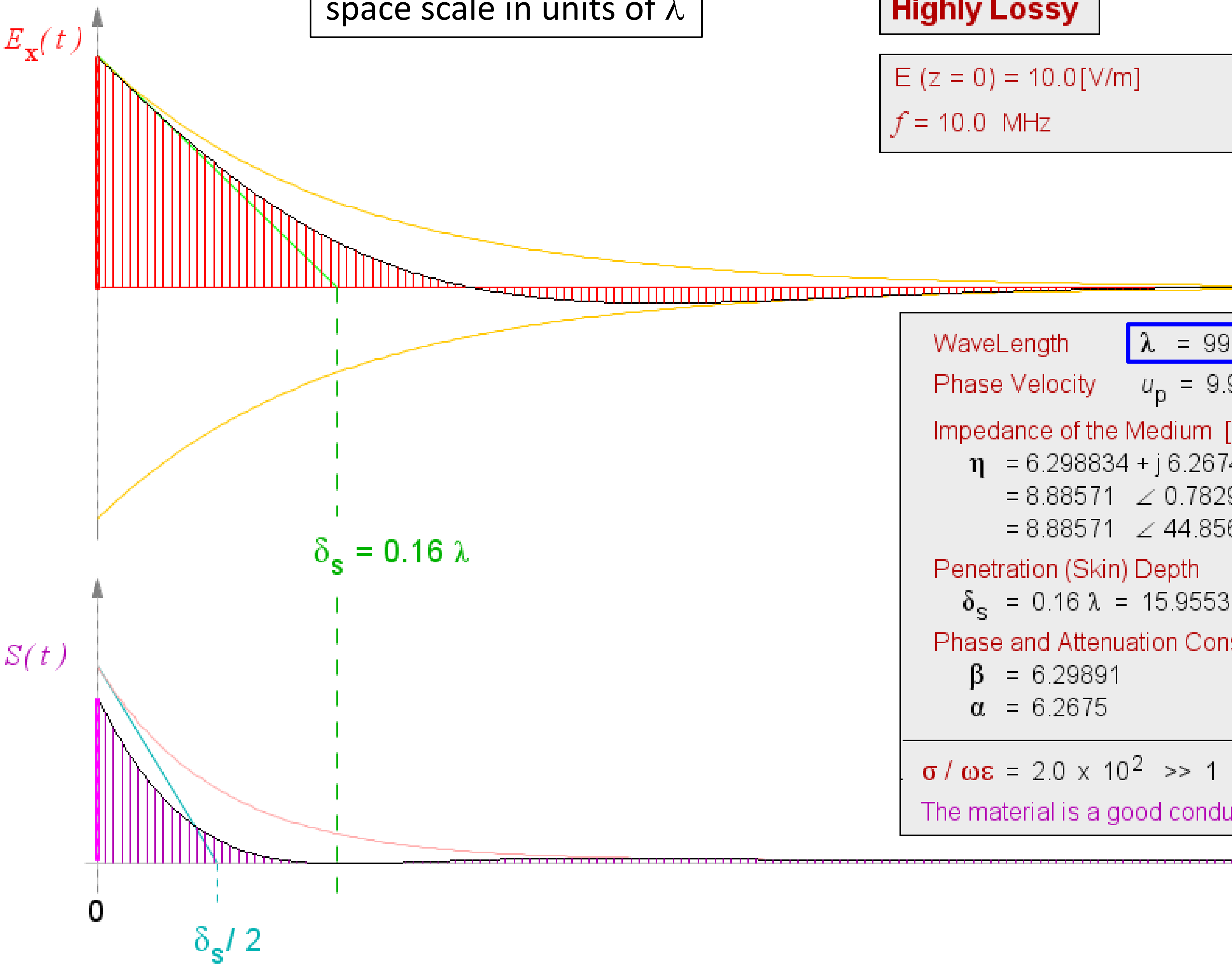
Significance of penetration (skin) depth

space scale in units of λ

Highly Lossy

$E\ (z = 0) = 10.0\ [\text{V/m}]$ $\quad \sigma = 1.0\ [\text{S/m}]$

$f = 10.0\ \text{MHz}$ $\quad \varepsilon_r = 9.0$



WaveLength $\boxed{\lambda = 99.75031\ [\text{cm}]}$

Phase Velocity $u_p = 9.97503 \times 10^6\ [\text{m/s}]$

Impedance of the Medium $[\Omega]$

$\eta = 6.298834 + j\,6.267419$

$= 8.88571 \angle 0.7829$ rad

$= 8.88571 \angle 44.8568°$

Penetration (Skin) Depth

$\delta_s = 0.16\ \lambda = 15.95533\ [\text{cm}]$

Phase and Attenuation Constants

$\beta = 6.29891$ $[\text{m}^{-1}]$

$\alpha = 6.2675$ $[\text{Ne/m}]$

$\sigma / \omega\varepsilon = 2.0 \times 10^2 \gg 1$

The material is a good conductor

Example – Current in a conductor

$$f = 1.0\ \mathrm{GHz}$$

$$\sigma = 10^7\ \mathrm{S/m}$$

$$\varepsilon_r = 1.0$$

$$\lambda = 31.6\ \mu\mathrm{m}$$

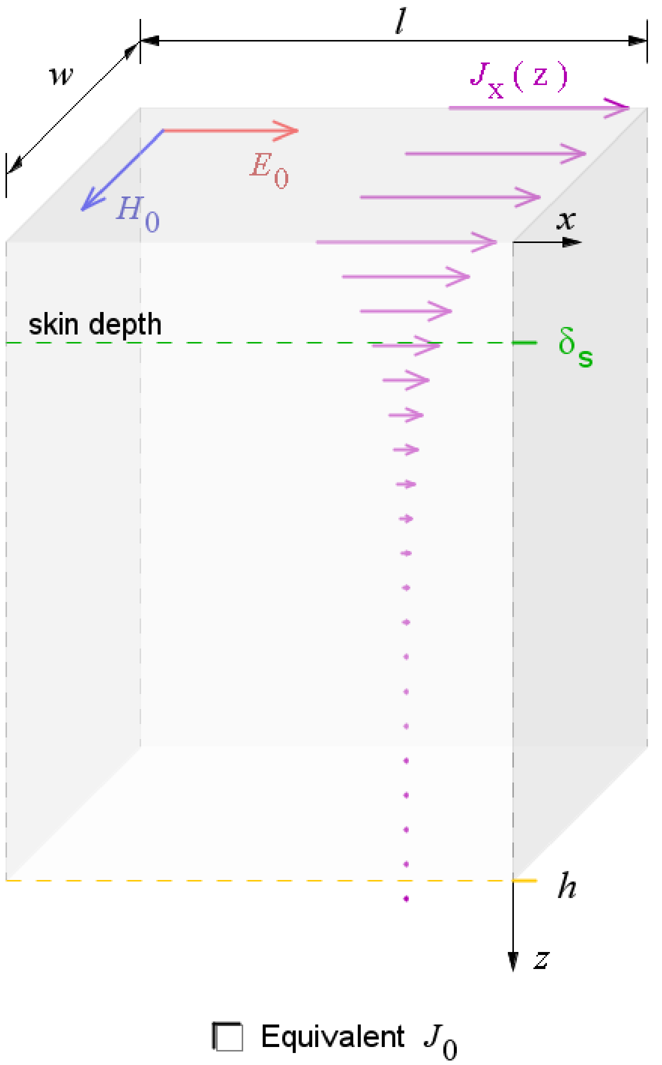
$$v_p = 3.16\ \mathrm{m/s}$$

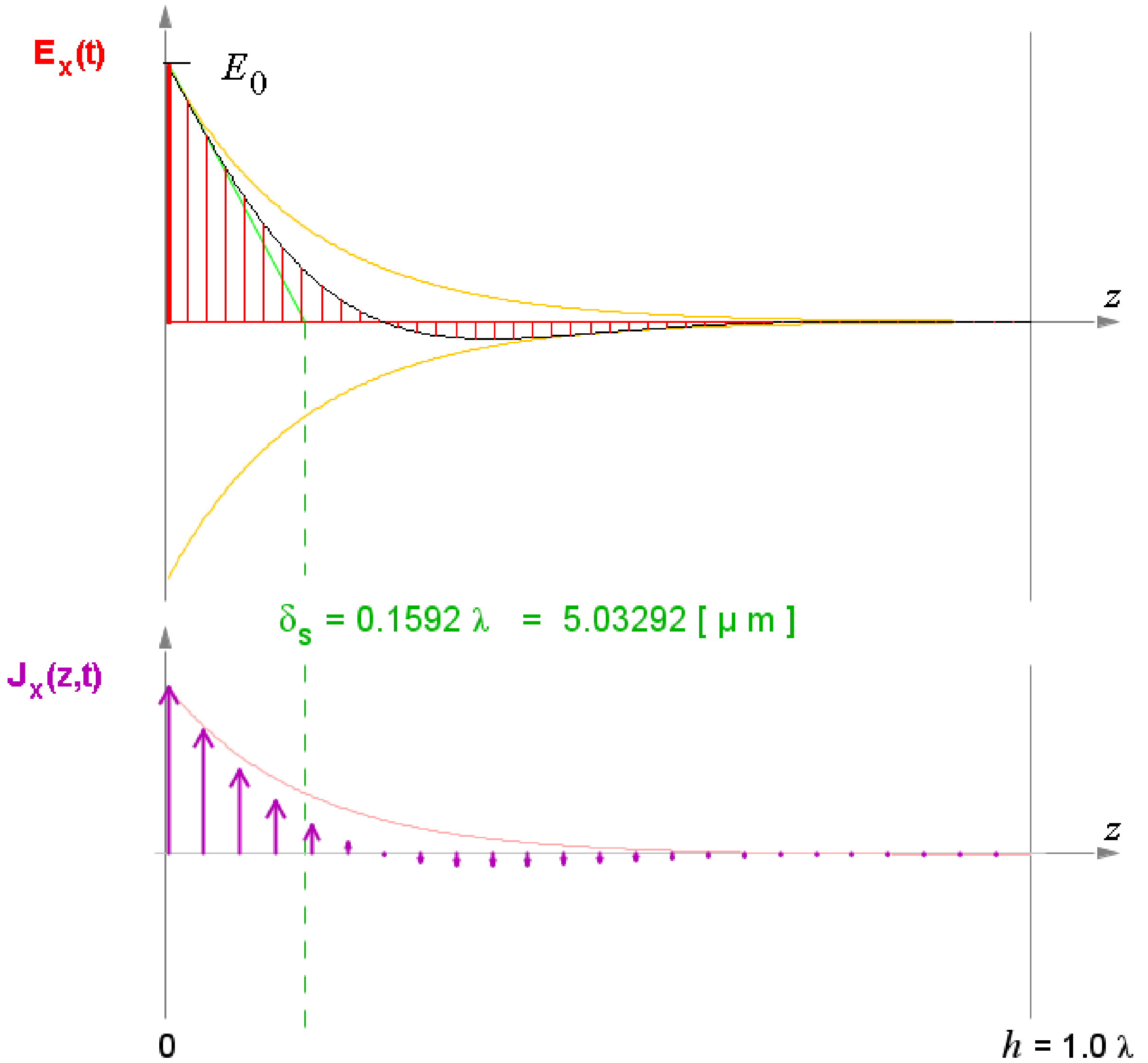
$$\eta = 0.0281\angle 45^\circ$$

$$\alpha = 1.98692\times 10^5\ \mathrm{Ne/m}$$

$$\beta = 1.98692\times 10^5\ \mathrm{m}^{-1}$$

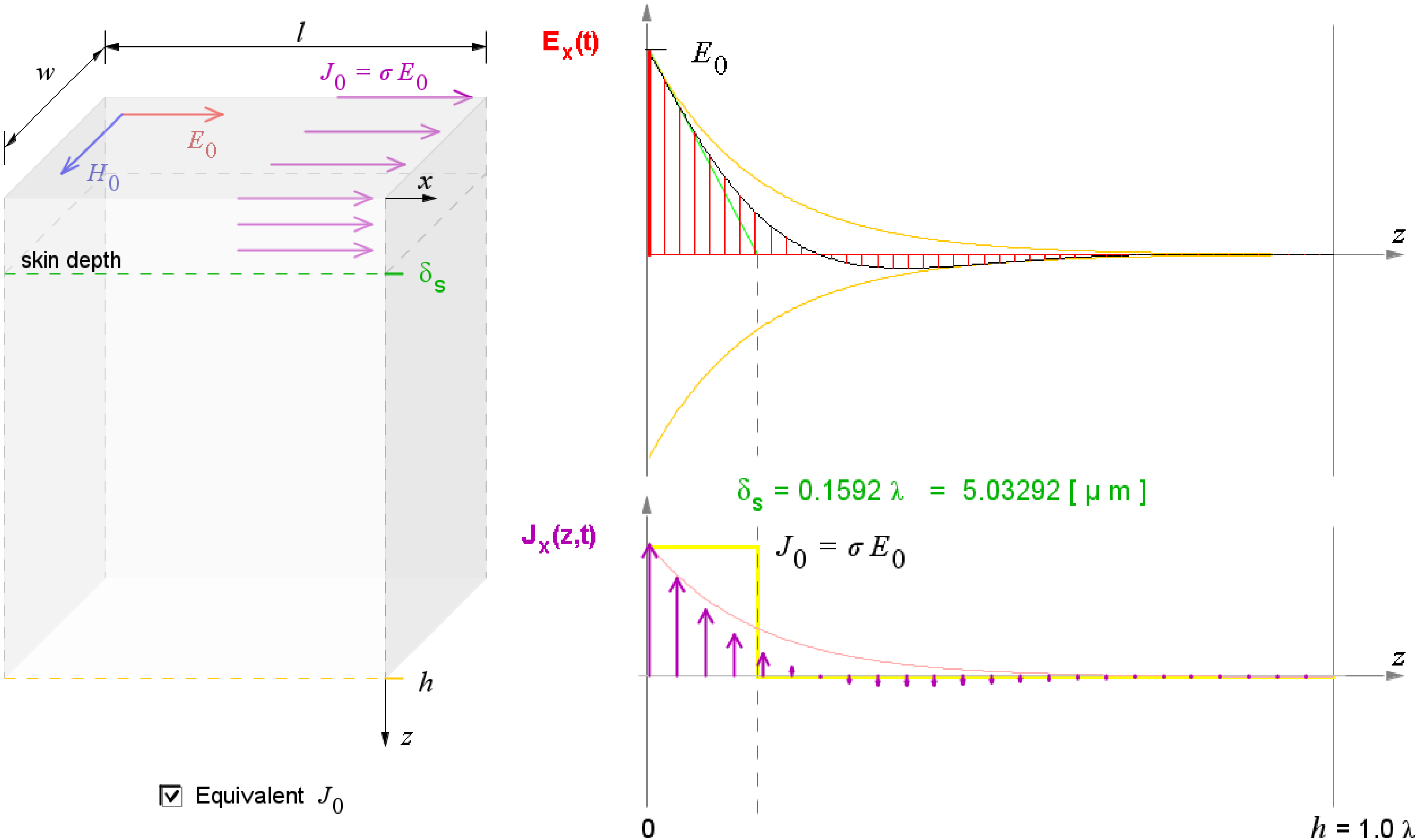
Current in a conductor





actual current distribution

Current in a conductor



equivalent uniform current model flowing only in the "skin" layer

Example: Poynting vector in imperfect dielectric

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$<\mathbf{S}> = \frac{1}{2}\,\Re e\left[\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}^*\right] \qquad (\mathrm{W/m^2})$$

Plane TEM wave at 1 GHz with $\sigma = 10^{-3}$ S/m and $\varepsilon = 4\epsilon_0$

$$\tilde{\mathbf{E}} = \hat{y}2e^{-\alpha z}e^{-j\beta z}\,\frac{\mathrm{V}}{\mathrm{m}}$$

$$\frac{\sigma}{\omega\epsilon} = \frac{10^{-3}\cdot 36\pi \times 10^9}{2\pi \cdot 10^9 \cdot 4} = \frac{9}{2}10^{-3} \ll 1$$

$$\tau \approx \frac{\sigma}{2\omega\epsilon} \approx \frac{9}{4}10^{-3}\,\mathrm{rad}$$

$$|\eta| \approx \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_o}{4\epsilon_o}} = \frac{\eta_o}{2} = 60\pi\,\Omega$$

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} = \frac{1}{2}10^{-3}60\pi = 30\pi\cdot 10^{-3}\,\frac{1}{\mathrm{m}}$$

Magnetic field

$$\tilde{\mathbf{H}} = -\hat{x}\frac{2}{\eta}e^{-\alpha z}e^{-j\beta z} \approx -\hat{x}\frac{2}{\sqrt{\mu/\epsilon}}e^{-\alpha z}e^{-j\beta z}e^{-j\tau}\ \frac{\mathrm{A}}{\mathrm{m}}$$

$$\boxed{\bar{\eta} = |\bar{\eta}|e^{j\tau}}$$

$$e^{-j\tau} = \cos(\tau) - j\sin(\tau) = \underbrace{0.9999975 - j0.00225}_{\approx \mathbf{1}}$$

Average Poynting vector

$$\begin{aligned}\langle \mathbf{S} \rangle &= \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\} = \frac{1}{2}\mathrm{Re}\{\hat{y}2e^{-\alpha z}e^{-j\beta z} \times (-\hat{x}\frac{2}{\sqrt{\mu/\epsilon}}e^{-\alpha z}e^{-j\beta z}e^{-j\tau})^*\} \\ &= -\frac{1}{2}\mathrm{Re}\{\hat{y}2e^{-\alpha z} \times \hat{x}\frac{2}{\sqrt{\mu/\epsilon}}e^{-\alpha z}e^{j\tau}\} = \hat{z}\frac{2}{\sqrt{\mu/\epsilon}}e^{-2\alpha z}\cos\tau.\end{aligned}$$

NOTE "$-2\alpha z$"

Average Poynting vector

$$\langle \mathbf{S} \rangle \;=\; \hat{z}\frac{2}{\sqrt{\mu/\epsilon}}e^{-2\alpha z}\cos\tau$$

$\boxed{z=0}$

$$\langle \mathbf{S} \rangle = \hat{z}\frac{2}{\sqrt{\mu/\epsilon}}\cos\tau \approx \hat{z}\frac{2}{60\pi} = \hat{z}\frac{1}{30\pi}\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

$\boxed{z=10\ \mathrm{m}}$

$$\langle \mathbf{S} \rangle = \hat{z}\frac{2}{\sqrt{\mu/\epsilon}}e^{-2\cdot 30\pi\cdot 10^{-3}\cdot 10}\cos\tau \approx \hat{z}\frac{2}{60\pi}e^{-6\pi/10} \approx \hat{z}\frac{0.15}{30\pi}\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

Even if the medium is an "imperfect dielectric" the wave has lost 85% of the power over 10 m. This power is transferred to the medium due to induced current density corresponding to a term

$$\langle \mathbf{J}\cdot\mathbf{E} \rangle$$

Example – Consider now the same $\mathbf{E}$ field in sea water but at frequency of 1kHz. Sea water has typically $\sigma = 4\,\mathrm{S/m}$ and $\varepsilon_r = 80$ to 81.

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times10^3\times80\times8.854\times10^{-12}} = 8.9\times10^5 \gg 1$$ **good conductor**

$$\tilde{\mathbf{E}} = \hat{y}2e^{-\alpha z}e^{-j\beta z}\,\frac{\mathrm{V}}{\mathrm{m}}$$

$$\tilde{\mathbf{H}} = -\hat{x}\frac{2}{\eta}e^{-\alpha z}e^{-j\beta z} \approx -\hat{x}\frac{2}{|\eta|}e^{-\alpha z}e^{-j\beta z}e^{-j\tau}\,\frac{\mathrm{A}}{\mathrm{m}}$$

$$\begin{aligned}\langle\mathbf{S}\rangle &= \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^*\} = \frac{1}{2}\mathrm{Re}\{\hat{y}2e^{-\alpha z}e^{-j\beta z}\times(-\hat{x}\frac{2}{|\eta|}e^{-\alpha z}e^{-j\beta z}e^{-j\tau})^*\} \\ &= -\frac{1}{2}\mathrm{Re}\{\hat{y}2e^{-\alpha z}\times\hat{x}\frac{2}{|\eta|}e^{-\alpha z}e^{j\tau}\} = \hat{z}\frac{2}{|\eta|}e^{-2\alpha z}\cos\tau.\end{aligned}$$

The expressions of magnetic field and Poynting vector are very similar, but now parameters are quite different

$$\tau \approx \frac{\pi}{4}\,\mathrm{rad}$$

$$|\eta| \approx \sqrt{\frac{\omega\mu}{\sigma}} = \sqrt{\frac{2\pi\cdot 10^3\cdot 4\pi\cdot 10^{-7}}{4}} = \pi\sqrt{2\times 10^{-4}} \approx \frac{\pi\sqrt{2}}{100}\,\Omega$$

$$\alpha \approx \sqrt{\pi f\mu\sigma} = \sqrt{\pi\cdot 10^3\cdot 4\pi\cdot 10^{-7}\cdot 4} = \sqrt{4^2\pi^2 10^{-4}} = \frac{\pi}{25}\,\frac{1}{\mathrm{m}}$$

$$\boxed{\alpha \propto \sqrt{f}}$$

lower attenuation at lower frequency but less communication bandwidth

$$\boxed{z = 0}$$

$$\langle \mathbf{S}\rangle = \hat{z}\frac{2}{|\eta|}\cos\tau \approx \hat{z}\frac{200}{\pi\sqrt{2}}\cos\frac{\pi}{4} = \hat{z}\frac{100}{\pi}\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

$$\boxed{z = 10\ \mathrm{m}}$$

$$\langle \mathbf{S}\rangle = \hat{z}\frac{100}{\pi}\,e^{-2\cdot\frac{\pi}{25}\cdot 10} \approx \hat{z}\frac{100}{\pi}0.081\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

$\approx$ 92% power reduction

Penetration depth for sea water at 1 kHz.

$$\delta \approx \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi 10^3 \cdot 4\pi \cdot 10^{-7} 4}}$$

$$= \frac{1}{\sqrt{4^2 \pi^2 \cdot 10^{-4}}} = \frac{100}{4\pi} = \frac{25}{\pi} \approx 7.95\,\mathrm{m}$$

This frequency would be suitable for communications with submarines at shallow depths.

- **Band for shallow depth communications is 3 kHz-300 kHz.**
- **For deeper communications (up to 100's of meters) the 3 to 300Hz band has been used. US SAGUINE System worked at 76 Hz. Signals are limited to a few characters per minute; transmitting antennas (buried in the ground) cover very large areas and need enormous power. This is a one-way technology since submarines cannot carry large enough antennas.**
- **Acoustic technologies are still being developed, including acoustic (in water) and radar (in air) hybrid systems.**