ECE 329 – Fall 2022

Prof. Ravaioli – Office: 2062 ECEB
Section E – 1:00pm
Lecture 26
Lecture 26 – Outline

• Final example from lecture 25
• Total Reflection
• Standing Waves
• Surface Current
• Surface Resistance
• Radiation Pressure

Reading assignment
Prof. Kudeki’s ECE 329 Lecture Notes on Fields and Waves:
26) Standing waves, radiation pressure
Exercise

A 50 MHz plane wave with electric field amplitude of 50 V/m is normally incident in air onto a semi-infinite, perfect dielectric medium with \( \varepsilon_r = 36 \). Determine the following:

(a) \( \Gamma \)

(b) The average power densities of the incident and reflected waves.

(c) The distance in the air medium from the boundary to the nearest minimum of the electric field intensity, \(|E|\).

\( (a) \quad \eta_1 = \eta_0 = 120\pi \ (\Omega) \)

\[ \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \frac{120\pi}{\sqrt{\varepsilon_r}} = \frac{120\pi}{6} = 20\pi \ (\Omega) \]

\[ \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{20\pi - 120\pi}{20\pi + 120\pi} = -0.71 \]
Boundary Conditions

medium 1
\( \varepsilon_r 1 = 1.0 \)
\( \mu_r 1 = 1.0 \)

medium 2
\( \varepsilon_r 2 = 36.0 \)
\( \mu_r 2 = 1.0 \)
Boundary Conditions

medium 1
$
\varepsilon_{r1} = 1.0
$
$
\mu_{r1} = 1.0
$

medium 2
$
\varepsilon_{r2} = 36.0
$
$
\mu_{r2} = 1.0
$
(b) 
\[ <S^i> = \frac{|E_0^i|^2}{2 \eta_1} = \frac{(50)^2}{2 \times 120\pi} = 3.32 \text{ (W/m}^2\text{)} \]
\[ <S^r> = |\Gamma|^2 <S^i> = (0.71)^2 \times 3.32 = 1.67 \text{ (W/m}^2\text{)} \]

(c) In medium 1 (air),
\[ \lambda_1 = \frac{c}{f} = \frac{3 \times 10^8}{5 \times 10^7} = 6 \text{ m} \]
\[ l_{\text{max}} = \frac{\angle \Gamma \lambda_1}{4\pi} = \frac{\pi \times 6}{4\pi} = 1.5 \text{ m} \]
\[ l_{\text{min}} = l_{\text{max}} - \frac{\lambda_1}{4} = 1.5 - 1.5 = 0 \text{ m (at the boundary)} \]
If the second medium is a perfect dielectric, the reflection coefficient at the interface is always a real number (because the intrinsic impedances are real).

\[ \epsilon_2 > \epsilon_1 \rightarrow \eta_2 < \eta_1 \]

\[ \Gamma < 0 \quad (\angle \Gamma = \pi) \]

\[ \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \]

\[ \epsilon_2 < \epsilon_1 \rightarrow \eta_2 > \eta_1 \]

\[ \Gamma > 0 \quad (\angle \Gamma = 0) \]
If $\Gamma < 0$: destructive interference at the interface between incoming and reflected wave. The combined wave has a minimum of the partial standing wave at the interface.
If $\Gamma > 0$: constructive interference at the interface between incoming and reflected electric field. The combined wave has a maximum of the partial standing wave at the interface.
(d) What is the maximum amplitude of the total electric field in the air medium and at what nearest distance from the boundary does it occur?

\[ |\vec{E}_1|_{\text{max}} = (1 + |\Gamma|)E_0^i = (1 + 0.71) \times 50 = 85.5 \text{ V/m} \]

\[ l_{\text{max}} = \frac{\angle \Gamma \lambda_1}{4\pi} = \frac{\pi \times 6}{4\pi} = 1.5 \text{ m} \]

(calculated earlier)
Electric Field - Standing Wave Pattern (Normal to Interface)
incident wave

transmitted wave

Interface
Magnetic Field - Standing Wave Pattern (Normal to Interface)
Animation of Electric Field waves in medium 1: Incident, Reflected, Total (see animation video clip)
Repeat (a) and (b) but replace the medium with 
\( \varepsilon_r = 1, \quad \mu_r = 1, \) and \( \sigma = 2.78 \times 10^{-3} \) S/m.

(a) **Medium 2:**

\[
\frac{\sigma_2}{\omega \varepsilon_2} = \frac{2.78 \times 10^{-3} \times 36\pi}{2\pi \times 5 \times 10^7 \times 10^{-9}} = 1
\]

\[
\eta_2 = 120\pi \left(1 - j \frac{\sigma_2}{\omega \varepsilon_2}\right)^{-1/2} = 120\pi (1 - j1)^{-1/2}
\]

\[
= 120\pi (\sqrt{2})^{-1/2} e^{j22.5^\circ} = (292.88 + j121.31)
\]

\[
\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{(292.88 + j121.31) - 377}{(292.88 + j121.31) + 377}
\]

\[
= -0.09 + j0.197 = 0.22 \angle 114.5^\circ
\]
(b) Time average Poynting vector

Incident and Reflected Power

$$< S^i > = \frac{|E_0^i|^2}{2\eta_1} = \frac{50^2}{2 \times 120\pi} = 3.32 \text{ (W/m}^2)$$

$$|< S^r >| = |\Gamma|^2; < S^i > = (0.22)^2 (3.32) = 0.16 \text{ (W/m}^2)$$

$$\langle S^i \rangle \quad | \quad \langle S^t \rangle$$

$$\langle S^r \rangle$$
Electric Field - Standing Wave Pattern (Normal to Interface)
Total Reflection

A perfect conductor boundary on the path of an EM wave acts as a mirror and causes total reflection. With no transmitted wave, the total EM field is represented by

Reflection coefficient at $z = 0$

$$\Gamma = -1$$

Incident wave

$$\tilde{E}_i = \hat{x} E_o e^{-j\beta_1 z}$$

$$\tilde{H}_i = \hat{y} \frac{E_o}{\eta_1} e^{-j\beta_1 z}$$

Reflected wave

$$\tilde{E}_r = -\hat{x} E_o e^{j\beta_1 z}$$

$$\tilde{H}_r = \hat{y} \frac{E_o}{\eta_1} e^{j\beta_1 z}$$

Diagram:

- Incident wave
- Reflected wave
- Perfect conductor

19
Standing Waves

The sum of incident and reflected wave give an interference pattern

\[ \tilde{E} = \tilde{E}_i + \tilde{E}_r = \hat{x} E_0 (e^{-j\beta_1 z} - e^{j\beta_1 z}) \]
\[ \tilde{H} = \tilde{H}_i + \tilde{H}_r = \hat{y} \frac{E_0}{\eta_1} (e^{-j\beta_1 z} + e^{j\beta_1 z}) \]

Using Euler’s identities, the results simplify to

\[ \tilde{E} = -j\hat{x} 2E_0 \sin(\beta_1 z) \]
\[ \tilde{H} = \hat{y} \frac{2E_0}{\eta_1} \cos(\beta_1 z) \]

These represent standing wave with time behavior

\[ E(z, t) = \hat{x} 2E_0 \sin(\beta_1 z) \sin(\omega t) \]
\[ H(z, t) = \hat{y} \frac{2E_0}{\eta_1} \cos(\beta_1 z) \cos(\omega t) \]
Standing Waves

\[ \mathbf{E}(0, t) = 0 \]

Perfect conductor
Standing Waves

\[ H(0, t) = \hat{y} \frac{2E_o}{\eta_1} \cos(\omega t) \]

\[ J_s = \hat{x} \frac{2E_o}{\eta_1} \cos(\omega t) \frac{A}{m} \]

\[-\hat{z} \times H(0, t) = J_s\]

Perfect conductor
The standing wave does not carry energy as can be seen from the Poynting vector

\[
\langle \mathbf{E} \times \mathbf{H} \rangle = \frac{1}{2} \text{Re}\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\}
\]

\[
= \frac{1}{2} \text{Re}\left\{ -j \hat{x} 2E_o \sin(\beta_1 z) \times \hat{y} \frac{2E_o}{\eta_1} \cos(\beta_1 z) \right\}
\]

\[
= \hat{z} \frac{2E_o^2}{\eta_1} \sin(\beta_1 z) \cos(\beta_1 z) \text{Re}\{-j\} = 0
\]

The same power is going in opposite directions and cancels out
Realistic good conductor

For a realistic conductor as medium 2, there is a transmitted EM field into the material

The relevant quantities in phasor form are expressed as

\[\begin{align*}
\vec{E}_t &= \hat{x} \tau E_o e^{-\gamma_2 z} \\
\vec{J}_t &= \sigma_2 \vec{E}_t = \hat{x} \sigma_2 \tau E_o e^{-\gamma_2 z} \\
\vec{H}_t &= \hat{y} \frac{\tau E_o}{\eta_2} e^{-\gamma_2 z} \\
\vec{B}_t &= \mu_2 \vec{H}_t = \hat{y} \frac{\mu_2 \tau E_o}{\eta_2} e^{-\gamma_2 z}
\end{align*}\]
In realistic conductors, there is actually current confined mostly to a layer close to the surface, several skin depths thick. Here we assume vacuum in Region 1.

For a good conductor

\[
\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\epsilon_2}} \approx \sqrt{\frac{j\omega\mu_2}{\sigma_2}}
\]

\[
\gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\epsilon_2)} \approx \sqrt{j\omega\mu_2\sigma_2}
\]

and the transmission coefficient is

\[
\tau = \frac{2\eta_2}{\eta_0 + \eta_2} \approx \frac{2\eta_2}{\eta_0}
\]
Realistic good conductor

The integral of the volumetric current gives the total current

\[ \int_0^\infty \tilde{J}_t dz = \hat{x} \int_0^\infty \sigma_2 \tau E_0 e^{-\gamma_2 z} dz \]

\[ = \hat{x} E_0 \frac{1}{\gamma_2} (\sigma_2 \tau) = \hat{x} E_0 \frac{2\gamma_2 \sigma_2}{\eta_0 \sqrt{j\omega \mu_2 \sigma_2}} \]

\[ = \hat{x} E_0 \frac{2\sigma_2}{\eta_0 \sqrt{j\omega \mu_2 \sigma_2}} \sqrt{\frac{j\omega \mu_2}{\sigma_2}} = \hat{x} \frac{2E_0}{\eta_0} \]

E field at interface on metal side

In the time-domain

\[ \mathbf{J}_s = \hat{x} \frac{2E_0}{\eta_0} \cos(\omega t) \frac{A}{m} \]
Realistic good conductor

Also note that

$$\gamma = \sqrt{\pi f \mu \sigma} (1 + j) = \frac{(1 + j)}{\delta}$$

$$\int_0^\infty \tilde{J}_t \, dz = \hat{x} E_o \frac{1}{\gamma_2} (\sigma_2 \tau) = \hat{x} \sigma_2 E_o \tau \frac{\delta}{(1 + j)}$$

Current density at the surface

This resembles a uniform equivalent current density flowing through a layer as thick as the skin depth.
Current in a conductor

\[ E_x(t) \]

\[ J_x(z,t) \]

\[ \delta_s = 0.1592 \lambda = 5.03292 \, [\, \mu m \,] \]

actual current distribution

\[ h = 1.0 \lambda \]
Current in a conductor

equivalent uniform current model flowing only in the “skin” layer
Consider the effective surface current of a good conductor

\[ \tilde{J}_s = \int_{z=0}^{\infty} \tilde{J}(z) \, dz = \int_{z=0}^{\infty} \tilde{J}(0) e^{-\gamma z} \, dz = \frac{\tilde{J}(0)}{\gamma} \]

where

\[ \gamma \approx \sqrt{j \omega \mu \sigma} = \alpha + j \beta = \alpha + j \alpha \]

We can express current density and electric field inside the conductor in terms of the effective surface current

\[ \tilde{J}(z) = \tilde{J}_s \gamma e^{-\gamma z} \]

\[ E(z) = \frac{\tilde{J}_s \gamma}{\sigma} e^{-\gamma z} \]
Surface Resistance

\[ \tilde{J}(z) = \tilde{J}_s \gamma e^{-\gamma z} \quad \tilde{E}(z) = \frac{\tilde{J}_s \gamma}{\sigma} e^{-\gamma z} \]

**Joule heating**

\[
\langle \mathbf{J}(z) \cdot \mathbf{E}(z) \rangle = \frac{1}{2} |\tilde{J}_s \gamma|^2 \frac{e^{-2\alpha z}}{\sigma} = \frac{1}{2} |\tilde{J}_s|^2 \frac{2\alpha^2 e^{-2\alpha z}}{\sigma}
\]

**Power dissipated per unit area**

\[
\int_0^\infty \langle \mathbf{J}(z) \cdot \mathbf{E}(z) \rangle \, dz = \frac{1}{2} R_s |\tilde{J}_s|^2
\]

with surface resistance

\[ R_s \equiv \frac{\alpha}{\sigma} = \sqrt{\frac{\pi f \mu \sigma}{\sigma}} = \sqrt{\frac{\pi f \mu}{\sigma}} \quad (\Omega) \]

(The resistance of a planar slab with square area)
Radiation Pressure

Consider \( N \) free charge carriers per unit volume inside the mirror. The force per unit volume on the mirror due to Lorentz force is

\[ NF = N q v \times B_t = J_t \times B_t \]

The integral over \( z \) is the total force per unit area of the mirror

\[ P_{rad} = \int_{0}^{\infty} J_t \times B_t \, dz \]

The magnitude of this quantity is defined as the radiation pressure of the reflecting wave. It is in time-dependent form, so let’s take the time-average
Radiation Pressure

Time-average of the radiation pressure

\[
\langle P_{rad} \rangle = \int_0^\infty \frac{1}{2} \text{Re}\{\mathbf{J} \times \mathbf{B}^*\} dz
\]

\[
= \hat{z} \int_0^\infty \frac{1}{2} \text{Re}\{(\sigma_2 \tau E_o)(\frac{\mu_2 \tau E_o}{\eta_2})\} e^{-2\alpha_2 z} dz
\]

\[
= \hat{z} \frac{|E_o|^2}{2} \text{Re}\{(\frac{2\gamma_2}{\eta_o})(\frac{\mu_2 2\eta_2}{\eta_o \eta_2})\} \frac{1}{2\alpha_2} = \hat{z} 2 \frac{|E_o|^2}{2\eta_o} \frac{\text{Re}\{\gamma_2\} \mu_2}{\alpha_2 \eta_o}
\]

\[
= \hat{z} 2 \frac{|E_o|^2 \mu_o}{2\eta_o \eta_o} = 2 \hat{z} \frac{|E_o|^2}{2\eta_o} \sqrt{\mu_o \epsilon_o} = 2 \langle S_i \rangle / c
\]

This factor accounts for wave recoil off the mirror

incident time-average Poynting vector

\[
\frac{1}{c}
\]
Radiation Pressure

The effect of radiation pressure is normally too weak for humans to feel it. For instance, the solar radiation power measured on Earth can be about 1.4 kW/m². The time average radiation pressure is

\[
\langle P_{rad} \rangle = \frac{1.4 \times 10^3}{3 \times 10^8} = 4.7 \times 10^{-6} \text{ N/m}^2
\]

We have assumed that the body absorbs the radiation and that reflection is negligible (no factor of 2). A person sunbathing on the beach with total cross-sectional area of 0.6 m² will feel a force of

\[
F = 4.7 \times 10^{-6} \times 0.6 = 2.82 \times 10^{-6} \text{ N}
\]

If the same person weighs 70 kg, the gravitational force would be approximately 686 newtons.
Radiation Pressure

However, radiation pressure could have practical applications in space. A thin polymer film coated with aluminum could be deployed to realize a “space sailboat”. Because it is very reflective, the factor of 2 can be included in the calculation.

For a sailboat with mass $m_1 = 10^3$ kg at a distance $r$ from the Sun (with mass $m_2 = 1.99 \times 10^{30}$ kg ) the gravitational force is

$$F_g = 6.67 \times 10^{-11} \left( \frac{m_1 m_2}{r^2} \right)$$

Estimate of solar radiation power density from previous example

$$\langle S_i \rangle = 1.4 \times 10^3 \frac{r_e^2}{r^2} \quad r_e = 1.5 \times 10^{11} \text{m}$$

Sun-Earth average distance

Radiation force on sail

$$F_r = 2 \times 1.4 \times 10^3 (\text{Area}/c) \left( \frac{r_e^2}{r^2} \right)$$

Gravitational pull is balanced by radiation pressure on the sail when

$$\text{Area} \approx 0.63 \times 10^6 \text{m}^2$$
A 200 MHz, left-hand circularly polarized plane wave with an electric field modulus of 5 V/m is normally incident in air on a dielectric medium with $\epsilon_r = 4$ in the region $z \geq 0$. (electric field is maximum at $z = 0$ and $t = 0$)

Calculate the reflection and transmission coefficients.

Determine the percentages of the incident average power reflected by the boundary and transmitted into the second medium.
Left-handed circular polarization

We can write the phasor of the electric field as

\[ \vec{E}^i = a_0 (\hat{x} + \hat{y} e^{j \pi/2}) e^{-j \beta z} = a_0 (\hat{x} + j \hat{y}) e^{-j \beta z} \]

In the time domain

\[ E^i(z, t) = \hat{x} a_0 \cos(\omega t - \beta z) - \hat{y} a_0 \sin(\omega t - \beta z), \]

\[ |E^i| = [a_0^2 \cos^2(\omega t - \beta z) + a_0^2 \sin^2(\omega t - \beta z)]^{1/2} \]

\[ \rightarrow a_0 = 5 \text{ (V/m)} \]

This is the radius of the circle described by the electric field.
Now get the wavenumbers

$$\beta_1 = \frac{\omega}{c} = \frac{2\pi \times 2 \times 10^8}{3 \times 10^8} = \frac{4\pi}{3} \text{ rad/m,}$$

$$\beta_2 = \frac{\omega}{u_{p2}} = \frac{\omega}{c} \sqrt{\varepsilon_{r2}} = \frac{4\pi}{3} \sqrt{4} = \frac{8\pi}{3} \text{ rad/m}$$

We can write the incident field as

$$\tilde{E}^i = 5(\hat{x} + j\hat{y}) e^{-j\frac{4\pi}{3}z} \text{ (V/m)}$$
Intrinsic impedances

\[ \eta_1 = \eta_0 = 120\pi \quad (\Omega) \]

\[ \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{\eta_0}{2} = 60\pi \quad (\Omega) \]

Reflection and Transmission coefficients

\[ \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{60\pi - 120\pi}{60\pi + 120\pi} = \frac{-60}{180} = -\frac{1}{3} \]

\[ \tau = 1 + \Gamma = 1 + \left(-\frac{1}{3}\right) = \frac{2}{3} \]

Also:

\[ \tau = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{2 \times 60\pi}{180\pi} = \frac{2}{3} \]
Electric Fields

\[ \vec{E}^i = 5(\hat{x} + j\hat{y})e^{-j4\pi z / 3} \quad (\text{V/m}) \]

\[ \vec{E}^r = 5\Gamma(\hat{x} + j\hat{y})e^{j4\pi z / 3} = -\frac{5}{3}(\hat{x} + j\hat{y})e^{j4\pi z / 3} \quad (\text{V/m}) \]

\[ \vec{E}^t = 5\tau(\hat{x} + j\hat{y})e^{-j8\pi z / 3} = \frac{10}{3}(\hat{x} + j\hat{y})e^{-j8\pi z / 3} \quad (\text{V/m}) \]

Total Field in medium 1

\[ \vec{E}_1 = \vec{E}^i + \vec{E}^r = 5(\hat{x} + j\hat{y}) \left[ e^{-j4\pi z / 3} - \frac{1}{3}e^{j4\pi z / 3} \right] \quad (\text{V/m}) \]
Power

\[
\langle S^i \rangle = \frac{|E_0^i|^2}{2\eta_1} \quad \langle S^r \rangle = \frac{|E_0^r|^2}{2\eta_1} \quad \langle S^t \rangle = \frac{|E_0^t|^2}{2\eta_2}
\]

incident \quad \text{reflected} \quad \text{transmitted}

For practice, calculate the time-average power densities above and verify that total power is conserved.

\[
\text{Reflectivity} = \frac{\langle S^r \rangle}{\langle S^i \rangle} = \frac{|E_0^r|^2}{2\eta_1} \cdot \frac{2\eta_1}{|E_0^i|^2} = |\Gamma|^2
\]

\[
\text{Transmissivity} = \frac{\langle S^t \rangle}{\langle S^i \rangle} = \frac{|E_0^t|^2}{2\eta_2} \cdot \frac{2\eta_1}{|E_0^i|^2} = |\tau|^2 \frac{\eta_1}{\eta_2}
\]
Power

% of reflected power = \( 100 \times |\Gamma|^2 = \frac{100}{9} = 11.11\% \)

Reflectivity

% of transmitted power =

\[
= 100 \times |\tau|^2 \frac{\eta_1}{\eta_2} = 100 \times \left(\frac{2}{3}\right)^2 \times \frac{120\pi}{60\pi} = 88.89\%
\]

Transmissivity