ECE 329 – Fall 2021

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

Lecture 22

Lecture 22&23 – Outline

- **Phasor wave equation**
- **Conductivity in realistic media**
- **Wave attenuation**
- **Poynting vector (power) attenuation**
- **Examples**

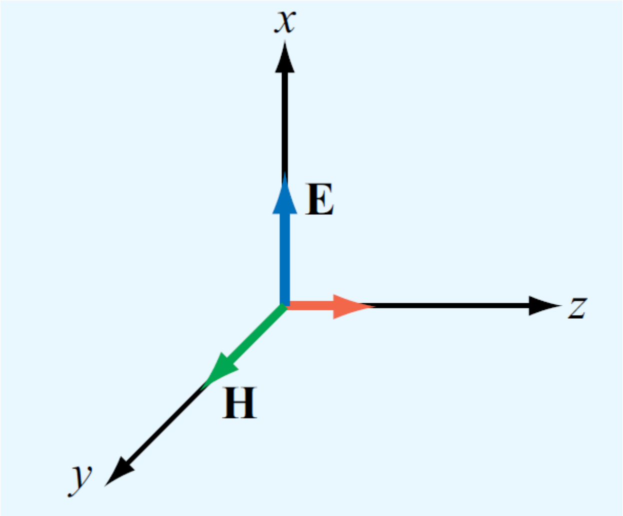
Reading assignment

Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:

22) Phasor form of Maxwell's equations and damped waves in conducting media

23) Imperfect dielectrics, good conductors

TEM wave in phasor notation



$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\widetilde{E}_x^+(z) = \hat{\mathbf{x}}E_{x0}^+ e^{-j\beta z}$$

$$\widetilde{\mathbf{H}}(z) = \hat{\mathbf{y}}\frac{\widetilde{E}_x^+(z)}{\eta} = \hat{\mathbf{y}}\frac{E_{x0}^+}{\eta}e^{-j\beta z}$$

The field amplitudes may be complex (there is an additional phase)

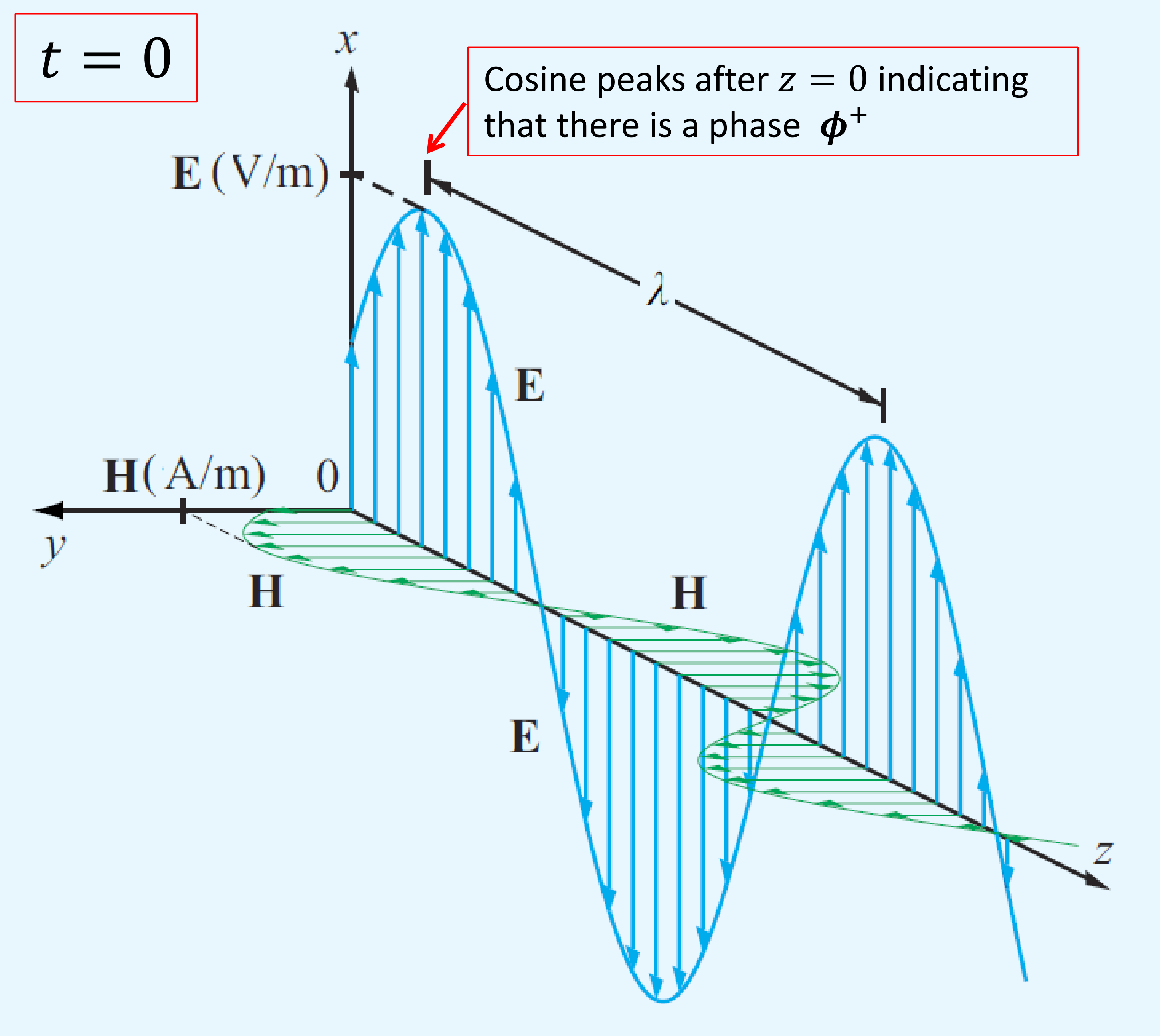
$$E_{x0}^+ = |E_{x0}^+|e^{j\phi^+}$$

The corresponding time-dependent fields are

$$\mathbf{E}(z,t) = \mathfrak{Re}\left[\widetilde{\mathbf{E}}(z)\, e^{j\omega t}\right] = \hat{\mathbf{x}}|E_{x0}^+|\cos(\omega t - \beta z + \phi^+) \quad \text{(V/m)}$$

$$\mathbf{H}(z,t) = \mathfrak{Re}\left[\widetilde{\mathbf{H}}(z)\, e^{j\omega t}\right] = \hat{\mathbf{y}}\frac{|E_{x0}^+|}{\eta}\cos(\omega t - \beta z + \phi^+) \quad \text{(A/m)}$$

Representation of Electric and Magnetic field in space



Example – The electric field of a **1 MHz** plane wave, traveling along $+z$ in air, is $\hat{x}$-polarized. This field reaches a peak value of 1.2π (mV/m) at t = 0 and z = 50 m. Obtain expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$ and then plot them as a function of z at t = 0.

At 1 MHz:

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{1 \times 10^6} = 300 \text{ m}$$

$$\beta = (2\pi/300) \text{ (rad/m)}$$

$$\mathbf{E}(z, t) = \hat{\mathbf{x}}|E_{x0}^+| \cos(\omega t - \beta z + \phi^+)$$

$$= \hat{\mathbf{x}}\, 1.2\pi \cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \phi^+\right) \text{ (mV/m)}$$

The field is maximum when the argument of the cosine equals 0 or 2π. Peak value was stated to be 1.2*π* (mV/m) at *t* = 0 and *z* = 50 m

$$\cos\left(2\pi\times 10^6 t - \frac{2\pi z}{300} + \phi^+\right)$$

$$-\frac{2\pi\times 50}{300} + \phi^+ = 0 \quad \Longrightarrow \quad \boxed{\phi^+ = \frac{\pi}{3}}$$

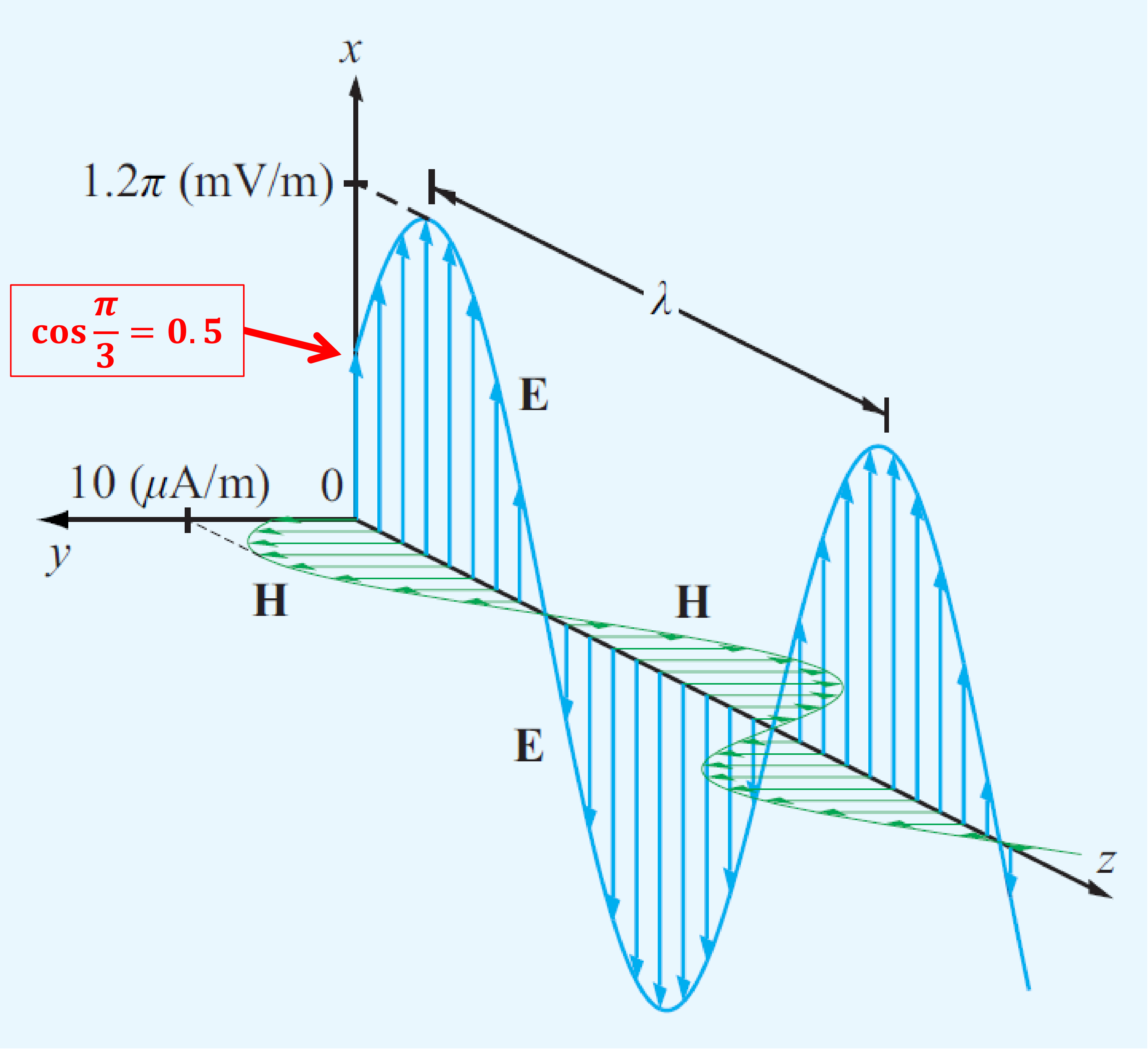
$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\, 1.2\pi \cos\left(2\pi\times 10^6 t \underbrace{- \frac{2\pi z}{300}}_{= -\pi/3 \text{ when } z = 50\text{m}} + \frac{\pi}{3}\right) \quad (\text{mV/m})$$

$$\mathbf{H}(z,t) = \hat{\mathbf{y}}\, \frac{E(z,t)}{\eta_0}$$

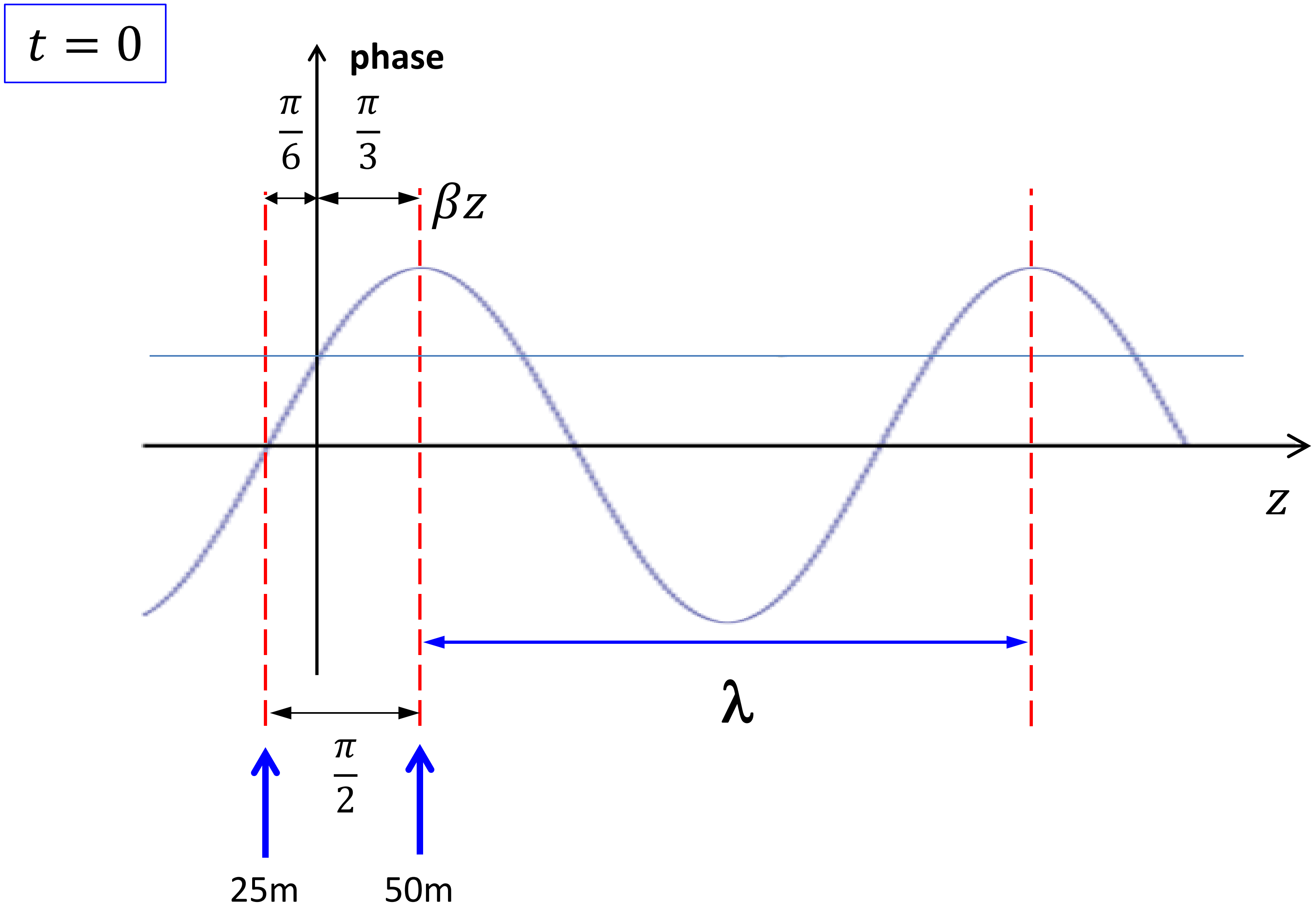
using the approximation $\eta_0 \approx 120\pi \ (\Omega)$

$$= \hat{\mathbf{y}}\, 10 \cos\left(2\pi\times 10^6 t - \frac{2\pi z}{300} + \frac{\pi}{3}\right) \quad (\mu\text{A/m})$$

note choice of units



x
1.2π (mV/m)
λ
$\cos\frac{\pi}{3} = 0.5$
E
10 (μA/m)
0
y
H
H
E
z


$t = 0$
phase
$\frac{\pi}{6}$
$\frac{\pi}{3}$
βz
z
λ
$\frac{\pi}{2}$
25m
50m

In the phasor domain, we can substitute

$$\frac{\partial}{\partial t} \rightarrow j\omega$$

so that Maxwell's equations are rewritten as

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot \tilde{\mathbf{D}} &= \tilde{\rho} \\
\nabla \cdot \tilde{\mathbf{B}} &= 0 \\
\nabla \times \tilde{\mathbf{E}} &= -j\omega\mathbf{B} \\
\nabla \times \tilde{\mathbf{H}} &= \tilde{\mathbf{J}} + j\omega\tilde{\mathbf{D}}
\end{aligned}$$

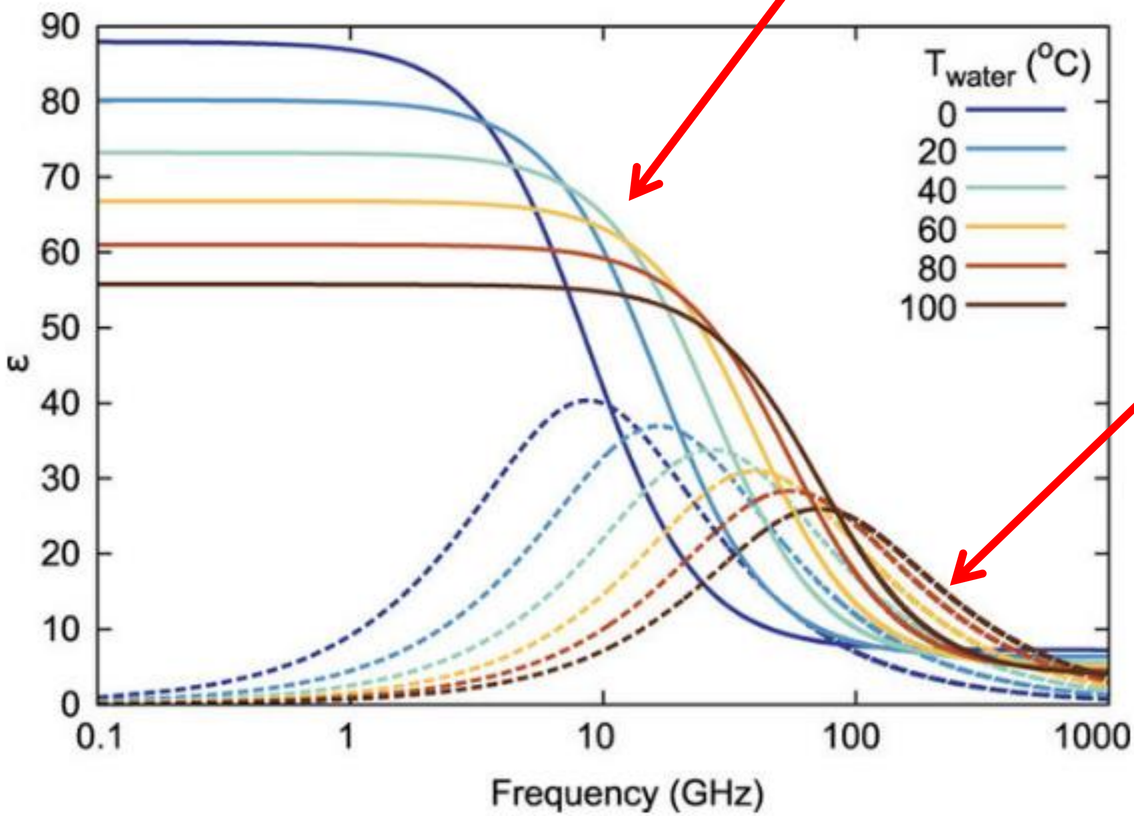
The auxiliary constitutive relations simply become

$$\tilde{\mathbf{D}} = \epsilon \tilde{\mathbf{E}}$$
$$\tilde{\mathbf{B}} = \mu \tilde{\mathbf{H}}$$
$$\tilde{\mathbf{J}} = \sigma \tilde{\mathbf{E}}$$

In general, the physical parameters ϵ, μ, σ are function of frequency.

this is the usual **real** relative permittivity

Example: relative permittivity of water



but there is also an **imaginary** part due to the conductivity losses

Nature – Scientific Reports 5(1):13535, August 2015

Wave equation in phasor form

Let's assume at first that we are away from charges and currents and that the medium has zero conductivity. We can transform directly equation in time-dependent form

d' Alembert equation

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \tilde{\mathbf{E}} = (-j\omega)(-j\omega)\,\mu\epsilon\tilde{\mathbf{E}} = -\omega^2\mu\epsilon\tilde{\mathbf{E}}$$

Helmholtz equation

$$\nabla^2 \tilde{\mathbf{E}} + \omega^2\mu\epsilon\tilde{\mathbf{E}} = 0$$

See in the notes equivalent derivation starting from $\nabla \times \tilde{\mathbf{E}}$

1-D model with x-polarized phasor electric field

$$\tilde{\mathbf{E}} = \hat{x}\tilde{E}_x(z)$$

1-D Helmholtz equation

$$\frac{\partial^2}{\partial z^2}\tilde{E}_x + \omega^2\mu\epsilon\tilde{E}_x = 0$$

Elementary solutions

$$\beta \equiv \omega\sqrt{\mu\epsilon}$$

$$\tilde{E}_x(z) = e^{\mp j\beta z}$$

General solution

$$\tilde{E}_x(z) = \mathrm{A}\, e^{-j\beta z} + \mathrm{B}\, e^{j\beta z}$$

forward wave **backward wave**

The coefficients A and B are the amplitudes of the wave solutions and can be complex due to a relative phase:

$$\mathrm{A} = \tilde{\mathrm{E}}_0 = E_0 e^{j\varphi} \quad \textbf{where} \quad E_0 = \left| \tilde{\mathrm{E}}_0 \right|$$

The complete solution for the forward wave can be written as

$$\tilde{\mathrm{E}}_x(z) = \tilde{\mathrm{E}}_0 e^{-j\beta z}\hat{x} = E_0 e^{j\varphi} e^{-j\beta z}\hat{x}$$

and we recover the time-dependent solution

$$E_x(z,t) = \Re e\left\{\tilde{\mathrm{E}}_x(z) e^{j\omega t}\hat{x}\right\} = \Re e\left\{E_0 e^{j\varphi} e^{-j\beta z} e^{j\omega t}\hat{x}\right\}$$

$$= E_0 \cos\left(\omega t - \beta z + \varphi\right)\hat{x}$$

The corresponding phasor of the magnetic field is obtained from the phasor form of Faraday's law

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}$$

$$\tilde{\mathbf{H}} = -\frac{\nabla \times \tilde{\mathbf{E}}}{j\omega\mu}$$

For our 1-D solution with x-polarized electric field

$$\nabla \times \left(E_0 e^{j\varphi} e^{-j\beta z}\hat{x}\right) = \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_0 e^{j\varphi} e^{-j\beta z} & 0 & 0 \end{bmatrix} =$$

$$\frac{\partial}{\partial z}\left(E_0 e^{j\varphi} e^{-j\beta z}\right)\hat{y} - \cancel{\frac{\partial}{\partial y}\left(E_0 e^{j\varphi} e^{-j\beta z}\right)}\hat{z} = -j\beta E_0 e^{j\varphi} e^{-j\beta z}\hat{y}$$

no y-dependence

Finally we obtain

$$\tilde{\mathrm{H}}_y(z) = -\frac{-j\beta E_0 e^{j\varphi} e^{-j\beta z}}{j\omega\mu}\hat{y}$$

$$= \frac{\omega\sqrt{\mu\varepsilon}}{\omega\mu} E_0 e^{j\varphi} e^{-j\beta z}\hat{y}$$

$$= \sqrt{\frac{\varepsilon}{\mu}} E_0 e^{j\varphi} e^{-j\beta z}\hat{y} = \frac{1}{\eta}\tilde{\mathrm{E}}_0 e^{-j\beta z}\hat{y}$$

Now we consider general media with conductivity due to free charges, but assumed to be neutral with zero net charge. Material properties are described by

$$\boxed{\tilde{\mathbf{J}} = \sigma\tilde{\mathbf{E}}} \qquad \boxed{\varepsilon = \varepsilon_r\varepsilon_0} \qquad \boxed{\mu = \mu_r\mu_0}$$

The curl Maxwell equations become

$$\nabla\times\tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}$$

$$\nabla\times\tilde{\mathbf{H}} = \sigma\tilde{\mathbf{E}} + j\omega\varepsilon\tilde{\mathbf{E}} = j\omega\left(\varepsilon - j\frac{\sigma}{\omega}\right)\tilde{\mathbf{E}}$$

It is as if the material conductivity introduces an imaginary part for the permittivity

We can now obtain the Helmholtz wave equation

$$\nabla \times \nabla \times \tilde{\mathbf{E}} = \cancel{\nabla \nabla \cdot \tilde{\mathbf{E}}} - \nabla^2 \tilde{\mathbf{E}} = -j\omega\mu \nabla \times \tilde{\mathbf{H}}$$

$$= -j\omega\mu\left(\tilde{\mathbf{J}} + j\omega\varepsilon\tilde{\mathbf{E}}\right)$$

$$\boxed{\nabla^2 \tilde{\mathbf{E}} = j\omega\mu(\sigma + j\omega\varepsilon)\tilde{\mathbf{E}}}$$

Note: we have set the divergence of the electric field to zero, even if there are free charges, because we assume the material to be charge neutral overall.

In 1D the wave equation is

$$\frac{\partial^2 \tilde{\mathrm{E}}_x}{\partial z^2} = j\omega\mu(\sigma + j\omega\varepsilon)\tilde{\mathrm{E}}_x = \gamma^2 \tilde{\mathrm{E}}_x$$

with general solution

$$\tilde{E}_x(z) = \mathbf{A}\, e^{-\gamma z} + \mathbf{B}\, e^{\gamma z}$$

The magnetic field becomes

$$\tilde{H}_y(z) = -\frac{1}{j\omega\mu}\frac{\partial \tilde{E}_x}{\partial z} = \sqrt{\frac{\sigma + j\omega\varepsilon}{j\omega\mu}}\left(\mathbf{A}\, e^{-\gamma z} - \mathbf{B}\, e^{\gamma z}\right)$$

$$= \frac{1}{\eta}\left(\mathbf{A}\, e^{-\gamma z} - \mathbf{B}\, e^{\gamma z}\right)$$

The propagation factor is complex

$$\gamma = \sqrt{(j\omega\mu)(\sigma + j\omega\varepsilon)} = \alpha + j\beta$$

and also the medium impedance is complex

$$\bar{\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\bar{\eta}|e^{j\tau}$$

These parameters describe the behavior of general media with conductivity which causes attenuation (damping) of electromagnetic waves. They are functions of frequency.