ECE 329 – Fall 2021

Prof. Ravaioli – Office: 2062 ECEB

Section E – 1:00pm

Lecture 24

Lecture 24 – Outline

- **Finish good conductors**
- **Examples of propagation in lossy medium**
- **Wave polarization**
- **Linear, Circular (and Elliptical Polarization)**

Reading assignment
Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:
24) Signal transmission, circular polarization

Good Conductor

$\sigma \gg$

$$\boxed{\frac{\sigma}{\omega\varepsilon} \gg 1}$$

$$\gamma = \sqrt{(j\omega\mu)(\sigma + \cancel{j\omega\varepsilon})} \approx \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma}\sqrt{j}$$

$$= \sqrt{\omega\mu\sigma}\, e^{j\frac{\pi}{4}} = \sqrt{\omega\mu\sigma}\left[\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right]$$

$$= \sqrt{\omega\mu\sigma}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = \sqrt{\pi f \mu\sigma}\,(1 + j)$$

$$\boxed{\alpha \approx \sqrt{\pi f \mu\sigma} \qquad \beta \approx \sqrt{\pi f \mu\sigma}}$$

Good Conductor

$$\sigma \gg$$

$$\frac{\sigma}{\omega\varepsilon} \gg 1$$

Phase Velocity

$$v_p = \frac{\omega}{\beta} \approx \frac{2\pi f}{\sqrt{\pi f \mu\sigma}} = \sqrt{\frac{4\pi f}{\mu\sigma}}$$

Wavelength

$$\lambda = \frac{2\pi}{\beta} \approx \frac{2\pi}{\sqrt{\pi f \mu\sigma}} = \sqrt{\frac{4\pi}{f\mu\sigma}}$$

Medium Impedance

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}} e^{j\frac{\pi}{4}}$$

$$= \sqrt{\frac{\omega\mu}{\sigma}}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{\pi f \mu}{\sigma}}(1+j)$$

Good Conductor

$\sigma \gg$

$$\frac{\sigma}{\omega\varepsilon} \gg 1$$

When this assumption is verified, the attenuation constant α and the propagation constant β are approximately equal.

The medium impedance η has nearly equal real and imaginary parts, with phase angle $\tau \approx 45° = \pi/4$.

Therefore, in a good conductor electric and magnetic field have always a phase difference $\tau \approx 45° = \pi/4$.

A strict rule of thumb is that approximations for good conductor can be applied when

$$\frac{\sigma}{\omega\varepsilon} \geq 100$$

Good Conductor

$\sigma \gg$

In a good conductor the fields attenuate rapidly. The distance over which fields are attenuated by a factor $\exp(-1)$ is

$$\frac{1}{\alpha} = \delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

For a good conductor, this is called "skin depth". In general, $1/\alpha$ can be called "penetration depth"

Perfect Conductor

$\sigma \to \infty$

For this ideal material, the attenuation is also infinite and the skin depth goes to zero. This means that the electromagnetic field must go to zero below the perfect conductor surface.

"General Medium"

In this case, we need to use the complete formulation, since approximations would be too inaccurate.

Following the previously introduced rules, we can define the general medium to be in the range:

$$0.01 \leq \frac{\sigma}{\omega\varepsilon} \leq 100$$

Of course, this is arbitrary. The full model is valid for all cases.

"General Medium"

Full result for the propagation constant

$$\gamma = \sqrt{(j\omega\mu)(\sigma + j\omega\varepsilon)} = \alpha + j\beta$$

It can be shown that

$$\alpha = \frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]^{\frac{1}{2}}$$

$$\beta = \frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]^{\frac{1}{2}}$$

"General Medium"

Phase Velocity

$$v_p = \frac{\omega}{\beta} = \frac{\sqrt{2}}{\sqrt{\mu\varepsilon}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{-\frac{1}{2}}$$

Wavelength

$$\lambda = \frac{2\pi}{\beta} = \frac{\sqrt{2}}{f\sqrt{\mu\varepsilon}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]^{-\frac{1}{2}}$$

Medium Impedance

$$\bar{\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\bar{\eta}|e^{j\tau}$$

Summary of the special cases (table in the notes)

	Condition	β	α	$\lvert\eta\rvert$	τ	$\lambda = \frac{2\pi}{\beta}$	$\delta = \frac{1}{\alpha}$
Perfect dielectric	$\sigma = 0$	$\omega\sqrt{\epsilon\mu}$	0	$\sqrt{\frac{\mu}{\epsilon}}$	0	$\frac{2\pi}{\omega\sqrt{\epsilon\mu}}$	∞
Imperfect dielectric	$\frac{\sigma}{\omega\epsilon} \ll 1$	$\sim \omega\sqrt{\epsilon\mu}$	$\beta\frac{1}{2}\frac{\sigma}{\omega\epsilon} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$	$\sim \sqrt{\frac{\mu}{\epsilon}}$	$\sim \frac{\sigma}{2\omega\epsilon}$	$\sim \frac{2\pi}{\omega\sqrt{\epsilon\mu}}$	$\frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$
Good conductor	$\frac{\sigma}{\omega\epsilon} \gg 1$	$\sim \sqrt{\pi f\mu\sigma}$	$\sim \sqrt{\pi f\mu\sigma}$	$\sqrt{\frac{\omega\mu}{\sigma}}$	$45°$	$\sim \frac{2\pi}{\sqrt{\pi f\mu\sigma}}$	$\sim \frac{1}{\sqrt{\pi f\mu\sigma}}$
Perfect conductor	$\sigma = \infty$	∞	∞	0	-	0	0

Example

The intrinsic impedance of the medium is complex as long as the conductivity is not zero. The phase angle of the intrinsic impedance indicates that electric field and magnetic field are out of phase.

Considering the forward wave solutions

$$\tilde{E}_x(z) = E_0 e^{j\varphi}\, e^{-\gamma z} = E_0 e^{j\varphi}\, e^{-\alpha z} e^{-j\beta z} = E_0\, e^{-\alpha z} e^{-j(\beta z-\varphi)}$$

$$\tilde{H}_y(z) = \frac{1}{\eta} E_0 e^{j\varphi}\, e^{-\gamma z} = \frac{1}{|\eta|} E_0 e^{j\varphi}\, e^{-\alpha z} e^{-j\beta z} e^{-j\tau}$$

$$= \frac{1}{|\eta|} E_0\, e^{-\alpha z} e^{-j(\beta z-\varphi+\tau)}$$

The time-dependent fields are recovered as:

$$E_x(z,t) = \Re e\left\{E_0\, e^{-\alpha z} e^{-j(\beta z - \varphi)}\, e^{j\omega t}\right\}$$

$$= E_0\, e^{-\alpha z} \cos\left(\omega t - \beta z + \varphi\right)$$

$$H_y(z,t) = \frac{1}{|\eta|} \Re e\left\{E_0\, e^{-\alpha z} e^{-j(\beta z - \varphi + \tau)}\, e^{j\omega t}\right\}$$

$$= \frac{1}{|\eta|} E_0\, e^{-\alpha z} \cos\left(\omega t - \beta z + \varphi - \tau\right)$$

Example: A typical good conductor is copper (Cu), which has the following parameters

$$\sigma = 5.80 \times 10^7 \,[\mathrm{S/m}]$$

$$\varepsilon \approx \varepsilon_o$$

$$\mu \approx \mu_o$$

What is the skin depth of copper at 30 GHz?

$$\frac{1}{\alpha} = \delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

$$= \frac{1}{\sqrt{\pi \times 30 \times 10^9 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}} = 381\,\mathrm{nm}$$

Example: Although copper is a good conductor, at what frequency it stops being that? At what frequency does it become an imperfect dielectric? Assume that the conductivity is always 5.8×10^7 S/m.

Conventionally, copper ceases to be a good conductor when

$$\frac{\sigma}{\omega\varepsilon}=100 \rightarrow f=\frac{\sigma}{100\times2\pi\times\varepsilon_r\varepsilon_0}=\frac{5.8\times10^7}{100\times2\pi\times8.854\times10^{-12}}\approx10^{17}\,\mathrm{Hz}$$

X-rays

Copper starts to be an imperfect dielectric when

$$\frac{\sigma}{\omega\varepsilon}=0.01 \rightarrow f=\frac{\sigma}{0.01\times2\pi\times\varepsilon_r\varepsilon_0}=\frac{5.8\times10^7}{0.01\times2\pi\times8.854\times10^{-12}}\approx10^{21}\,\mathrm{Hz}$$

Gamma rays

Example: Consider sea water, with parameters

$$\sigma \approx 4.0 \text{ [S/m]}$$

$$\varepsilon \approx 80\varepsilon_o$$

$$\mu \approx \mu_o$$

What type of material is sea water at 25 kHz?

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi \times 25 \times 10^3 \times 80 \times 8.854 \times 10^{-12}} = 35{,}958$$

$$\boxed{\frac{\sigma}{\omega\varepsilon} \gg 1}$$

good conductor

Example: Poynting vector in imperfect dielectric

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$<\mathbf{S}> = \frac{1}{2}\,\Re e\left[\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}^*\right] \qquad (\mathrm{W/m^2})$$

Plane TEM wave at 1 GHz with $\sigma = 10^{-3}$ S/m and $\varepsilon = 4\epsilon_0$

$$\tilde{\mathbf{E}} = \hat{y}2e^{-\alpha z}e^{-j\beta z}\,\frac{\mathrm{V}}{\mathrm{m}}$$

$$\frac{\sigma}{\omega\epsilon} = \frac{10^{-3}\cdot 36\pi \times 10^9}{2\pi\cdot 10^9\cdot 4} = \frac{9}{2}10^{-3} \ll 1$$

$$\tau \approx \frac{\sigma}{2\omega\epsilon} \approx \frac{9}{4}10^{-3}\,\mathrm{rad}$$

$$|\eta| \approx \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_o}{4\epsilon_o}} = \frac{\eta_o}{2} = 60\pi\,\Omega$$

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} = \frac{1}{2}10^{-3}60\pi = 30\pi\cdot 10^{-3}\,\frac{1}{\mathrm{m}}$$

Magnetic field

$$\tilde{\mathbf{H}} = -\hat{x}\frac{2}{\eta}e^{-\alpha z}e^{-j\beta z} \approx -\hat{x}\frac{2}{\sqrt{\mu/\epsilon}}e^{-\alpha z}e^{-j\beta z}e^{-j\tau}\ \frac{\mathrm{A}}{\mathrm{m}}$$

$$\boxed{\bar{\eta} = |\bar{\eta}|e^{j\tau}}$$

$$e^{-j\tau} = \cos(\tau) - j\sin(\tau) = \underbrace{0.9999975 - j0.00225}_{\approx \mathbf{1}}$$

Time Average Poynting vector

$$\begin{aligned}\langle \mathbf{S} \rangle &= \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\} = \frac{1}{2}\mathrm{Re}\{\hat{y}2e^{-\alpha z}e^{-j\beta z} \times (-\hat{x}\frac{2}{\sqrt{\mu/\epsilon}}e^{-\alpha z}e^{-j\beta z}e^{-j\tau})^*\} \\ &= -\frac{1}{2}\mathrm{Re}\{\hat{y}2e^{-\alpha z} \times \hat{x}\frac{2}{\sqrt{\mu/\epsilon}}e^{-\alpha z}e^{j\tau}\} = \hat{z}\frac{2}{\sqrt{\mu/\epsilon}}e^{-2\alpha z}\cos\tau.\end{aligned}$$

NOTE "$-2\alpha z$"

Time Average Poynting vector

$$\langle \mathbf{S} \rangle = \hat{z}\frac{2}{\sqrt{\mu/\epsilon}}e^{-2\alpha z}\cos\tau$$

$\boxed{z = 0}$

$$\langle \mathbf{S} \rangle = \hat{z}\frac{2}{\sqrt{\mu/\epsilon}}\cos\tau \approx \hat{z}\frac{2}{60\pi} = \hat{z}\frac{1}{30\pi}\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

$\boxed{z = 10\ \mathrm{m}}$

$$\langle \mathbf{S} \rangle = \hat{z}\frac{2}{\sqrt{\mu/\epsilon}}e^{-2\cdot 30\pi\cdot 10^{-3}\cdot 10}\cos\tau \approx \hat{z}\frac{2}{60\pi}e^{-6\pi/10} \approx \hat{z}\frac{0.15}{30\pi}\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

Even if the medium is an "imperfect dielectric" the wave has lost 85% of the power over 10 m. This power is transferred to the medium due to induced current density corresponding to a term

$$\langle \mathbf{J} \cdot \mathbf{E} \rangle$$

Some examples follow for you to explore more.

We do not have time to cover them in class but we will point out some important concepts.

Example – Consider now the same $\mathbf{E}$ field in sea water but at frequency of 1kHz. Sea water has typically $\sigma = 4\,\mathrm{S/m}$ and $\varepsilon_r = 80$ to 81.

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times10^{3}\times80\times8.854\times10^{-12}} = 8.9\times10^{5} \gg 1$$

good conductor

$$\tilde{\mathbf{E}} = \hat{y}2e^{-\alpha z}e^{-j\beta z}\,\frac{\mathrm{V}}{\mathrm{m}}$$

$$\tilde{\mathbf{H}} = -\hat{x}\frac{2}{\eta}e^{-\alpha z}e^{-j\beta z} \approx -\hat{x}\frac{2}{|\eta|}e^{-\alpha z}e^{-j\beta z}e^{-j\tau}\,\frac{\mathrm{A}}{\mathrm{m}}$$

$$\begin{aligned}\langle\mathbf{S}\rangle &= \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^{*}\} = \frac{1}{2}\mathrm{Re}\{\hat{y}2e^{-\alpha z}e^{-j\beta z}\times(-\hat{x}\frac{2}{|\eta|}e^{-\alpha z}e^{-j\beta z}e^{-j\tau})^{*}\}\\ &= -\frac{1}{2}\mathrm{Re}\{\hat{y}2e^{-\alpha z}\times\hat{x}\frac{2}{|\eta|}e^{-\alpha z}e^{j\tau}\} = \hat{z}\frac{2}{|\eta|}e^{-2\alpha z}\cos\tau.\end{aligned}$$

The expressions of magnetic field and Poynting vector are very similar, but now parameters are quite different

$$
\begin{aligned}
\tau &\approx \frac{\pi}{4}\,\mathrm{rad} \\
|\eta| &\approx \sqrt{\frac{\omega\mu}{\sigma}} = \sqrt{\frac{2\pi\cdot 10^3\cdot 4\pi\cdot 10^{-7}}{4}} = \pi\sqrt{2\times 10^{-4}} \approx \frac{\pi\sqrt{2}}{100}\,\Omega \\
\alpha &\approx \sqrt{\pi f\mu\sigma} = \sqrt{\pi\cdot 10^3\cdot 4\pi\cdot 10^{-7}\cdot 4} = \sqrt{4^2\pi^2 10^{-4}} = \frac{\pi}{25}\,\frac{1}{\mathrm{m}}
\end{aligned}
$$

$$\boxed{\alpha \propto \sqrt{f}}$$

lower attenuation at lower frequency but less communication bandwidth

$\boxed{z = 0}$

$$\langle \mathbf{S} \rangle = \hat{z}\frac{2}{|\eta|}\cos\tau \approx \hat{z}\frac{200}{\pi\sqrt{2}}\cos\frac{\pi}{4} = \hat{z}\frac{100}{\pi}\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

$\approx 92\%$ power reduction

$\boxed{z = 10\ \mathrm{m}}$

$$\langle \mathbf{S} \rangle = \hat{z}\frac{100}{\pi}\,e^{-2\cdot\frac{\pi}{25}\cdot 10} \approx \hat{z}\frac{100}{\pi}0.081\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

Penetration depth for sea water at 1 kHz.

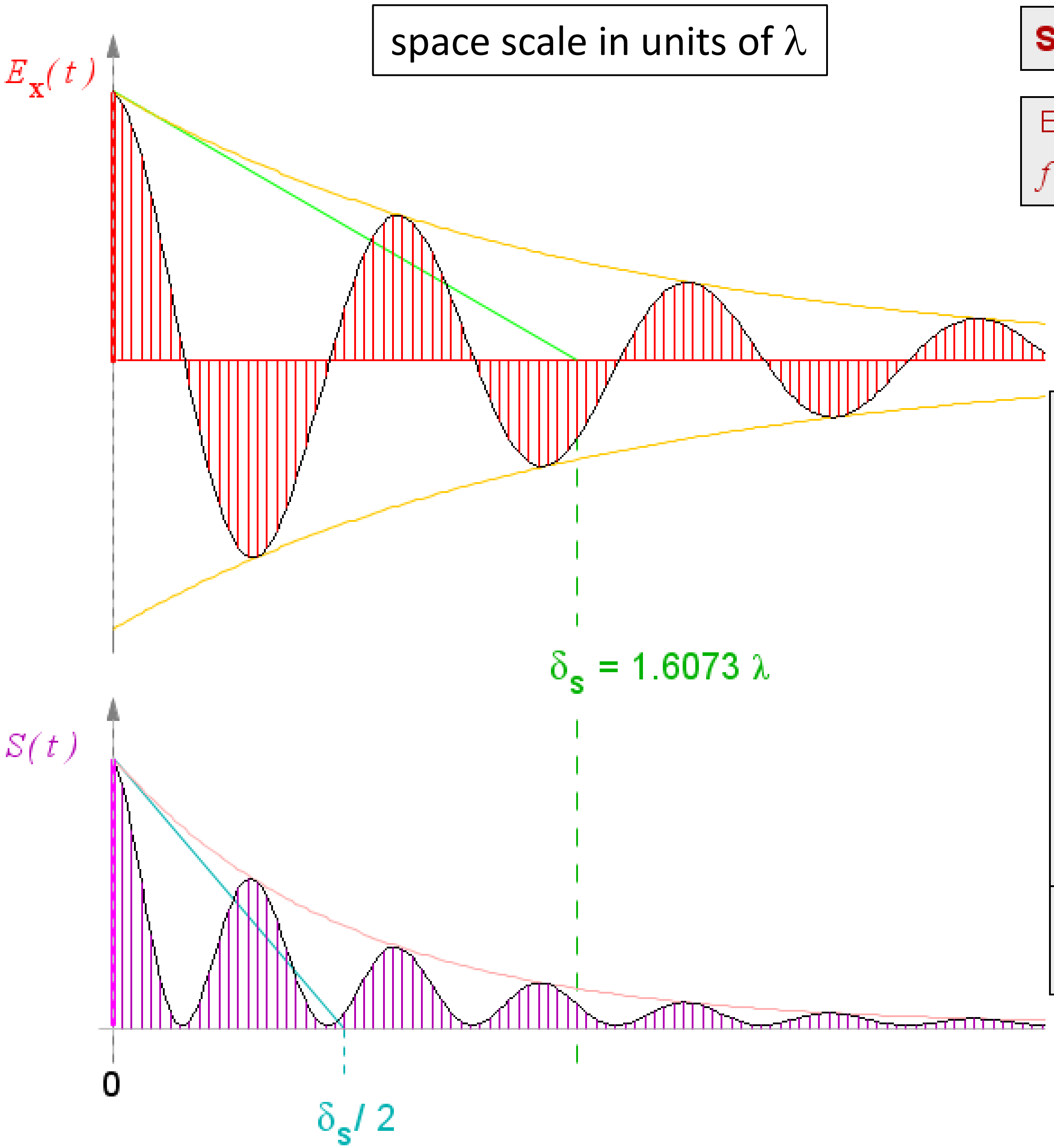
$$\delta \approx \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi 10^3 \cdot 4\pi \cdot 10^{-7} 4}}$$

$$= \frac{1}{\sqrt{4^2 \pi^2 \cdot 10^{-4}}} = \frac{100}{4\pi} = \frac{25}{\pi} \approx 7.95\,\mathrm{m}$$

This frequency would be suitable for communications with submarines at shallow depths.

- **Band for shallow depth communications is 3 kHz-300 kHz.**
- **For deeper communications (up to 100's of meters) the 3 to 300Hz band has been used. US SAGUINE System worked at 76 Hz. Signals are limited to a few characters per minute; transmitting antennas (buried in the ground) cover very large areas and need enormous power. This is a one-way technology since submarines cannot carry large enough antennas.**
- **Acoustic technologies are still being developed, including acoustic (in water) and radar (in air) hybrid systems.**

Significance of penetration (skin) depth



Slightly Lossy

E (z = 0) = 10.0 [V/m] σ = 0.001 [S/m]

f = 10.0 MHz $\varepsilon_r = 9.0$

WaveLength λ = 9.95085 [m]

Phase Velocity $u_p = 9.95085 \times 10^7$ [m / s]

Impedance of the Medium [Ω]

η = 123.831975 + j 12.261782
= 124.437572 ∠ 0.0987 rad
= 124.437572 ∠ 5.655 °

Penetration (Skin) Depth

$\delta_s = 1.6073\ \lambda = 15.9941$ [m]

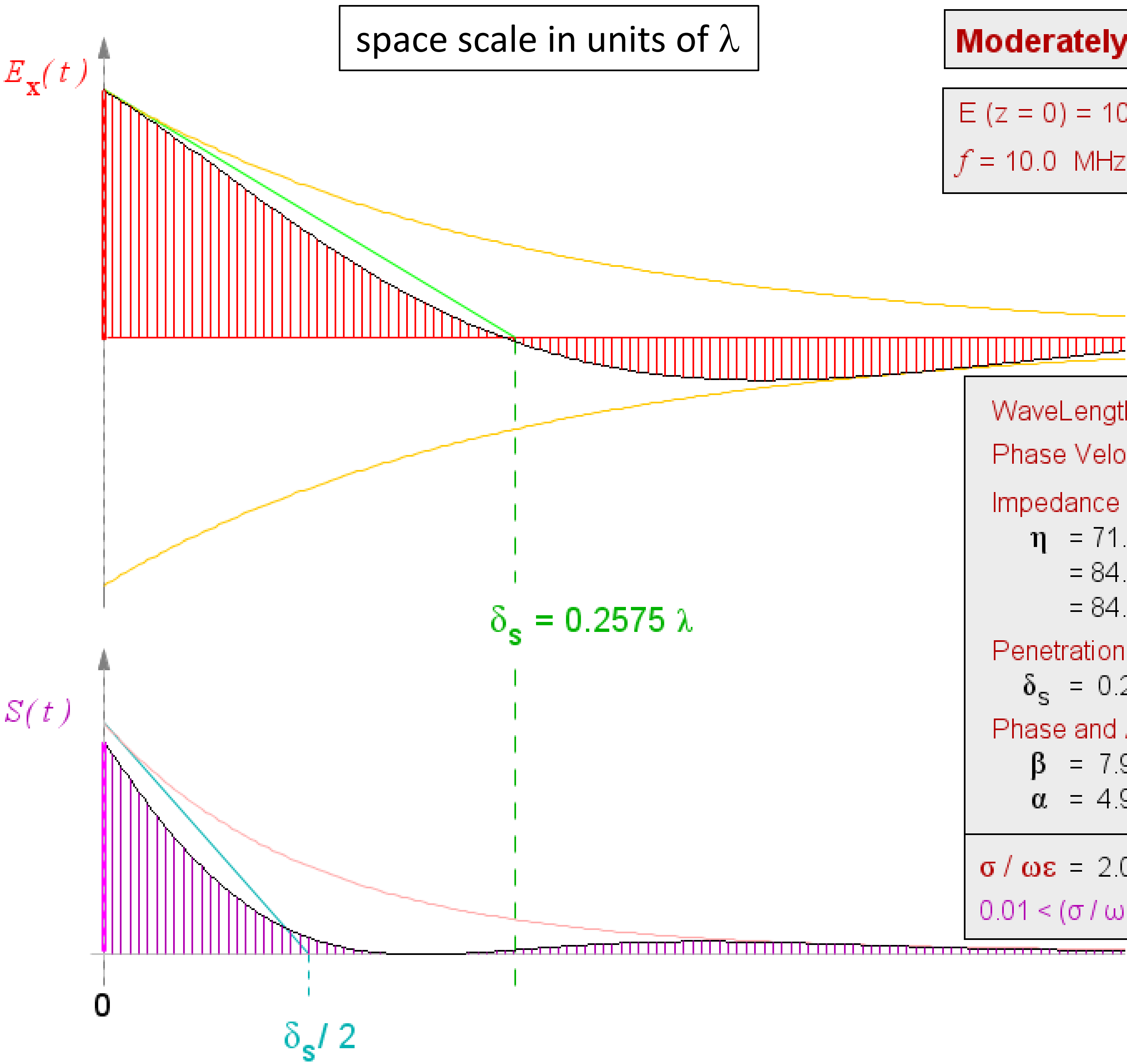
Phase and Attenuation Constants

$\beta = 6.31422 \times 10^{-1}$ [m^{-1}]
$\alpha = 6.25231 \times 10^{-2}$ [Ne / m]

$\sigma / \omega\varepsilon = 0.2$

$0.01 < (\sigma / \omega\varepsilon) < 100.0$ General medium

Significance of penetration (skin) depth



Moderately Lossy

E (z = 0) = 10.0 [V/m] σ = 0.01 [S/m]
f = 10.0 MHz $\varepsilon_r = 9.0$

WaveLength λ = 7.86151 [m]
Phase Velocity $u_p = 7.86151 \times 10^7$ [m / s]
Impedance of the Medium [Ω]
η = 71.485619 + j 44.180542
= 84.036385 ∠ 0.5536 rad
= 84.036385 ∠ 31.7175 °
Penetration (Skin) Depth
δ_s = 0.2575 λ = 2.02448 [m]
Phase and Attenuation Constants
$\beta = 7.99234 \times 10^{-1}$ [m^{-1}]
$\alpha = 4.93953 \times 10^{-1}$ [Ne / m]

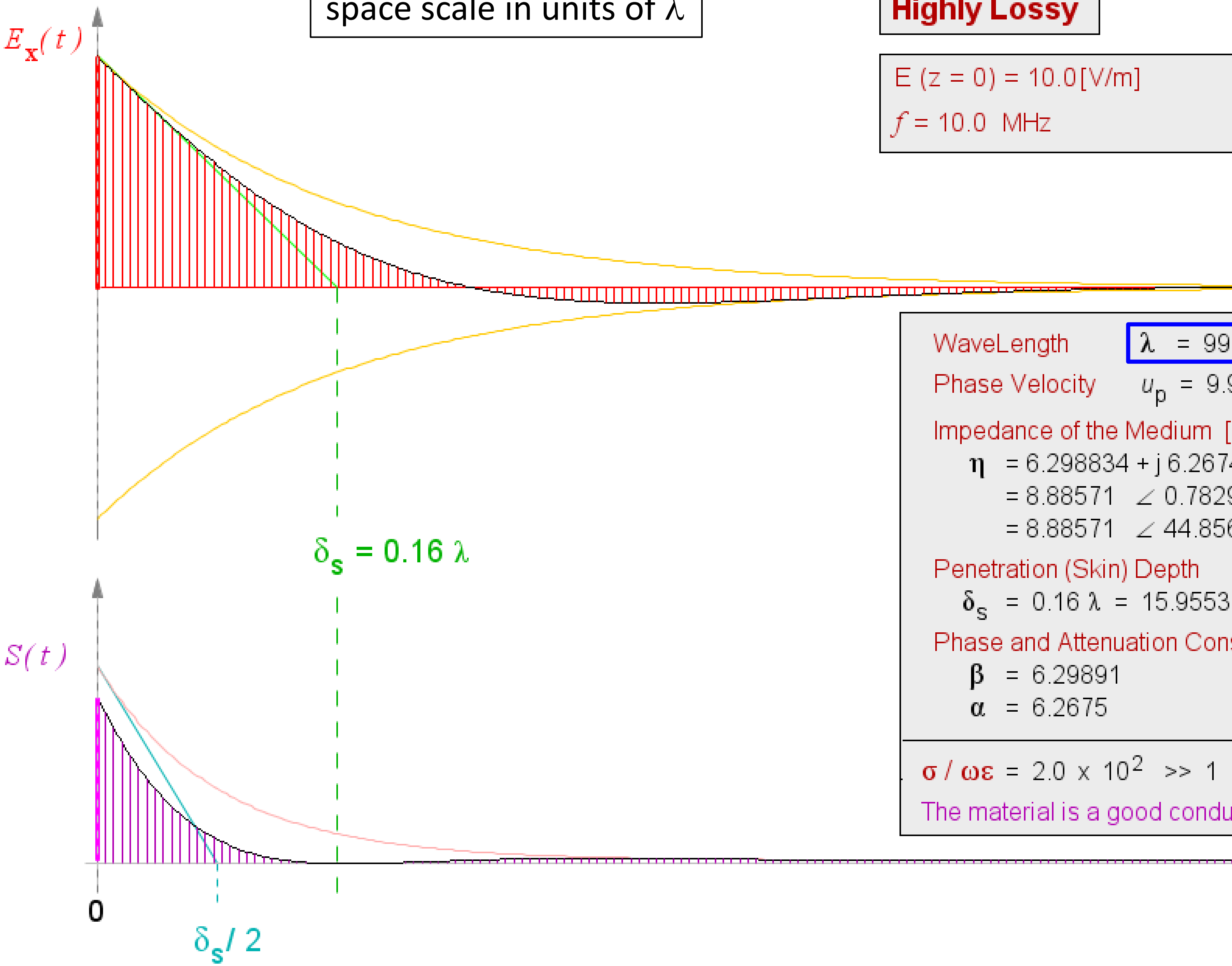
$\sigma / \omega\varepsilon$ = 2.0
$0.01 < (\sigma / \omega\varepsilon) < 100.0$ General medium

Significance of penetration (skin) depth

space scale in units of λ

Highly Lossy

E (z = 0) = 10.0 [V/m]	σ = 1.0 [S/m]
f = 10.0 MHz	ε_r = 9.0



WaveLength $\boxed{\lambda = 99.75031 \ [\text{cm}]}$

Phase Velocity $u_p = 9.97503 \times 10^6$ [m / s]

Impedance of the Medium [Ω]

η = 6.298834 + j 6.267419
= 8.88571 $\angle$ 0.7829 rad
= 8.88571 $\angle$ 44.8568 °

Penetration (Skin) Depth

δ_s = 0.16 λ = 15.95533 [cm]

Phase and Attenuation Constants

β = 6.29891 [m^{-1}]

α = 6.2675 [Ne / m]

$\sigma / \omega\varepsilon = 2.0 \times 10^2 >> 1$

The material is a good conductor

Example – Current in a conductor

$$f = 1.0\ \text{GHz}$$

$$\sigma = 10^7\ \text{S/m}$$

$$\varepsilon_r = 1.0$$

$$\lambda = 31.6\ \mu\text{m}$$

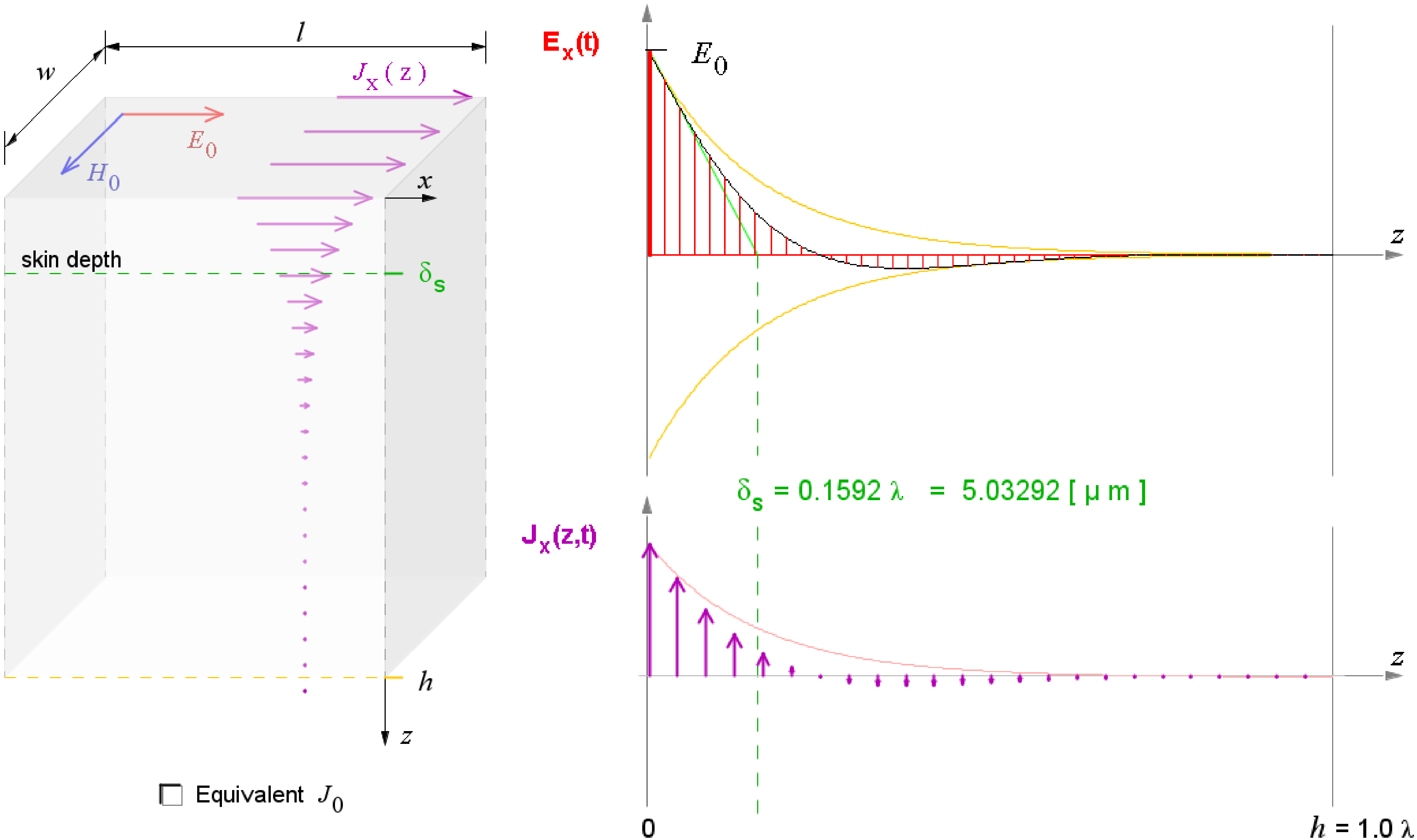
$$v_p = 3.16\ \text{m/s}$$

$$\eta = 0.0281\angle 45^\circ$$

$$\alpha = 1.98692\times 10^5\ \text{Ne/m}$$

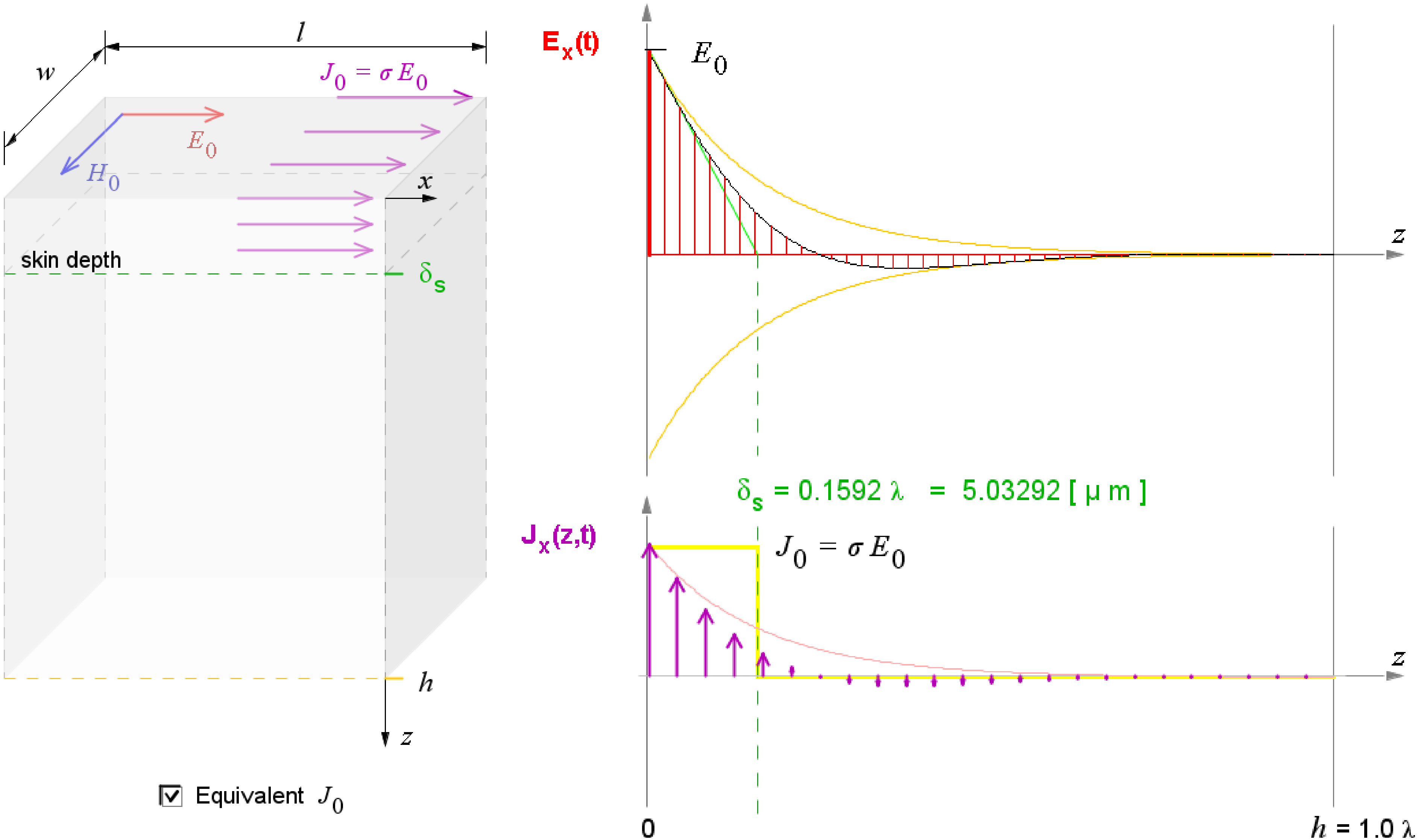
$$\beta = 1.98692\times 10^5\ \text{m}^{-1}$$

Current in a conductor



actual current distribution

Current in a conductor



equivalent uniform current model flowing only in the "skin" layer

Wave Polarization

Consider two TEM waves transmitted simultaneously and amplitude modulated with the same signal

$$f_1(t) = m(t)\cos(\omega t)$$

$$f_2(t) = m(t)\sin(\omega t)$$

polarized along x

$$\mathbf{E} = m(t - \frac{z}{v})\cos(\omega t - \beta z)\hat{x}$$

polarized along y

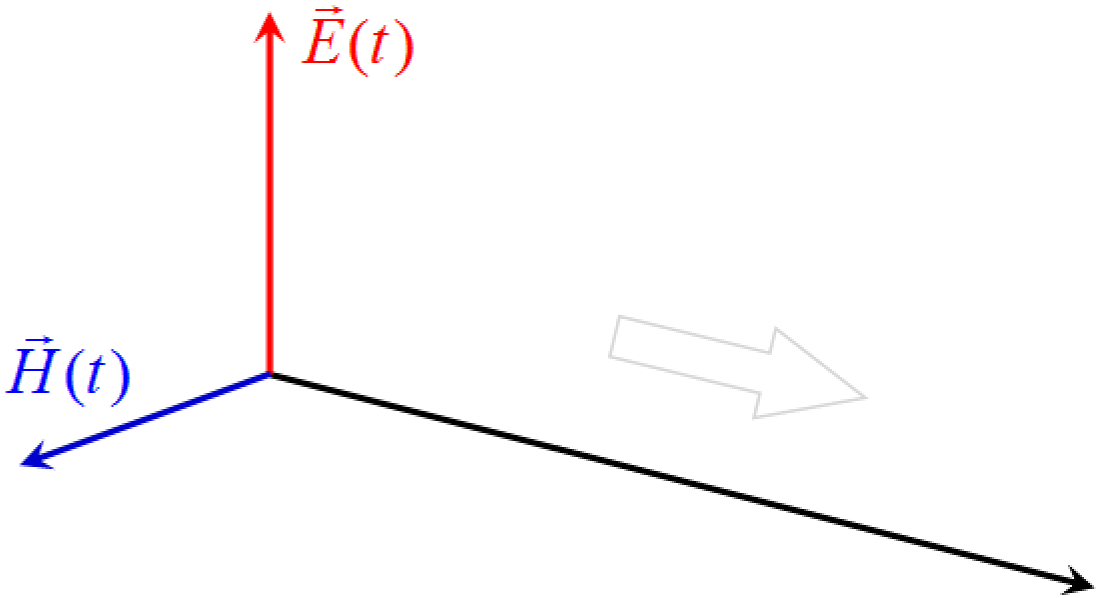
$$\mathbf{E} = m(t - \frac{z}{v})\sin(\omega t - \beta z)\hat{y}$$

combined wave

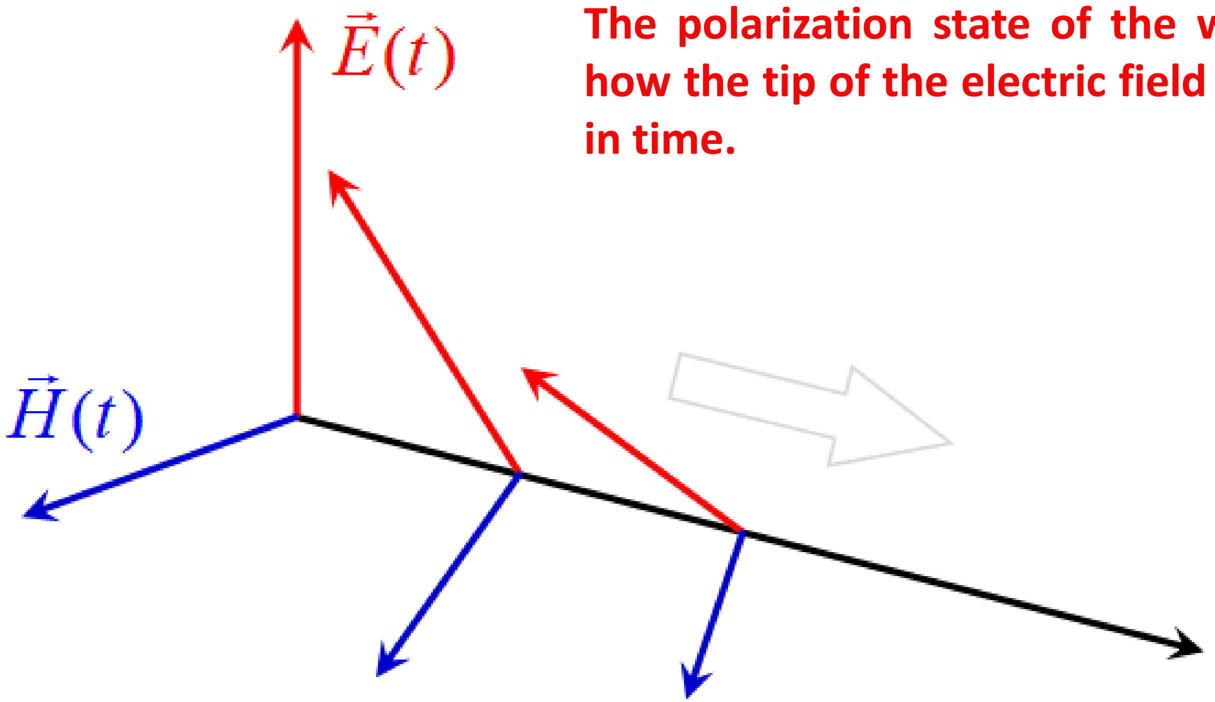
$$\boxed{\mathbf{E} = m(t - \frac{z}{v})[\cos(\omega t - \beta z)\hat{x} + \sin(\omega t - \beta z)\hat{y}]}$$

$$\boxed{\mathbf{H} = m(t - \frac{z}{v})[\cos(\omega t - \beta z)\hat{y} - \sin(\omega t - \beta z)\hat{x}]/\eta}$$

The resultant vector for the combined fields rotates in time. The tip of the vector describes a circle rotating counter-clockwise for an observer facing the incoming wave along z. This is called a right-handed circular polarization.

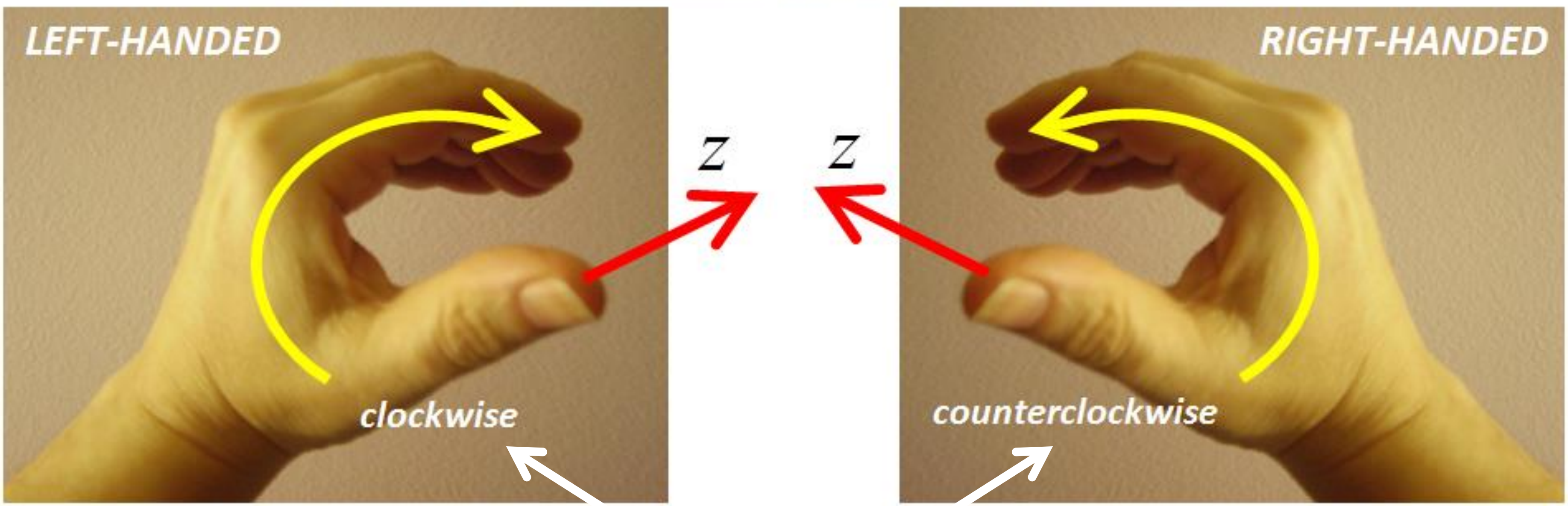


The polarization state of the wave identifies how the tip of the electric field vector evolves in time.



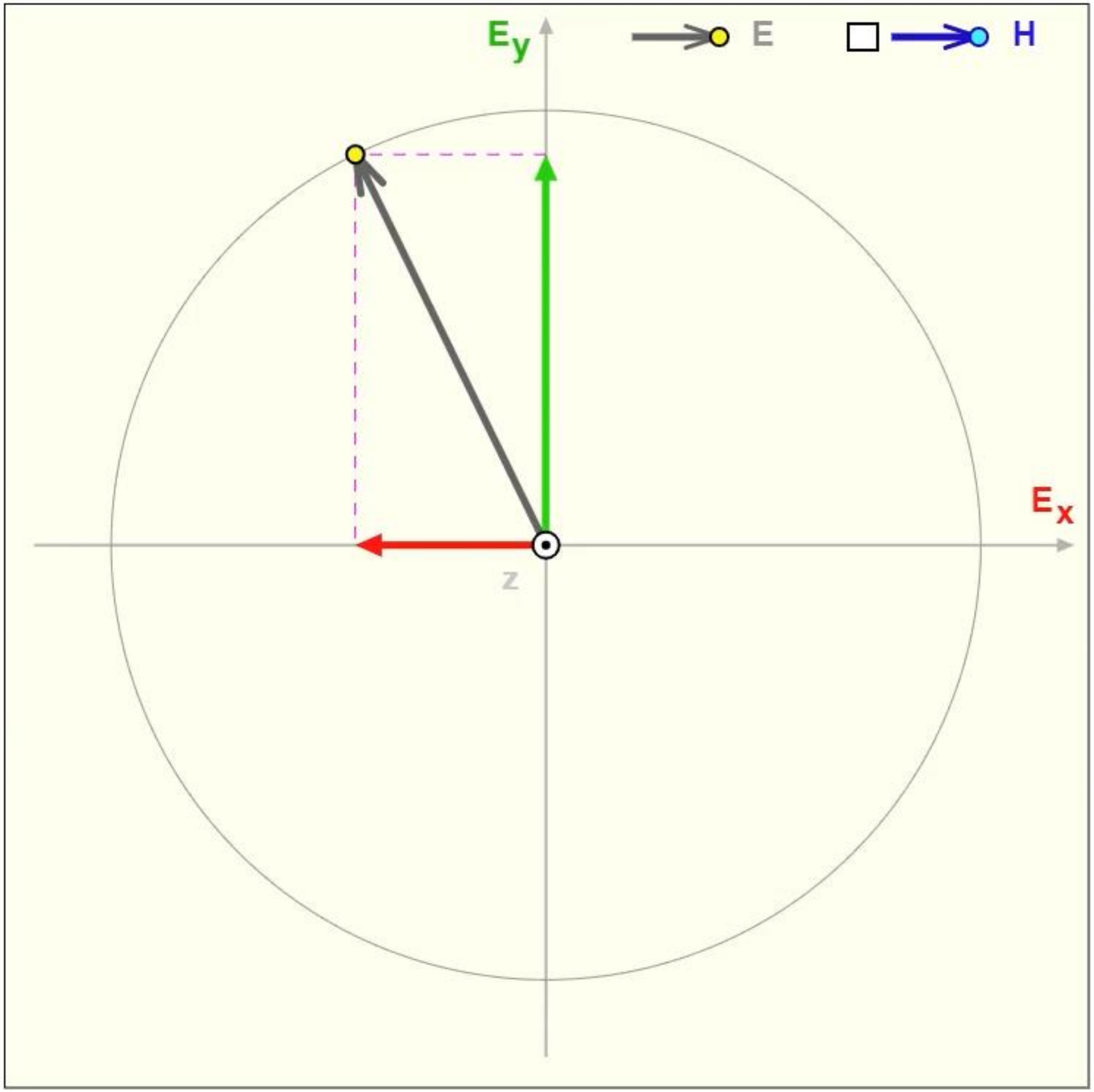
The motion of the *electric field* vector tip on the $\{x, y\}$ plane illustrates the polarization state (*linear*, *elliptical*, or *circular*).

The positive z–direction is the reference for rotation.



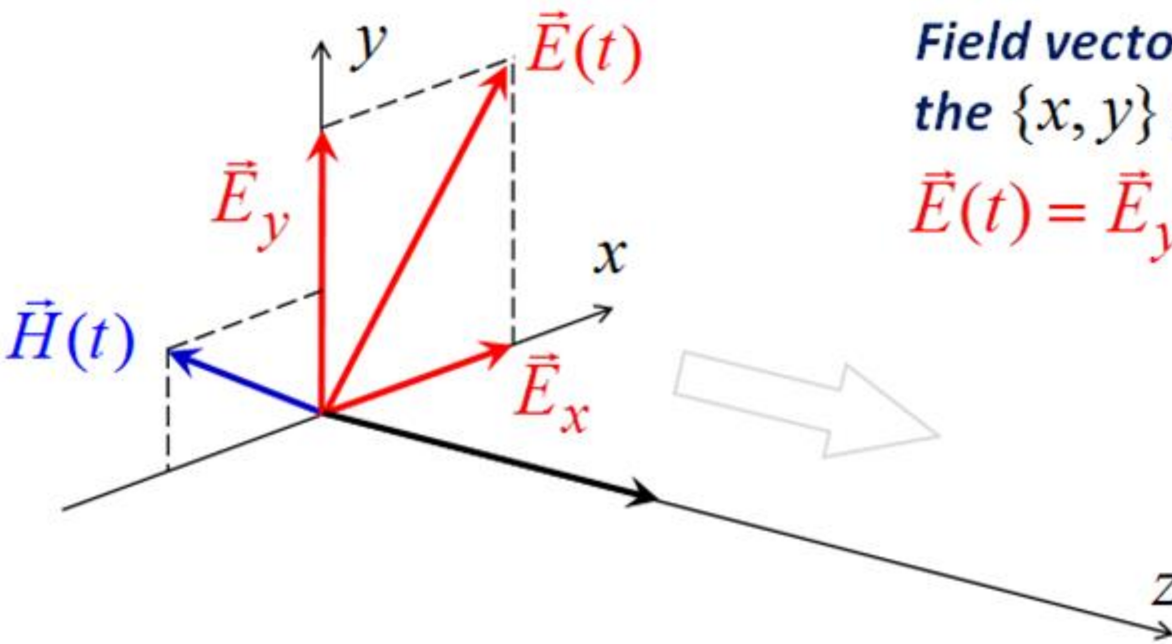
looking into the thumb

Right-Handed Circular Polarization



Polarization of Plane Waves

The electric field of a general electromagnetic wave can be decomposed into two orthogonal components. The resultant field vector propagates with a pattern which depends on the relative amplitude and phase of the components.



Field vectors are on the $\{x, y\}$ plane

$$\vec{E}(t) = \vec{E}_y(t) + \vec{E}_x(t)$$

The electric field is usually the reference for analysis of polarization states. For simplicity, we assume that the propagation direction is parallel to the z-axis.

The general expression for the electric field in the $\{x, y\}$ plane is

amplitude ***phase***

$$\mathbf{E}(z,t) = E_x \cos(\omega t - \beta z + \phi_x)\,\hat{x} + E_y \cos(\omega t - \beta z + \phi_y)\,\hat{y}$$

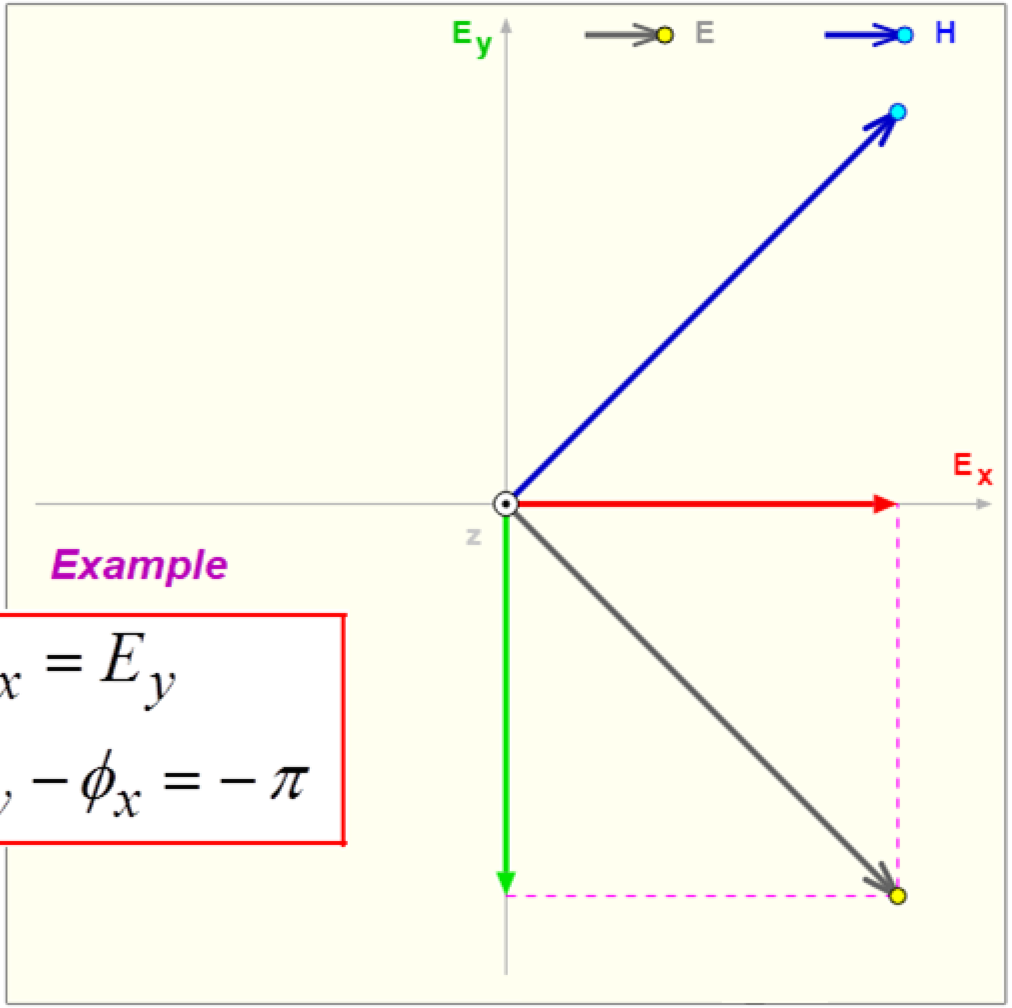
amplitude ***phase***

wave number

$$\boxed{\beta = \frac{2\pi}{\lambda}}$$

phase difference

$$\boxed{\Delta\phi = \phi_y - \phi_x \in [-\pi, \pi]}$$



Example

$$E_x = E_y$$

$$\phi_y - \phi_x = -\pi$$

Linear Polarization

$$\phi_y - \phi_x = \pm n\pi$$

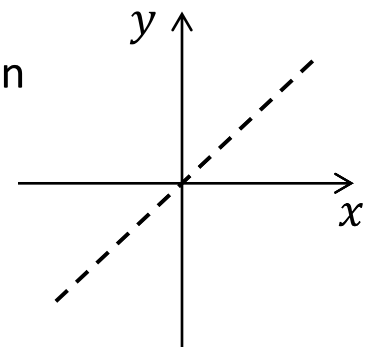
$$n = 0, 1$$

Field components are exactly in phase or in opposition of phase.

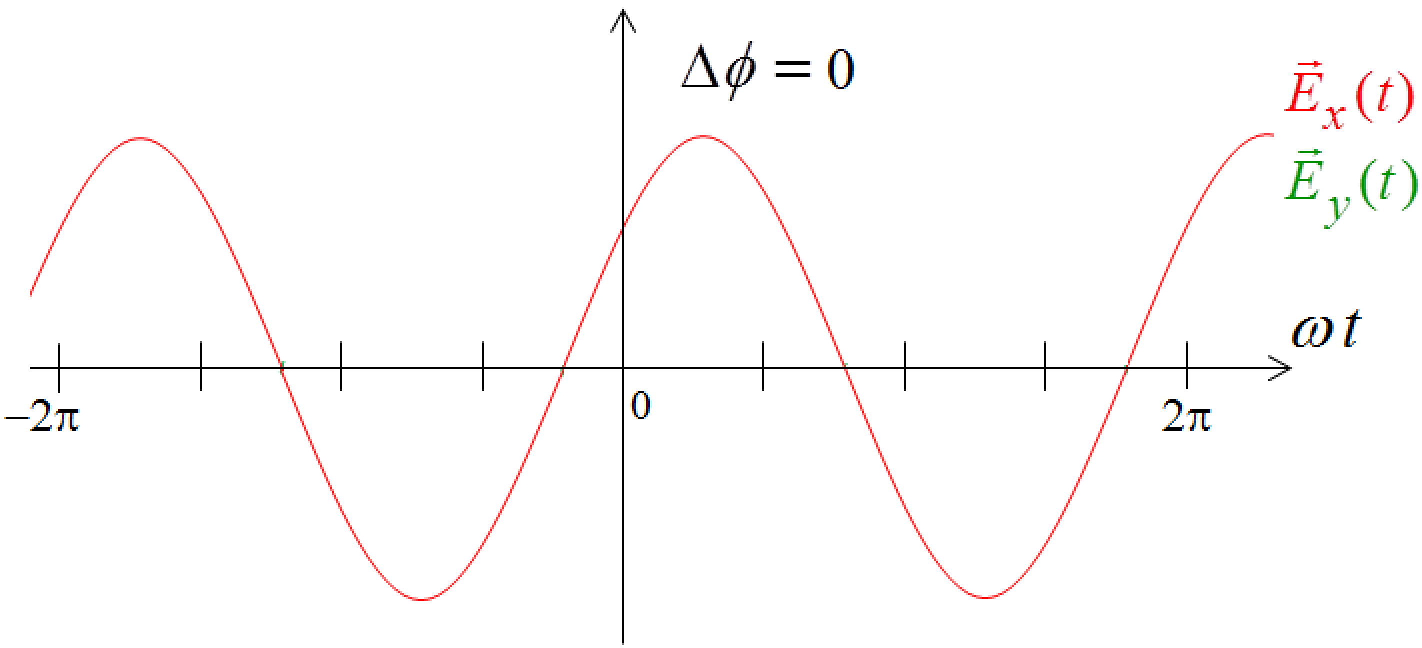
Special Cases

$E_x = 0$ ***Vertical Polarization***

$E_y = 0$ ***Horizontal Polarization***

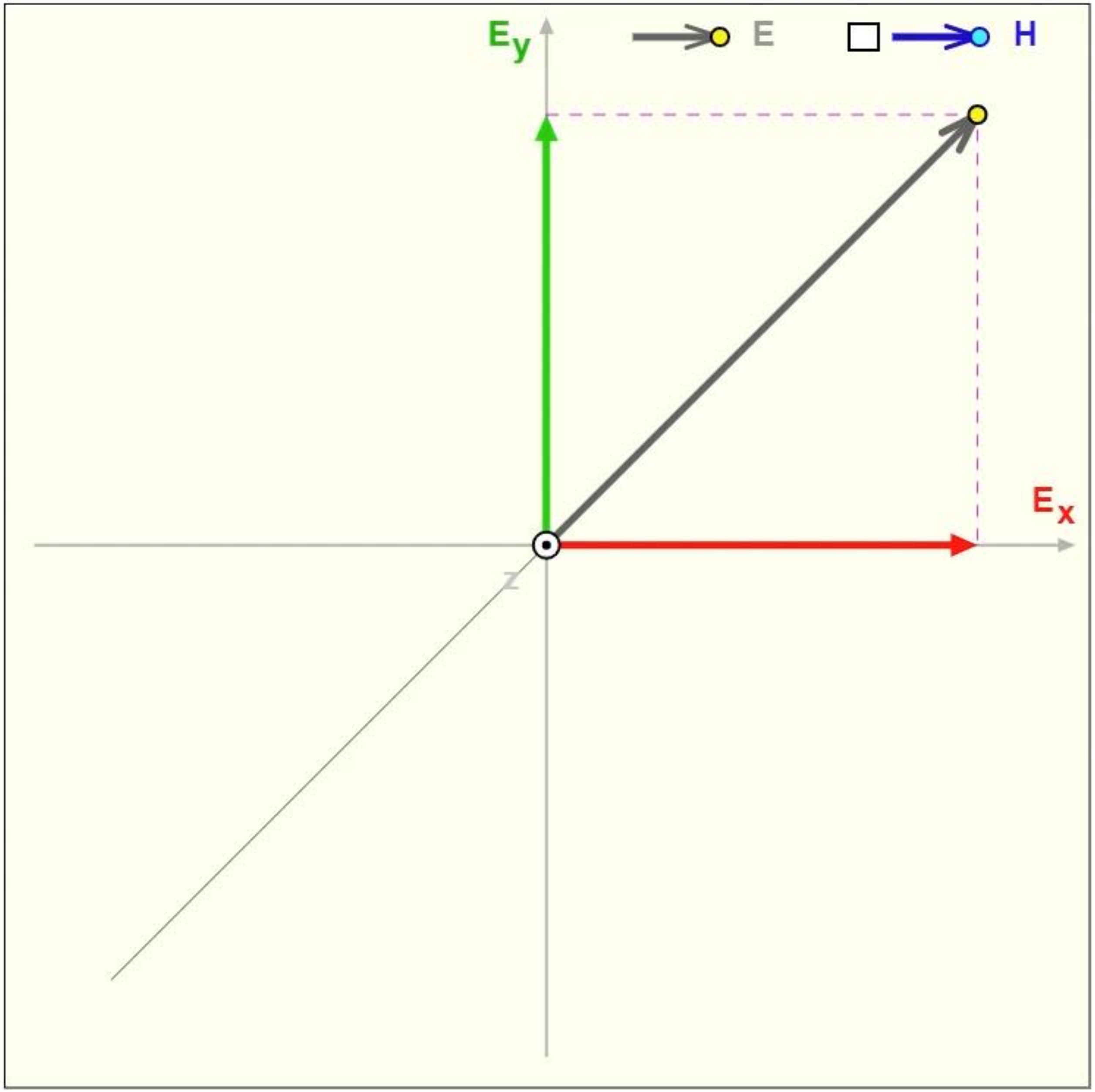


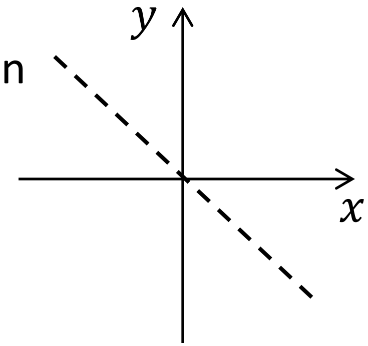
$\vec{E}_x(t)$ ***and*** $\vec{E}_y(t)$ ***are in phase***



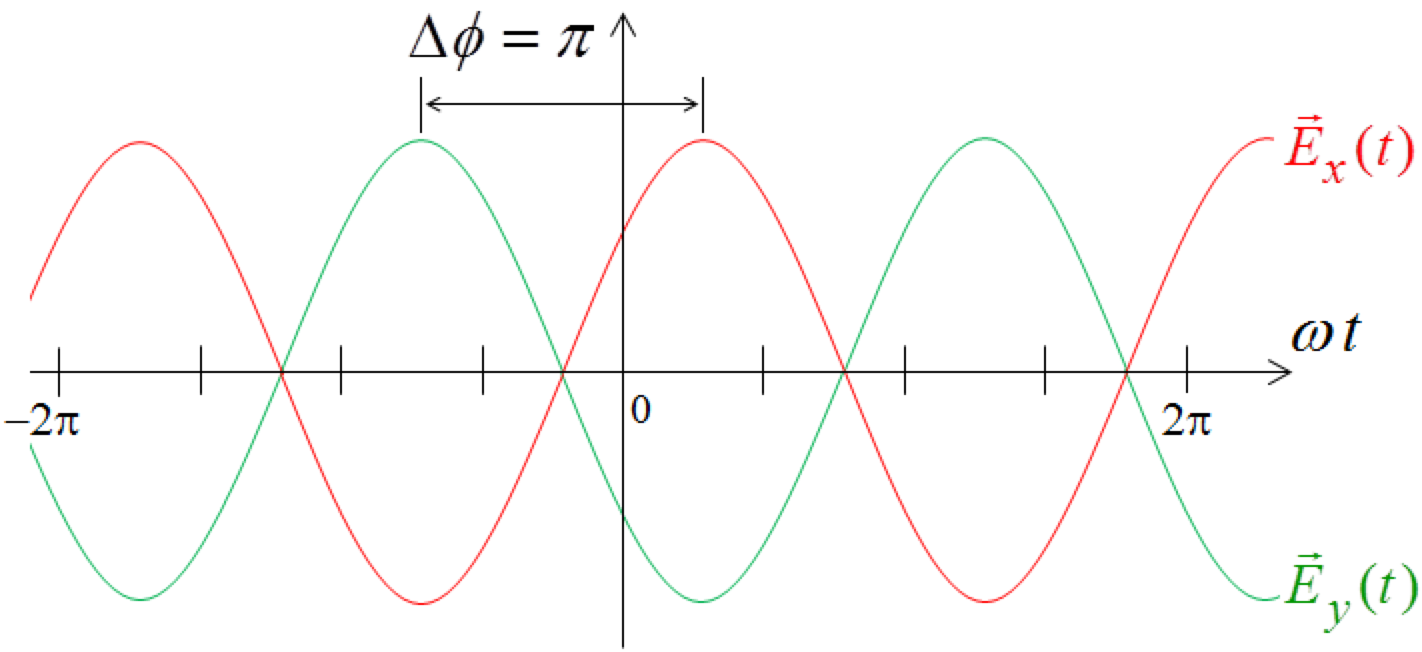
fixed location

Linear Polarization - E_x and E_y in phase with same magnitude



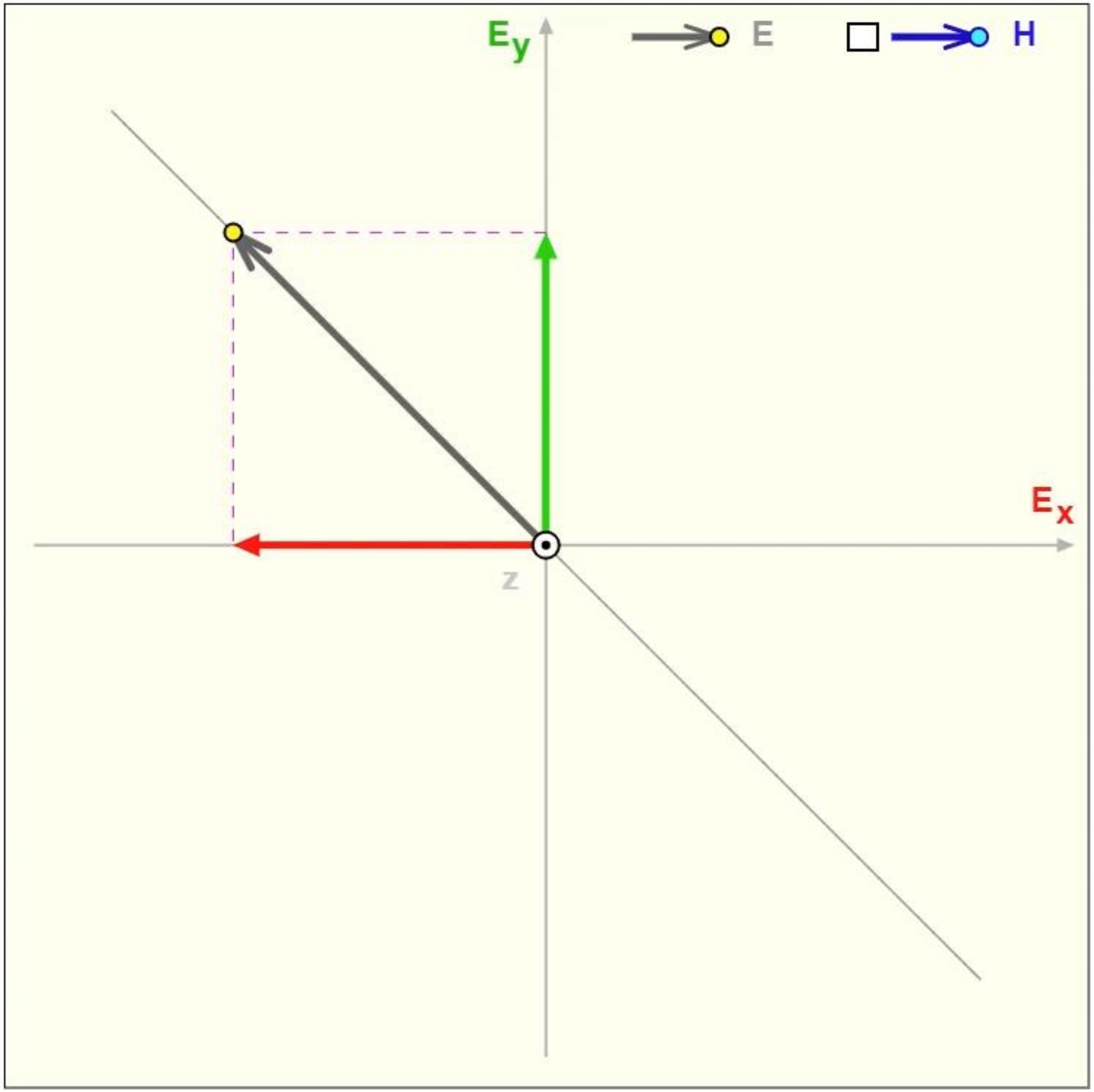


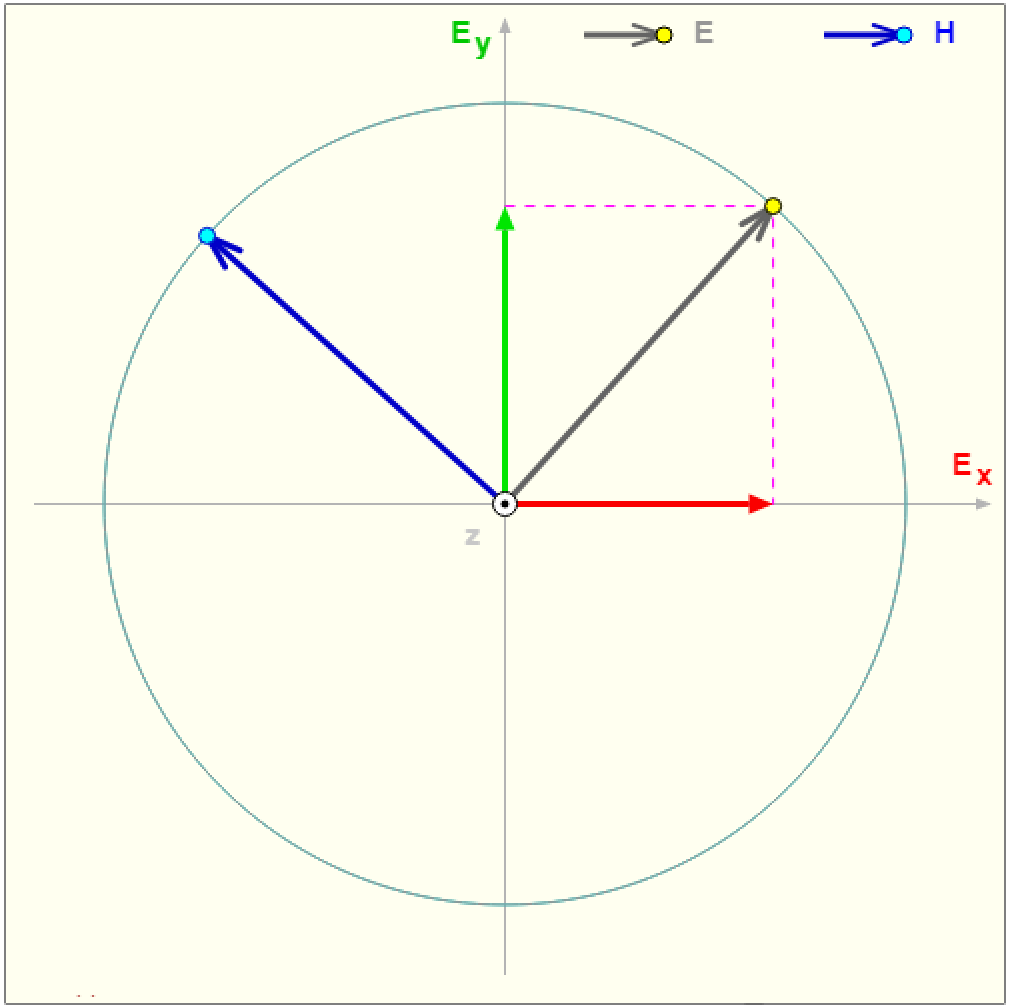
$\vec{E}_x(t)$ ***and*** $\vec{E}_y(t)$ ***are in opposition of phase***



fixed location

Linear Polarization - E_x and E_y in opposition of phase with same magnitude





Circular Polarization

$$E_x = E_y$$

Left Handed

$$\phi_y - \phi_x = \pi/2$$

Right Handed

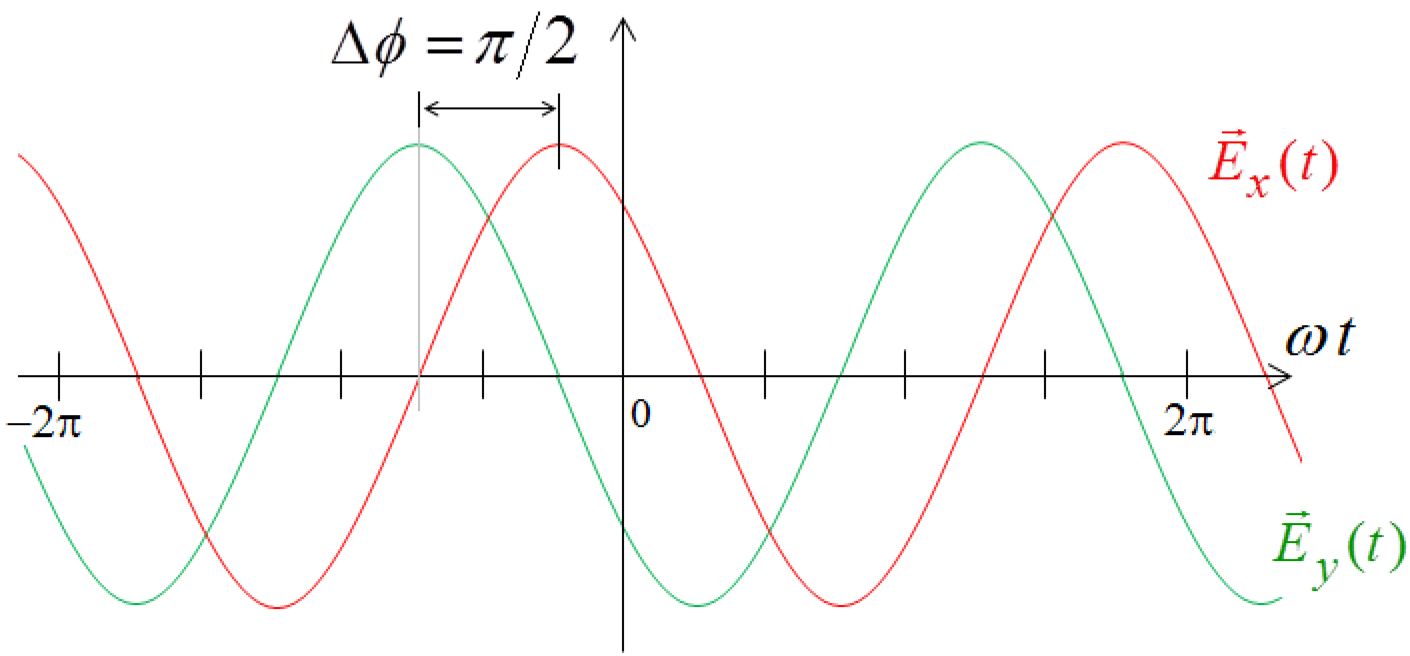
$$\phi_y - \phi_x = -\pi/2$$

Field components are in quadrature.

Circular Polarization is a special case of Elliptical Polarization.

circular
polarization

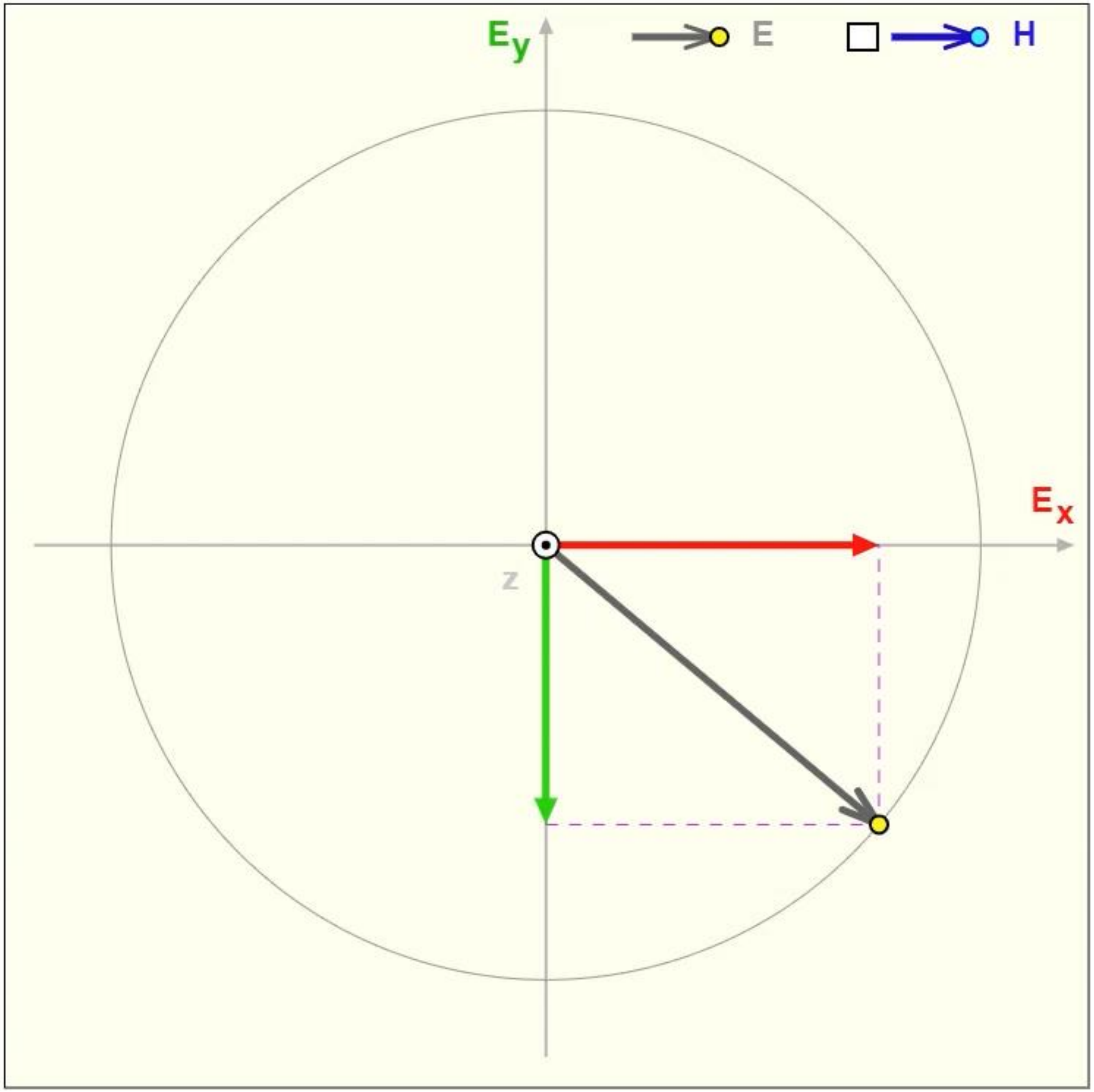
$\vec{E}_x(t)$ ***and*** $\vec{E}_y(t)$ ***are in quadrature***



fixed location

left-handed

Left-Handed Circular Polarization



Example 1 - Find the polarization of

$$\mathbf{E}(z,t) = A\cos(\omega t - kz + \phi_o)\,\hat{x} - A\sin(\omega t - kz + \phi_o)\,\hat{y}$$

$$\mathbf{E}(z,t) = A\cos(\omega t - kz + \phi_o)\,\hat{x} - A\cos(\omega t - kz + \phi_o - \pi/2)\,\hat{y}$$

$$= A\cos(\omega t - kz + \phi_o)\,\hat{x} + A\cos(\omega t - kz + \phi_o + \pi/2)\,\hat{y}$$

$$\boxed{\phi_y - \phi_x = \phi_o + \pi/2 - \phi_o = \pi/2} \qquad \boxed{E_x = E_y}$$

circular left-handed

Note: $|\vec{E}_x|^2 + |\vec{E}_y|^2 = A^2[\underbrace{\cos^2(\omega t - kz + \phi_o) + \sin^2(\omega t - kz + \phi_o)}_{=1}] = A^2$

equation of a circle

Representation with two circular polarizations

An alternative way to synthesize any polarization state is to combine two orthogonal right-handed and left-handed circular polarization fields as a basis function, instead of two linear polarizations, such as for example

right-handed circular

$$(\hat{x} - j\hat{y})e^{-j\beta z}$$

left-handed circular

$$(\hat{x} + j\hat{y})e^{-j\beta z}$$

By assigning variable weights to real and imaginary parts of the two basis functions, it is possible to reproduce any ellipse, including linear polarizations which correspond to very thin ellipses.

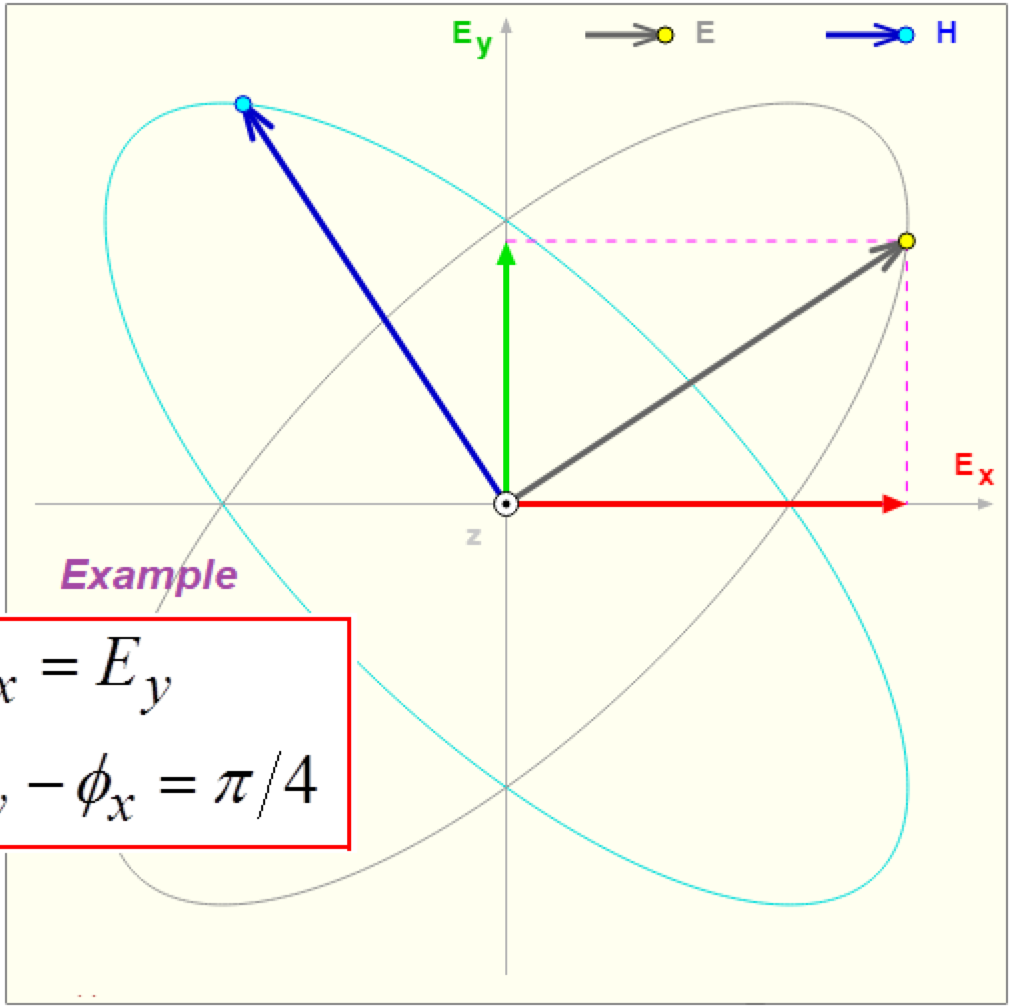
In this formalism one can map the polarization states to a sphere (called the Poincaré Sphere) – This is shown in the EXTRA material that follows.

All other possible cases with arbitrary amplitude and phase generate “elliptical polarization”

Some examples are illustrated in the video animations (posted on the website).

EXTRA MATERIAL
FOR THE INTERESTED STUDENTS

Elliptical Polarization



$$E_x = E_y$$
$$\phi_y - \phi_x = \pi/4$$

$$E_x \neq 0$$
$$E_y \neq 0$$
$$\phi_y - \phi_x \neq \pm n\,\pi$$
$$n = 0, 1$$

$\phi_y - \phi_x > 0$ **Left Handed**

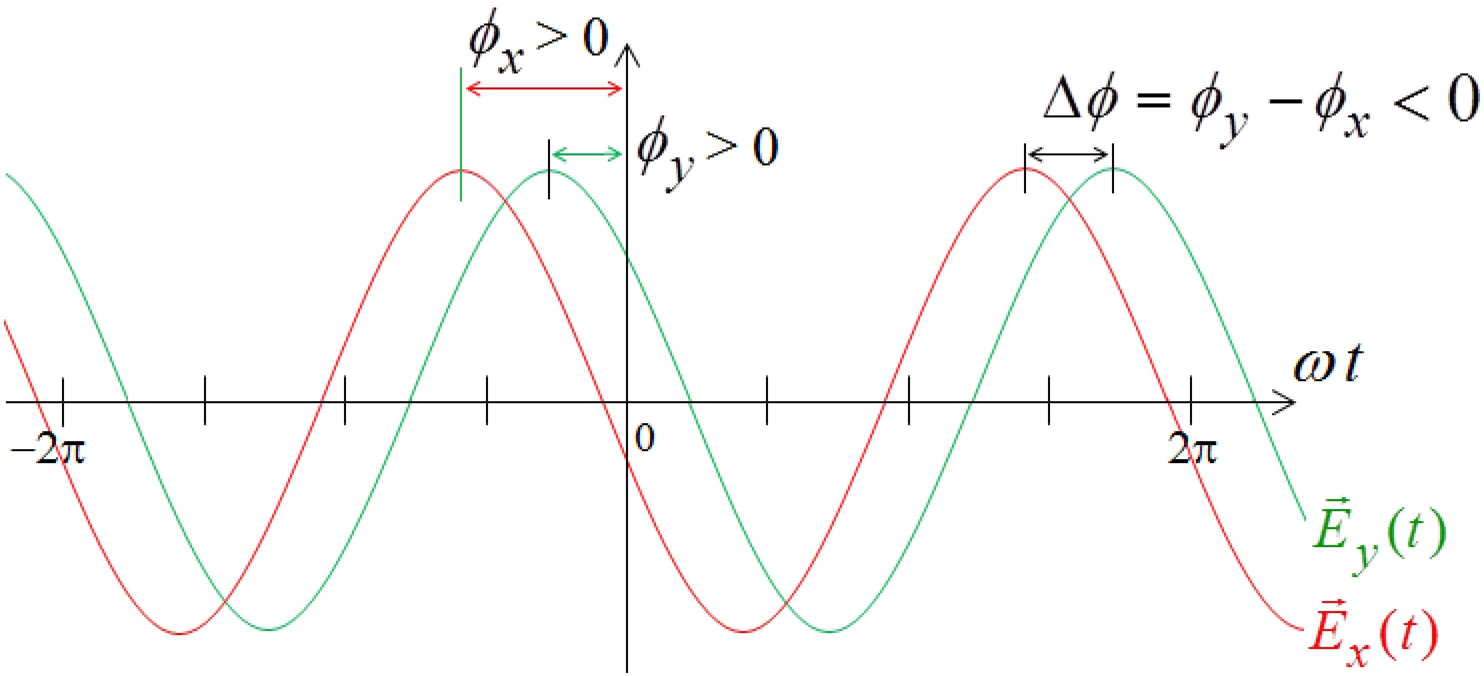
$\phi_y - \phi_x < 0$ **Right Handed**

with

$$\Delta\phi = \phi_y - \phi_x \in [-\pi, \pi]$$

elliptical polarization

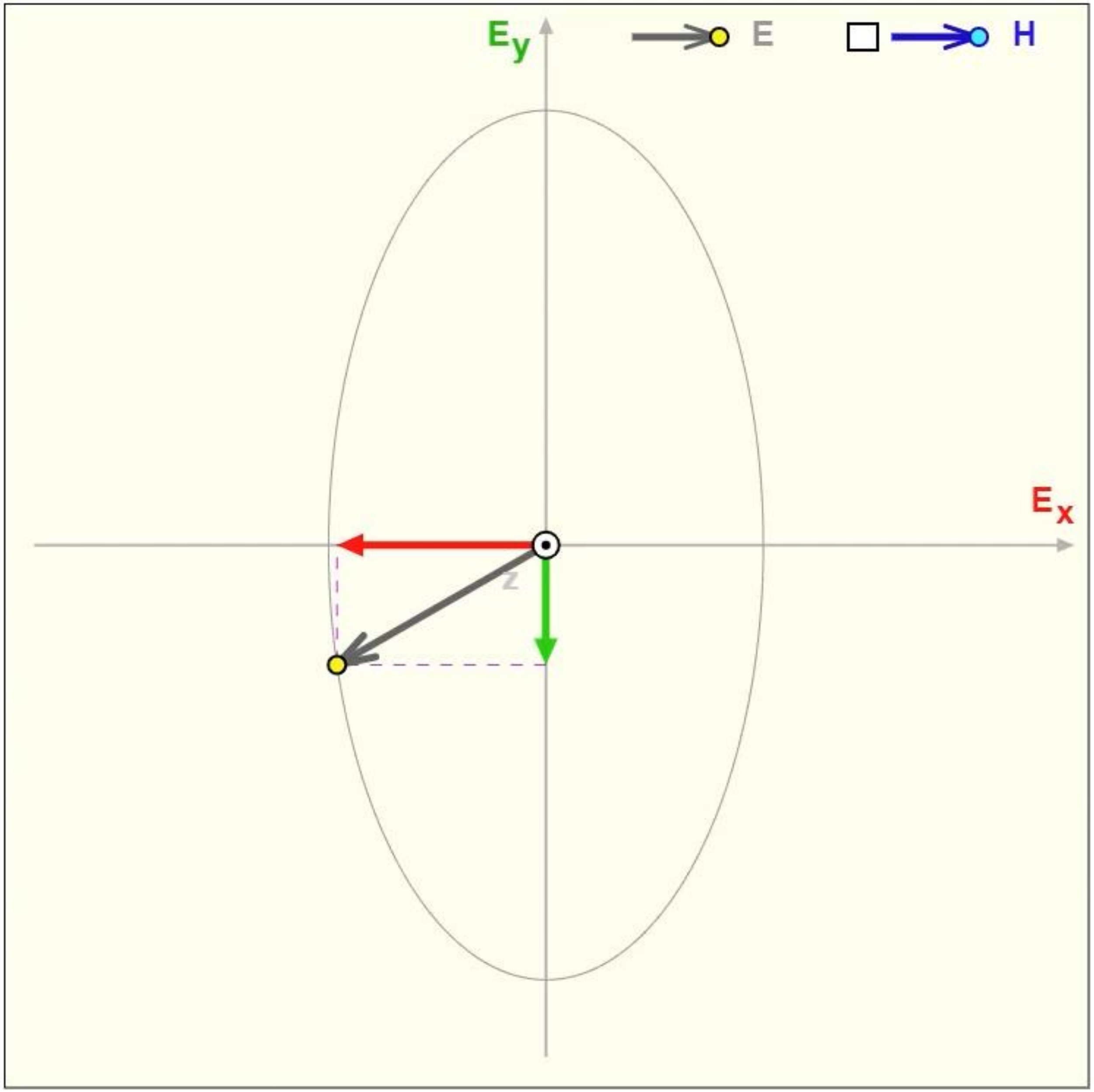
$\vec{E}_x(t)$ ***crests before*** $\vec{E}_y(t)$ $\Rightarrow$ $\vec{E}_x(t)$ ***leads & *** $\vec{E}_y(t)$ ***lags***



fixed location

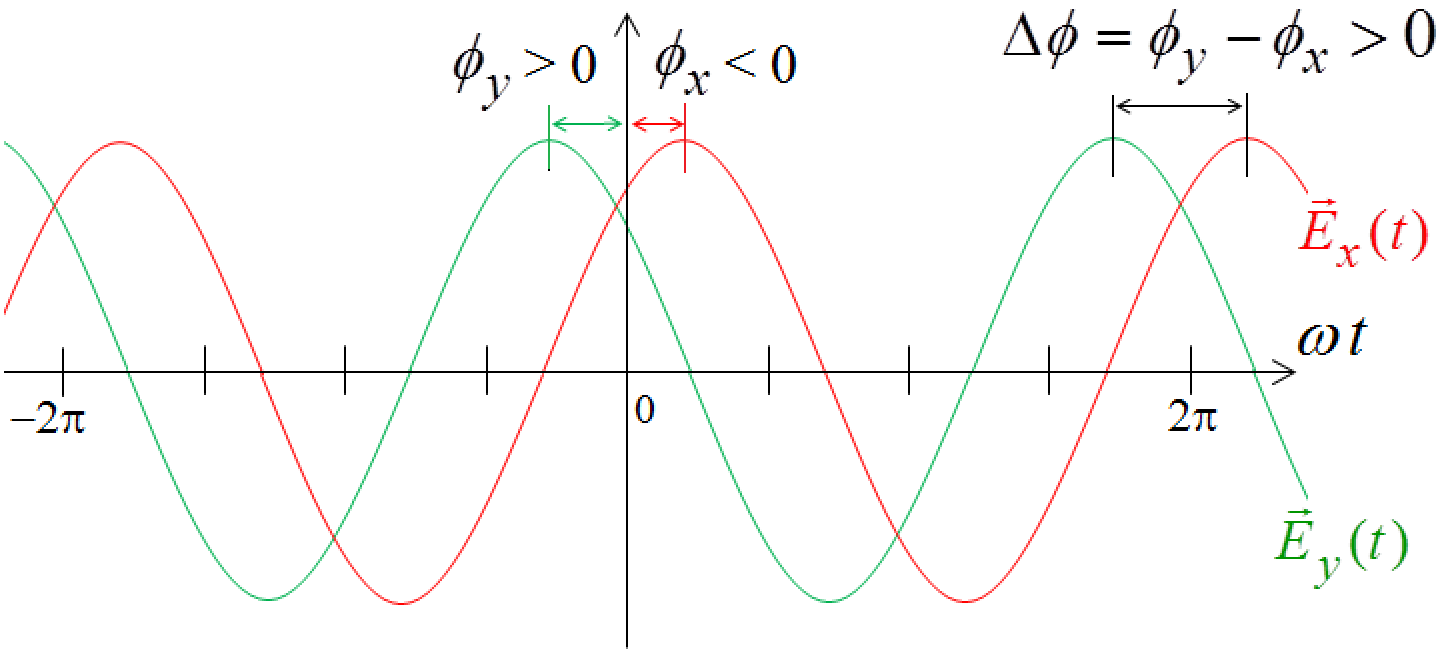
right-handed

Right-Handed Elliptical Polarization in Quadrature and $|E_x| = 0.5|E_y|$



elliptical polarization

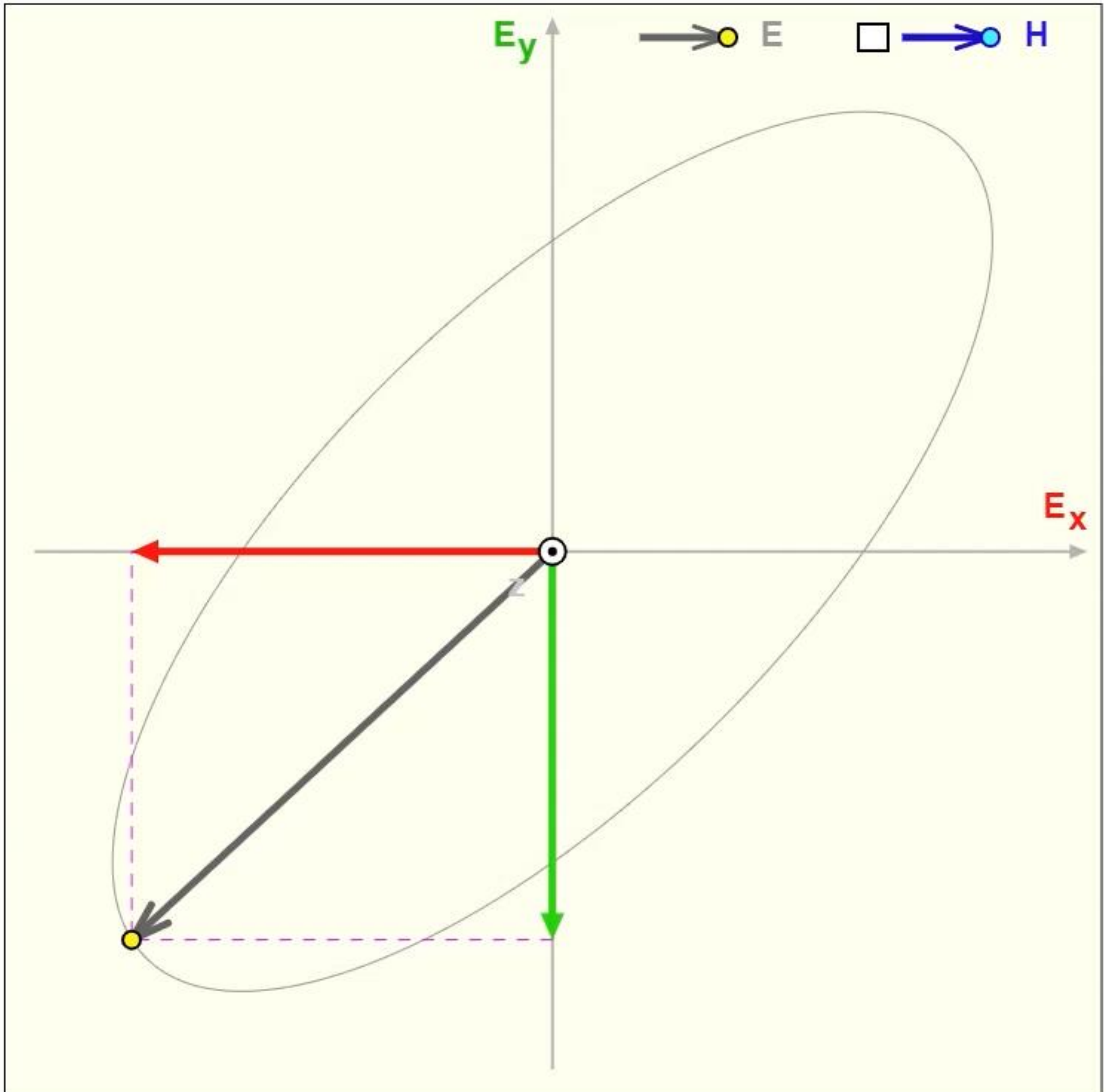
$\vec{E}_y(t)$ ***crests before*** $\vec{E}_x(t)$ $\Rightarrow$ $\vec{E}_y(t)$ ***leads & *** $\vec{E}_x(t)$ ***lags***



fixed location

left-handed

Left-Handed Elliptical Polarization with $\Delta\varphi = 45°$ and $|E_x| = |E_y|$



Example 2 - Find the polarization of

$$\mathbf{E}(z,t) = A\cos(\omega t - kz + \phi_o)\,\hat{x} + 3A\sin(\omega t - kz + \phi_o)\,\hat{y}$$

$$\mathbf{E}(z,t) = A\cos(\omega t - kz + \phi_o)\,\hat{x} + 3A\cos(\omega t - kz + \phi_o - \pi/2)\,\hat{y}$$

$$\phi_y - \phi_x = \phi_o - \pi/2 - \phi_o = -\pi/2$$

$$E_y = 3E_x$$

elliptical right-handed

Note:

$$\left(\frac{|\vec{E}_x|}{A}\right)^2 + \left(\frac{|\vec{E}_y|}{3A}\right)^2 = \cos^2(\omega t - kz + \phi_o) + \sin^2(\omega t - kz + \phi_o) = 1$$

equation of an ellipse

Phasor representation

The phasor of the total polarized field has the form

$$\tilde{\mathbf{E}} = \tilde{\mathrm{E}}_x \hat{x} + \tilde{\mathrm{E}}_y \hat{y} = \left[E_x \hat{x} + E_y \hat{y} \right] e^{-j\beta z}$$

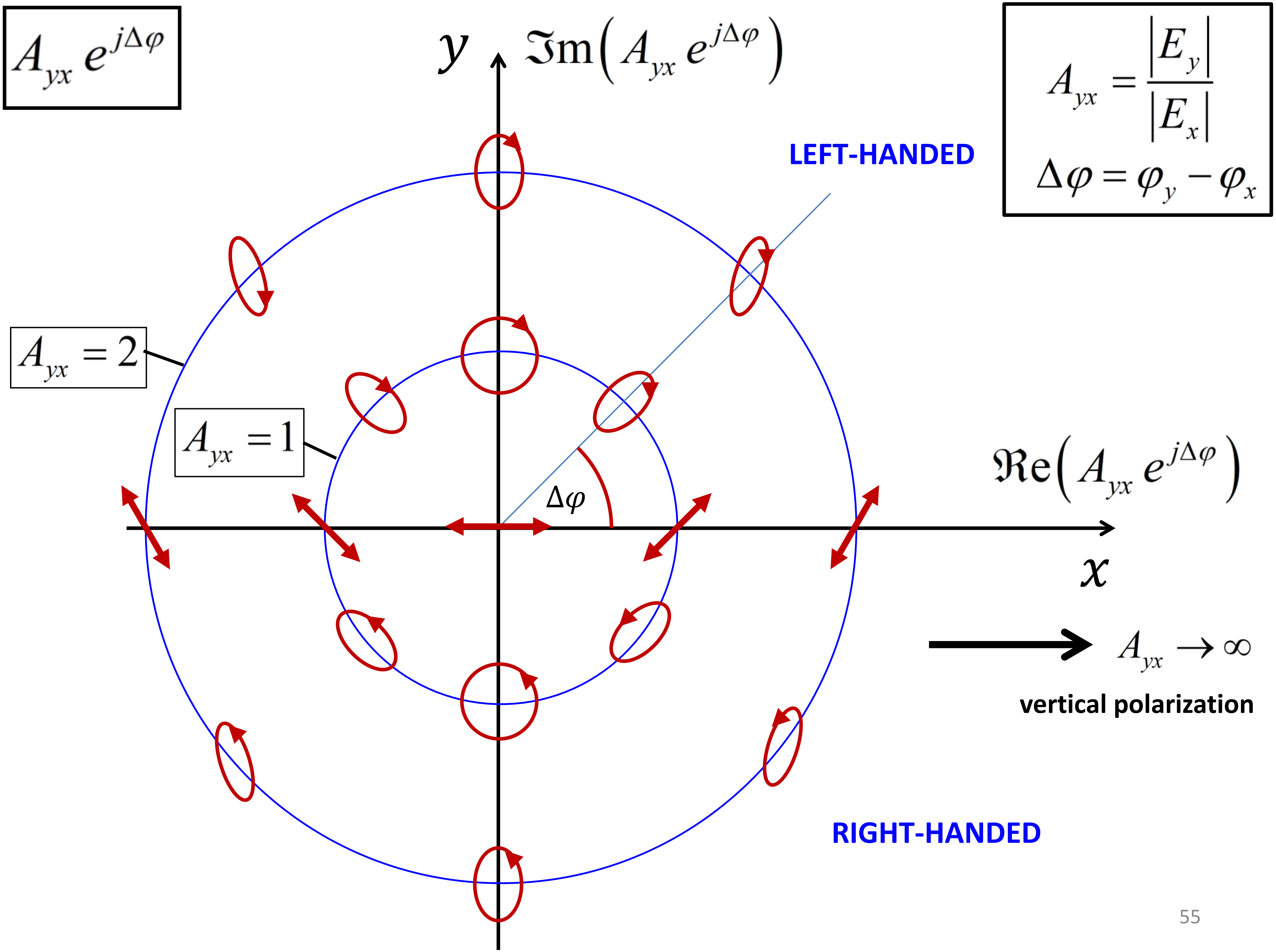
where

$$\boxed{E_x = \left| E_x \right| \, e^{j\varphi_x}} \qquad \boxed{E_y = \left| E_y \right| \, e^{j\varphi_y}}$$

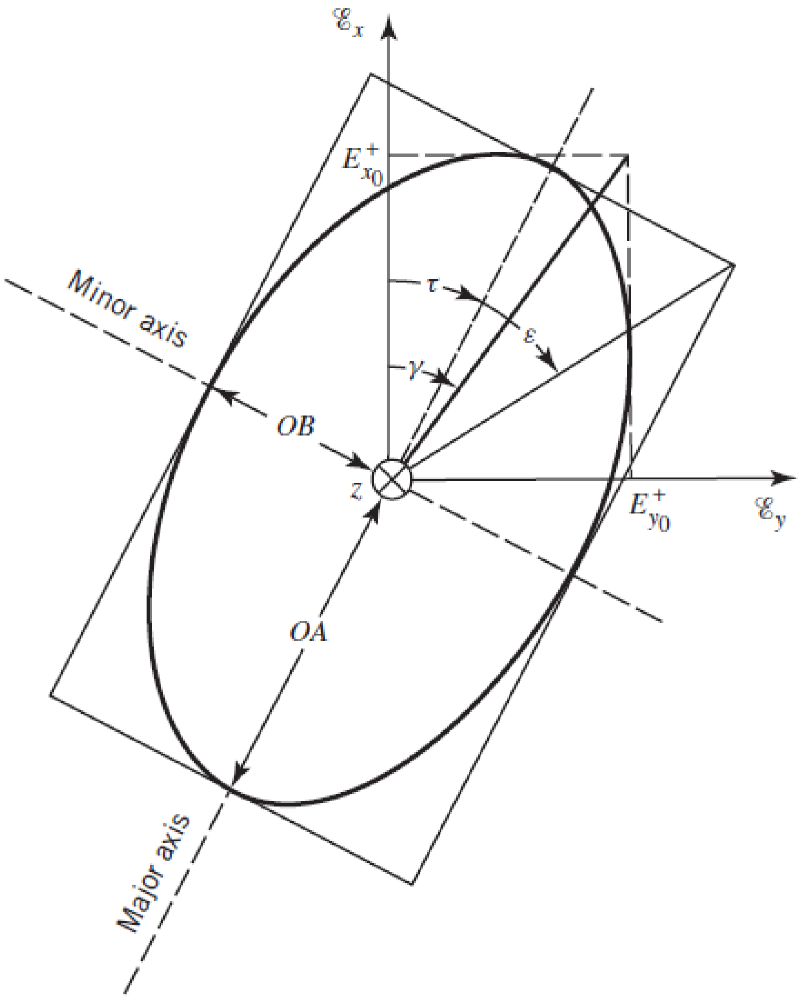
The ratio between the amplitudes of the two components gives

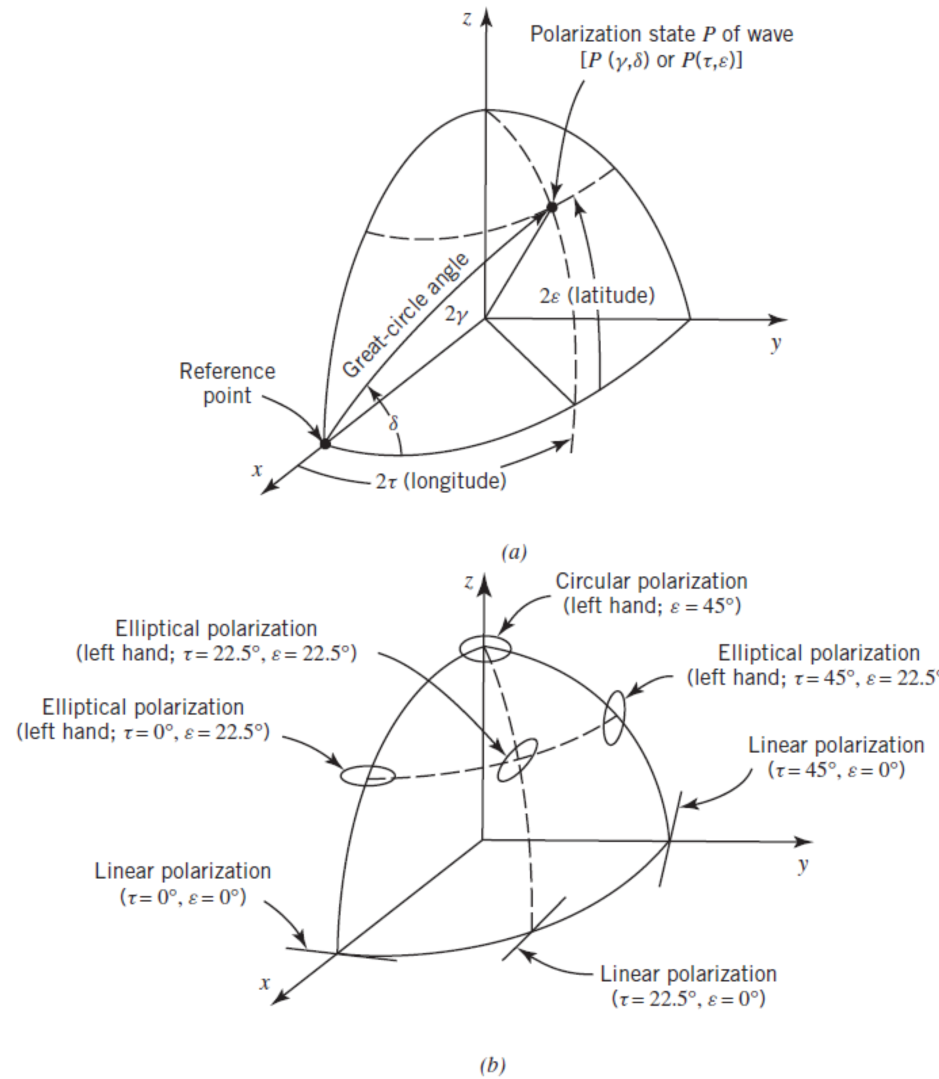
$$\boxed{\frac{E_y}{E_x} = \frac{\left| E_y \right|}{\left| E_x \right|} \, e^{j\left(\varphi_y - \varphi_x\right)} = A_{yx} \, e^{j\Delta\varphi}} \qquad \boxed{\Delta\varphi = \varphi_y - \varphi_x}$$

The complex ratio in polar form provides a simple guideline to classify the polarization state, when mapped to its complex plane.


$A_{yx}\, e^{j\Delta\varphi}$
y
$\Im m\left(A_{yx}\, e^{j\Delta\varphi} \right)$
$A_{yx} = \dfrac{|E_y|}{|E_x|}$
$\Delta\varphi = \varphi_y - \varphi_x$
LEFT-HANDED
$A_{yx} = 2$
$A_{yx} = 1$
$\Delta\varphi$
$\Re e\left(A_{yx}\, e^{j\Delta\varphi} \right)$
x
$A_{yx} \to \infty$
vertical polarization
RIGHT-HANDED

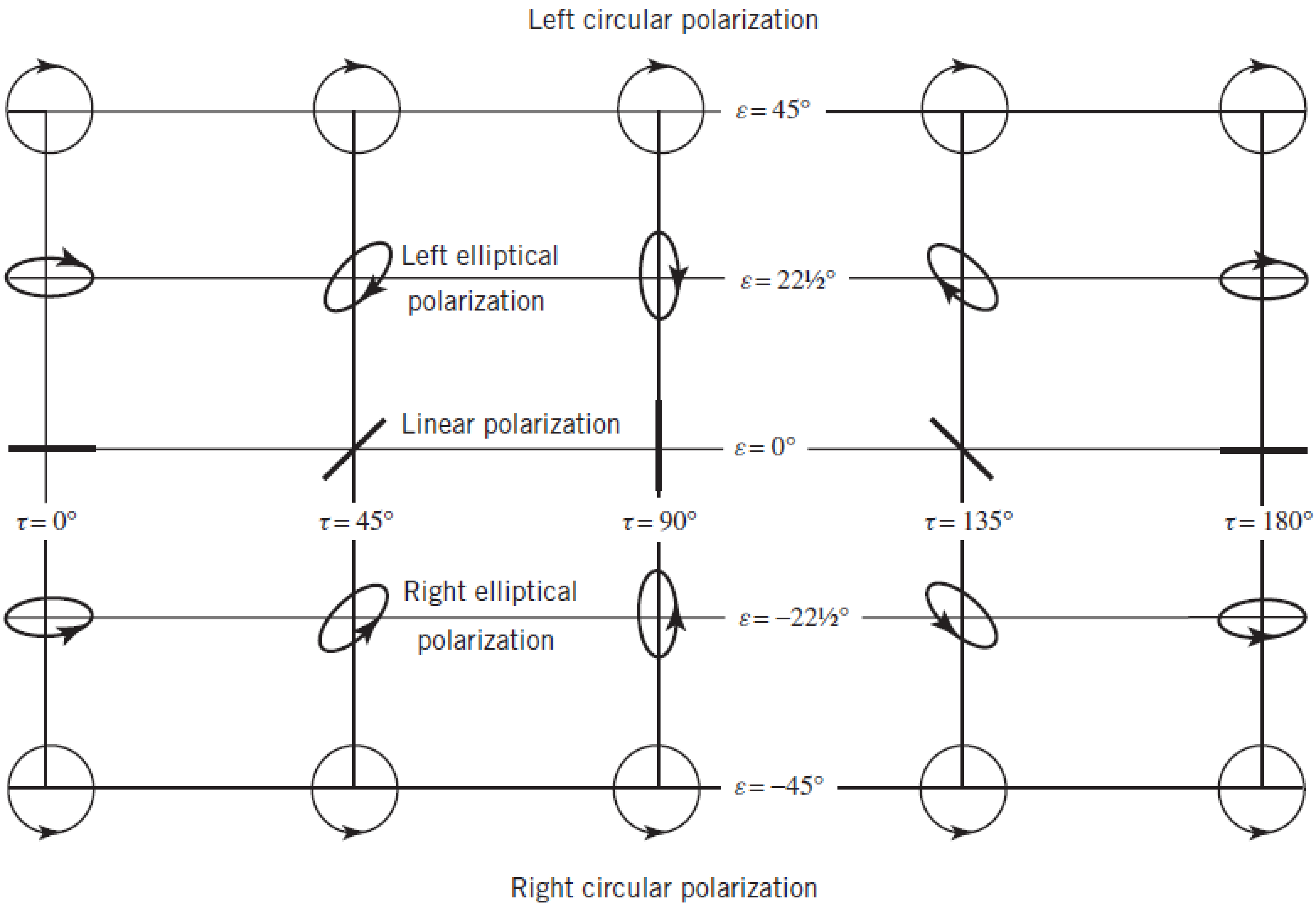
EXTRA Parameterization for Poincaré Sphere





Poincaré sphere for the polarization state of an electromagnetic wave. (Source: J. D. Kraus, *Electromagnetics*, 1984, McGraw-Hill Book Co.). (*a*) Poincaré sphere. (*b*) Polarization state.

EXTRA Projection onto a plane of Poincaré Sphere



Polarization states of electromagnetic waves on a planar surface projection of a Poincaré sphere. (Source: J. D. Kraus, *Electromagnetics*, 1984, McGraw-Hill Book Co.).

Example: Poynting vector of polarized signal – Lossless medium

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\, \widetilde{E}_x(z) + \hat{\mathbf{y}}\, \widetilde{E}_y(z)$$
$$= (\hat{\mathbf{x}}\, E_{x0} + \hat{\mathbf{y}}\, E_{y0}) e^{-j\beta z}$$

$$\widetilde{\mathbf{H}}(z) = (\hat{\mathbf{x}}\, \widetilde{H}_x + \hat{\mathbf{y}}\, \widetilde{H}_y) e^{-j\beta z} = \frac{1}{\eta}\, \hat{\mathbf{z}} \times \widetilde{\mathbf{E}}$$
$$= \frac{1}{\eta}\, (-\hat{\mathbf{x}}\, E_{y0} + \hat{\mathbf{y}}\, E_{x0}) e^{-j\beta z}$$

$$\boxed{<\mathbf{S}> = \frac{1}{2}\, \Re e \left[\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}^* \right]}$$

Getting the Magnetic Field from the Electric Field (propagation along z)

$$\frac{1}{\eta}\,\hat{\mathbf{z}} \times \widetilde{\mathbf{E}} = \frac{1}{\eta}\det\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \mathbf{1} \\ E_{x0}e^{-j\beta z} & E_{y0}e^{-j\beta z} & 0 \end{vmatrix}$$

$$= \frac{1}{\eta}\left(-\hat{\mathbf{x}}\,E_{y0} + \hat{\mathbf{y}}\,E_{x0}\right)e^{-j\beta z}$$

Poynting vector of polarized signal – Lossless medium

$$\langle \mathbf{S} \rangle = \tfrac{1}{2}\,\Re e\left[\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}^*\right]$$

$$\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* = \det \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ E_{x0} e^{-j\beta z} & E_{y0} e^{-j\beta z} & 0 \\ -\dfrac{E_{y0}^*}{\eta} e^{j\beta z} & \dfrac{E_{x0}^*}{\eta} e^{j\beta z} & 0 \end{vmatrix}$$

$$= \hat{\mathbf{z}} \left(\frac{E_{x0} e^{-j\beta z} E_{x0}^* e^{j\beta z}}{\eta} - \frac{E_{y0} e^{-j\beta z} \left(-E_{x0}^* e^{j\beta z} \right)}{\eta} \right)$$

Modulus of the electric field

$$|\widetilde{\mathbf{E}}| = (\widetilde{\mathbf{E}} \cdot \widetilde{\mathbf{E}}^*)^{1/2} = [|E_{x0}|^2 + |E_{y0}|^2]^{1/2}$$

$$\langle \mathbf{S} \rangle = \frac{1}{2} \Re e\left[\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* \right] = \hat{\mathbf{z}} \frac{1}{2\eta} \left(|E_{x0}|^2 + |E_{y0}|^2 \right) = \hat{\mathbf{z}} \frac{|\tilde{\mathbf{E}}|^2}{2\eta} \quad \left(\mathrm{W/m^2} \right)$$

Example: Poynting vector of polarized signal – Lossy medium

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\,\widetilde{E}_x(z) + \hat{\mathbf{y}}\,\widetilde{E}_y(z)$$

$$= (\hat{\mathbf{x}}\,E_{x0} + \hat{\mathbf{y}}\,E_{y0})e^{-\alpha z}e^{-j\beta z}$$

$$\widetilde{\mathbf{H}}(z) = \frac{1}{\eta}(-\hat{\mathbf{x}}\,E_{y0} + \hat{\mathbf{y}}\,E_{x0})e^{-\alpha z}e^{-j\beta z}$$

$$\boxed{<\mathbf{S}> = \frac{1}{2}\,\Re\mathfrak{e}\left[\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}^{*}\right]}$$

Poynting vector of polarized signal – Lossy medium

$$<\mathbf{S}> = \frac{1}{2}\,\Re e\left[\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\right]$$

$$\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* = \det\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ E_{x0}e^{-\alpha z}e^{-j\beta z} & E_{y0}e^{-\alpha z}e^{-j\beta z} & 0 \\ -\dfrac{E_{y0}^*}{\eta^*}e^{-\alpha z}e^{j\beta z} & \dfrac{E_{x0}^*}{\eta^*}e^{-\alpha z}e^{j\beta z} & 0 \end{vmatrix}$$

$$= \hat{\mathbf{z}}\left(\frac{E_{x0}E_{x0}^*e^{-2\alpha z}}{\eta^*} - \frac{E_{y0}\left(-E_{x0}^*\right)e^{-2\alpha z}}{\eta^*}\right)$$

$$\langle \mathbf{S} \rangle = \hat{\mathbf{z}}\frac{e^{-2\alpha z}}{2}\Re e\left\{\frac{\left|E_{x0}\right|^2 + \left|E_{y0}\right|^2}{\eta^*}\right\} = \hat{\mathbf{z}}\frac{e^{-2\alpha z}\,|\,\tilde{\mathbf{E}}(0)\,|^2}{2}\Re e\left\{\frac{e^{j\tau}}{|\,\eta\,|}\right\}$$

$$= \hat{\mathbf{z}}\frac{e^{-2\alpha z}\,|\,\tilde{\mathbf{E}}(0)\,|^2}{2\,|\,\eta\,|}\cos(\tau) \quad \left(\mathrm{W/m^2}\right)$$