ECE 329 – Fall 2022

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Lecture 27

Lecture 27 – Outline

- **Guiding of EM waves**
- **Transmission lines as guides of TEM waves**
- **Parallel Plate structure**
- **Examples of transmission lines**
- **Fundamental equations**
- **Characteristic impedance**

Reading assignment
Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:
27) Guided TEM waves on TL systems

Last time we considered good conductors and we obtained the effective surface current

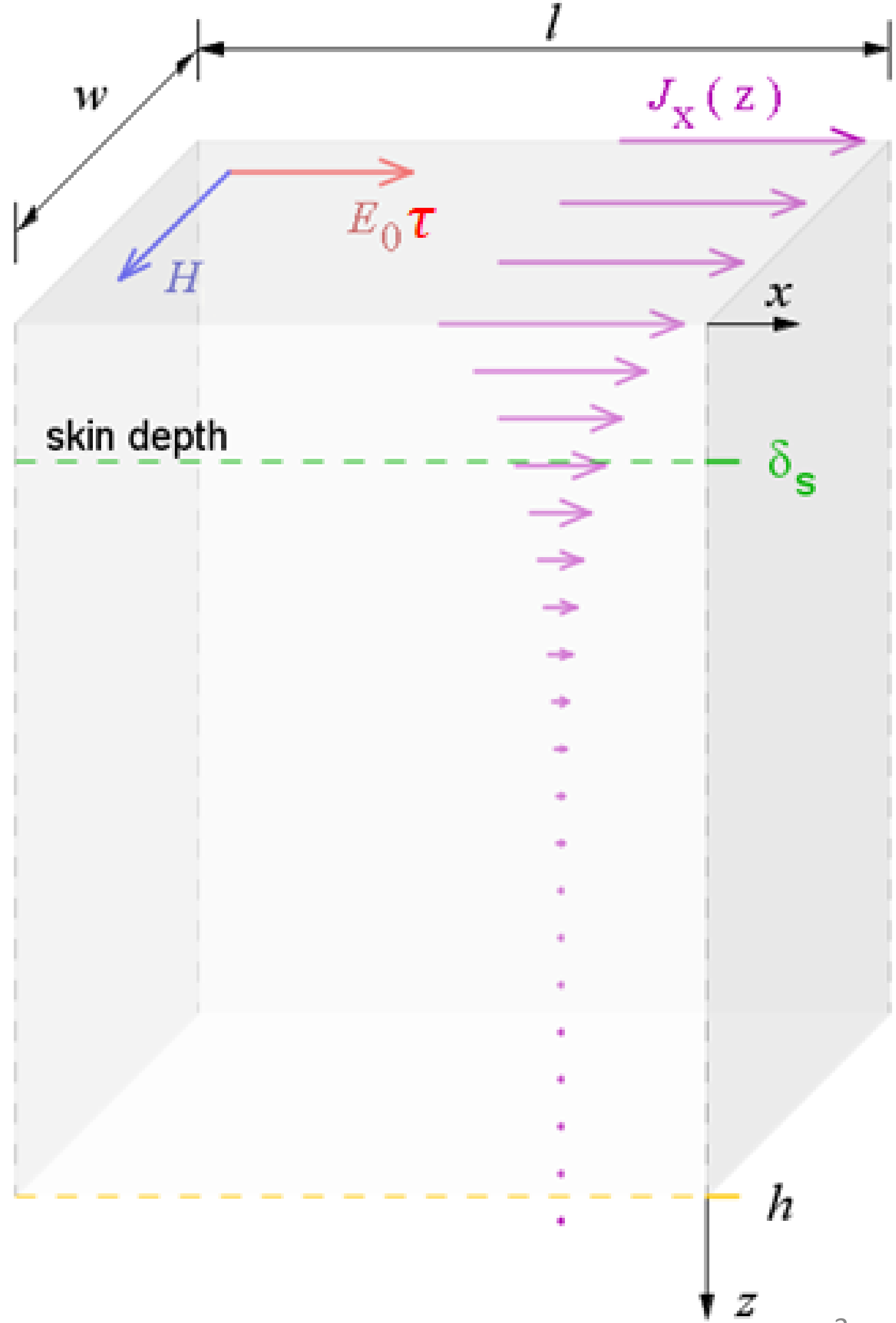
$$\int_0^\infty \tilde{\mathbf{J}}_t dz = \hat{x}\frac{2E_o}{\eta_o}$$

$$= \hat{x}\,\frac{\mathbf{J}_o}{(1+j)}\,\delta\,\frac{\mathrm{A}}{\mathrm{m}}$$

where

$$\tilde{\mathbf{J}}(0) = \hat{x}\,\mathbf{J}_o$$

$$\gamma \approx \alpha + j\alpha = \frac{\delta}{(1+j)}$$



Surface Resistance

Consider the effective surface current of a good conductor

$$\tilde{\mathbf{J}}_s = \int_{z=0}^{\infty} \tilde{\mathbf{J}}(z)dz = \int_{z=0}^{\infty} \tilde{\mathbf{J}}(0)e^{-\gamma z}dz = \frac{\tilde{\mathbf{J}}(0)}{\gamma}$$

where

$$\gamma \approx \sqrt{j\omega\mu\sigma} = \alpha + j\beta = \alpha + j\alpha$$

We can express current density and electric field inside the conductor in terms of the effective surface current

$$\boxed{\tilde{\mathbf{J}}(0) = \tilde{\mathbf{J}}_s\gamma}$$

$$\tilde{\mathbf{J}}(z) = \tilde{\mathbf{J}}_s\gamma e^{-\gamma z}$$

$$\mathbf{E}(z) = \frac{\tilde{\mathbf{J}}_s\gamma}{\sigma}e^{-\gamma z}$$

Surface Resistance

$$\tilde{\mathbf{J}}(z) = \tilde{\mathbf{J}}_s \gamma e^{-\gamma z} \qquad \mathbf{E}(z) = \frac{\tilde{\mathbf{J}}_s \gamma}{\sigma} e^{-\gamma z}$$

Joule heating

$$\langle \mathbf{J}(z) \cdot \mathbf{E}(z) \rangle = \frac{1}{2} |\tilde{\mathbf{J}}_s \gamma|^2 \frac{e^{-2\alpha z}}{\sigma} = \frac{1}{2} |\tilde{\mathbf{J}}_s|^2 \frac{2\alpha^2 e^{-2\alpha z}}{\sigma}$$

Note: $|\gamma|^2 = 2\alpha^2$

Power dissipated per unit area

$$\int_0^\infty \langle \mathbf{J}(z) \cdot \mathbf{E}(z) \rangle dz = \frac{1}{2} R_s |\tilde{\mathbf{J}}_s|^2$$

$$R_s \equiv \frac{\alpha}{\sigma} = \frac{\sqrt{\pi f \mu \sigma}}{\sigma} = \sqrt{\frac{\pi f \mu}{\sigma}} \quad (\Omega)$$

(The resistance of a planar slab with square area)

Radiation Pressure

Consider N free charge carriers per unit volume inside the mirror. The force per unit volume on the mirror due to Lorentz force is

$$N\mathbf{F} = \underbrace{Nq\mathbf{v}}_{\boxed{\text{current density}}} \times \mathbf{B}_t = \mathbf{J}_t \times \mathbf{B}_t$$

The integral over z is the total force per unit area of the mirror

$$\mathbf{P}_{rad} = \int_0^\infty \mathbf{J}_t \times \mathbf{B}_t \, dz$$

The magnitude of this quantity is defined as the radiation pressure of the reflecting wave. It is in time-dependent form, so let's take the time-average

Radiation Pressure

Time-average of the radiation pressure

$$
\begin{aligned}
\langle \mathbf{P}_{rad} \rangle &= \int_0^\infty \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{J}} \times \tilde{\mathbf{B}}^*\} dz \\
&= \hat{z} \int_0^\infty \frac{1}{2}\mathrm{Re}\{(\sigma_2 \tau E_o)(\frac{\mu_2 \tau E_o}{\eta_2})\} e^{-2\alpha_2 z} dz \\
&= \hat{z}\frac{|E_o|^2}{2}\mathrm{Re}\{(\frac{2\gamma_2}{\eta_o})(\frac{\mu_2}{\eta_2}\frac{2\eta_2}{\eta_o})\}\frac{1}{2\alpha_2} = \hat{z}2\frac{|E_o|^2}{2\eta_o}\frac{\mathrm{Re}\{\gamma_2\}}{\alpha_2}\frac{\mu_2}{\eta_o} \\
&= \hat{z}2\frac{|E_o|^2}{2\eta_o}\frac{\mu_o}{\eta_o} = 2\ \hat{z}\underbrace{\frac{|E_o|^2}{2\eta_o}}_{\text{incident time-average Poynting vector}}\ \underbrace{\sqrt{\mu_o \epsilon_o}}_{\frac{1}{c}} = 2\langle \mathbf{S}_i \rangle / c
\end{aligned}
$$

This factor accounts for wave recoil off the mirror

Radiation Pressure

The effect of radiation pressure is normally too weak for humans to feel it. For instance, the solar radiation power measured on Earth can be about 1.4 kW/m^2. The time average radiation pressure is

$$\langle P_{rad}\rangle = \frac{1.4\times 10^{3}}{3\times 10\hat{}8} = 4.7\times 10^{-6}\ \ \frac{\mathrm{N}}{\mathrm{m}^2}$$

We have assumed that the body absorbs the radiation and that reflection is negligible (no factor of 2). A person sunbathing on the beach with total cross-sectional area of 0.6 m^2 will feel a force of

$$F = 4.7\times 10^{-6}\times 0.6 = 2.82\times 10^{-6}\ \mathrm{N}$$

If the same person weighs 70 kg, the gravitational force would be approximately 686 newtons.

Radiation Pressure

However, radiation pressure could have practical applications in space. A thin polymer film coated with aluminum could be deployed to realize a "space sailboat". Because it is very reflective, the factor of 2 can be included in the calculation.

For a sailboat with mass $m_1 = 10^3$ kg at a distance r from the Sun (with mass $m_2 = 1.99 \times 10^{30}$ kg) the gravitational force is

$$F_g = 6.67 \times 10^{-11}(m_1 m2\ / r^2)$$

Estimate of solar radiation power density from previous example

$$\langle S_i \rangle = 1.4 \times 10^3\ r_e^2 / r^2 \qquad r_e = 1.5 \times 10^{11}\mathrm{m}$$

Sun-Earth average distance

Radiation force on sail

$$F_r = 2 \times 1.4 \times 10^3 (Area/c)(r_e^2/r^2)$$

Gravitational pull is balanced by radiation pressure on the sail when

$$Area \approx 0.63 \times 10^6 \mathrm{m}^2$$

Guiding EM waves for a purpose

We have learned so far that electromagnetic waves carry energy. They can also carry information if we encode it in the wave amplitude, frequency or phase.

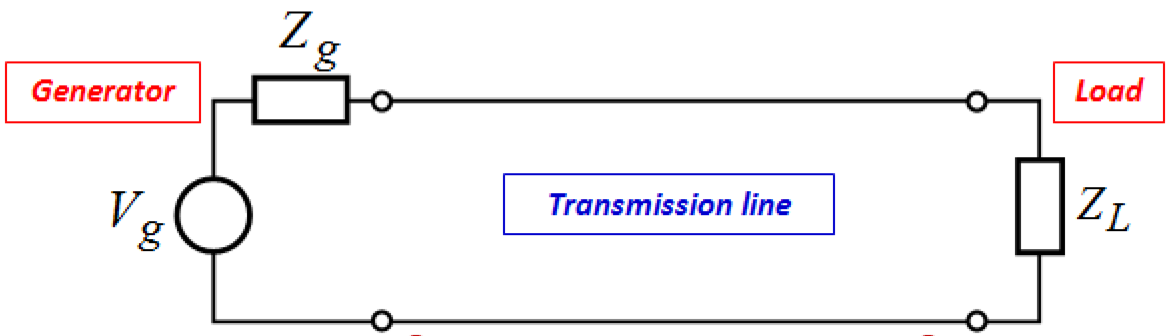
On a large scale, EM waves carry signals between a transmitting antenna and one or many receiving antennas (e.g., radio and TV broadcasting, cellular telephones, communication satellites).

In many situations, however, we need to carry information using EM waves from a "generator" to one or many receiving "loads" through electrical cables, optical fibers, circuit boards.

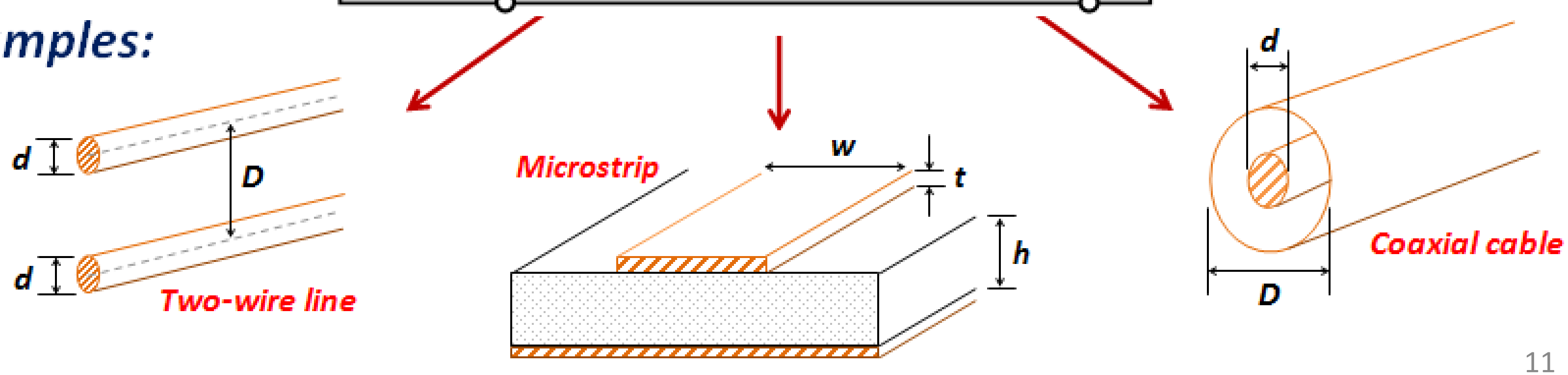
Whenever a "signal" is sent electrically or optically we are using EM waves (even when we think that voltages and currents are doing the job).

Transmission Lines

Transmission lines are the circuit components which support propagation of a signal from a generator to a load. At sufficiently high frequencies, when the wavelength is comparable to the length of the conductors forming the transmission line, propagation of the signal must be treated with wave theory.

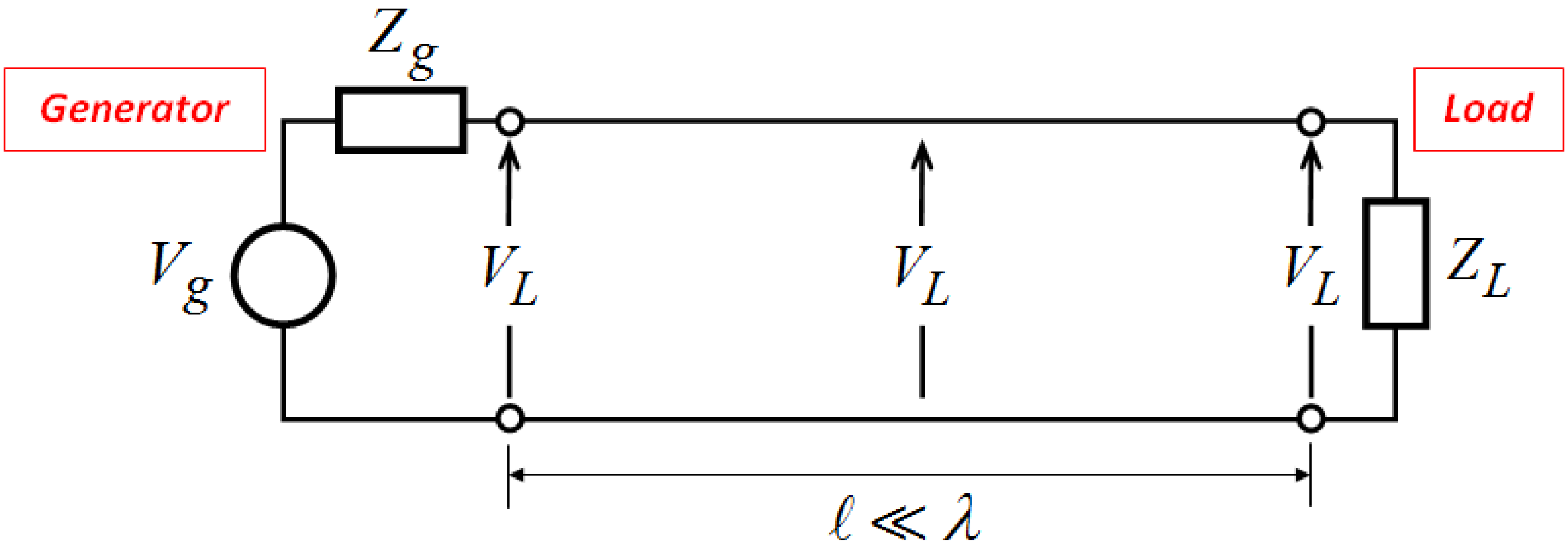


Examples:



It all depends on the frequency

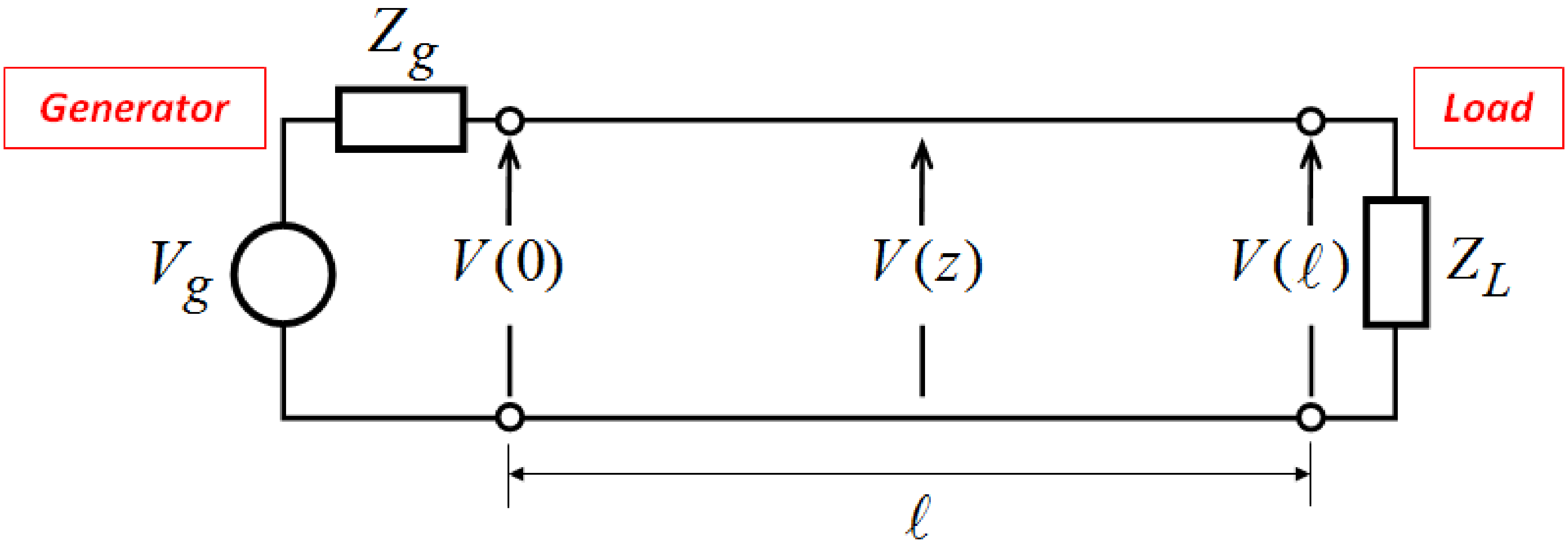
At low frequency the line elements may be approximated as equipotential conductors. This is reasonable as long as the length of the wires is much smaller than the wavelength of the signal.



$$V_L = V_g \frac{Z_L}{Z_g + Z_L}$$

It all depends on the frequency

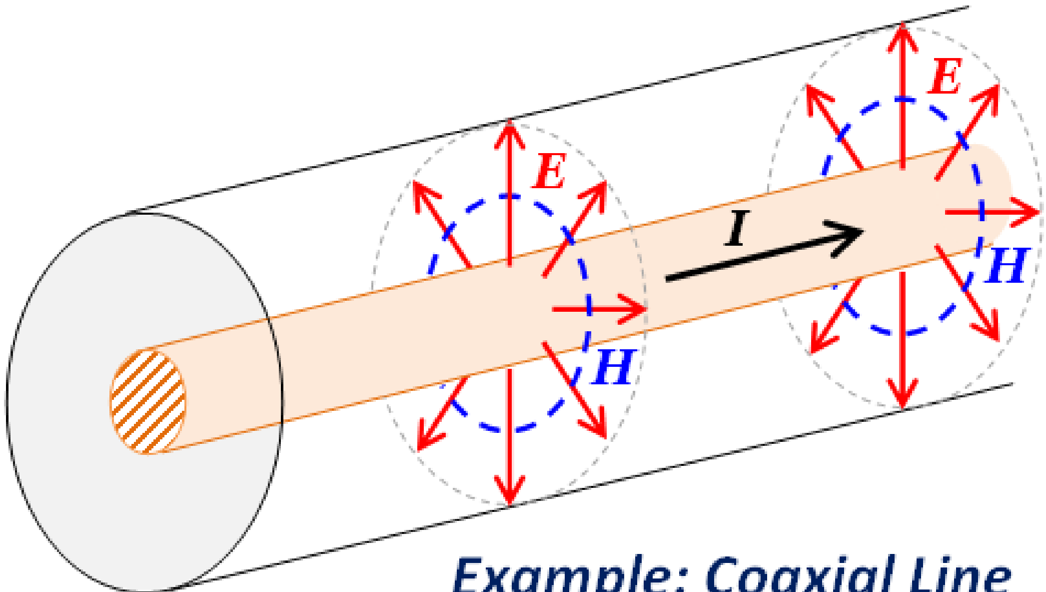
At higher frequencies the wavelength is comparable with the transmission line length. The signal cannot change instantaneously at all locations and it propagates as a wave of voltage and current.



$$V(z) = V^{+} e^{-j\beta z} + V^{-} e^{j\beta z}$$

Fundamental assumption of transmission line theory –

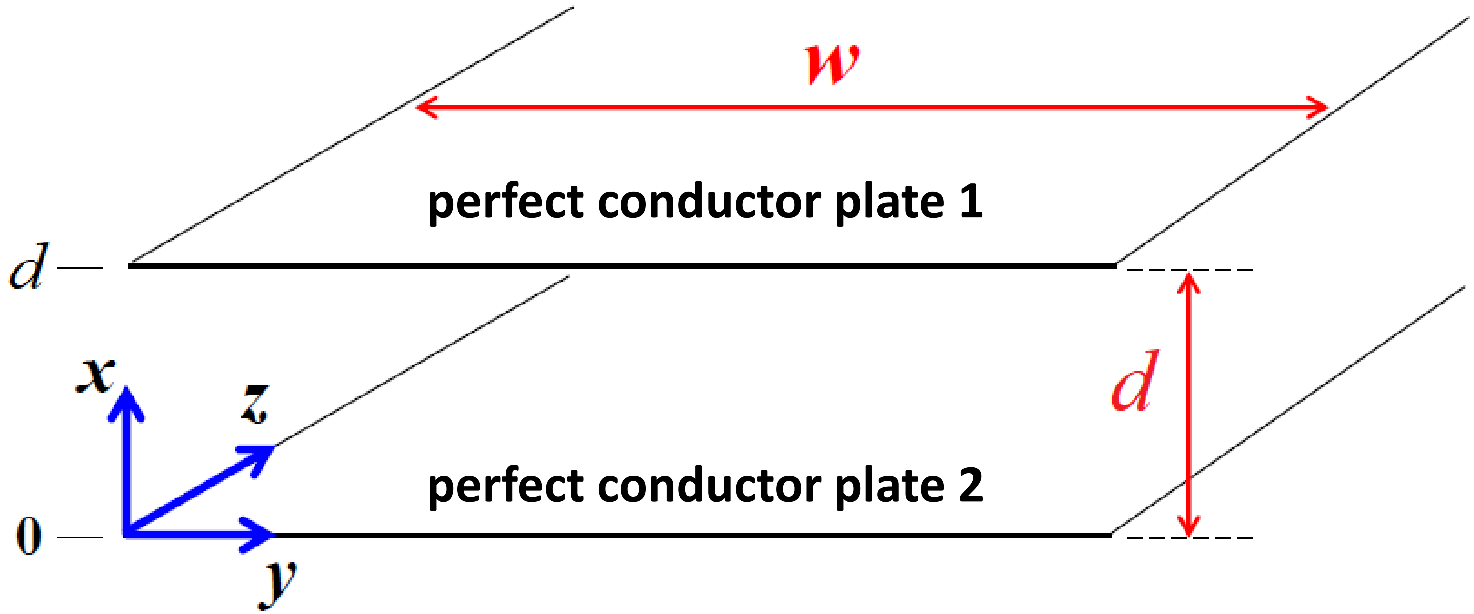
The electromagnetic field vectors always lie on the cross-section normal to the direction of propagation. This is the Transverse Electro-Magnetic (TEM) configuration which exists at any frequency.



Example: Coaxial Line

At sufficiently high frequency the TEM assumption is no longer valid because other field configurations (modes) appear and waveguide theory should be used.

The parallel plate transmission line (wave guide)



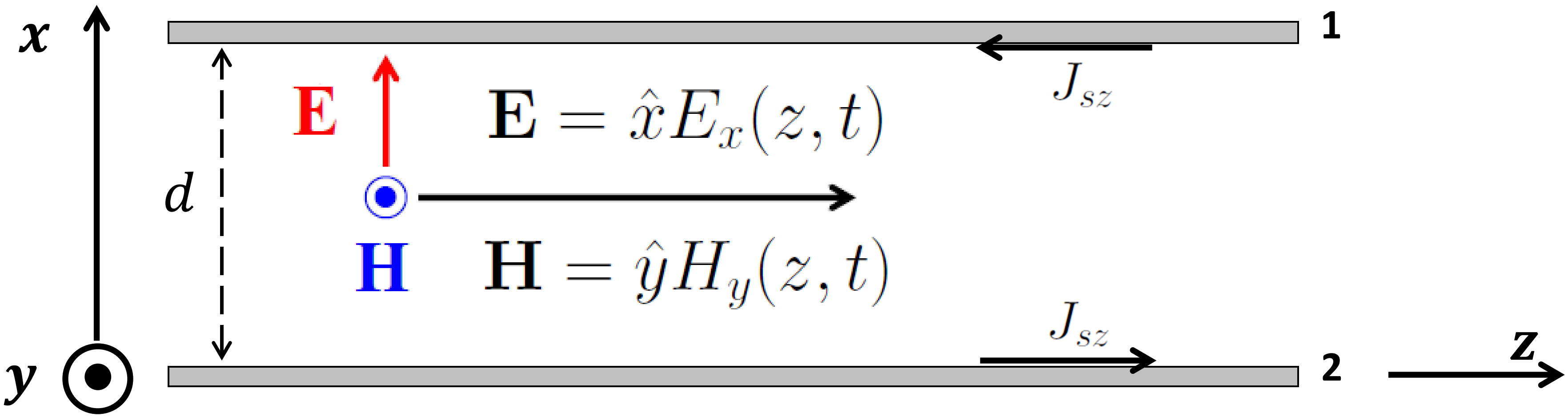
Assume uniform waves along the y-direction $\Rightarrow \frac{\partial}{\partial y}(\ ) = 0$

Assume no fringing effects $\Rightarrow w >> d$

Propagation along the z-direction

In these condition we can assume TEM wave fields

TEM propagation



Obtain scalar forms (1D model)

Faraday's Law

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \Rightarrow \quad \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$

Ampere's Law

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad \Rightarrow \quad -\frac{\partial H_y}{\partial z} = \sigma E_x + \epsilon \frac{\partial E_x}{\partial t}$$

conductivity of dielectric between plates

Transform directly into circuit equations

- **Multiply by d and W both equations**
- **Define the voltage from plate 2 to 1**

$$V = E_x d$$

- **Define the current on plate 2**

$$I = J_{sz} W = H_y W$$

- **Obtain**

$$W \frac{\partial V}{\partial z} = -\mu d \frac{\partial I}{\partial t}$$

$$-d \frac{\partial I}{\partial z} = \epsilon W \frac{\partial V}{\partial t} + \sigma W V$$

$$W\frac{\partial V}{\partial z} = -\mu d\frac{\partial I}{\partial t}$$

$$\boxed{-\frac{\partial V}{\partial z} = \mathcal{L}\frac{\partial I}{\partial t}}$$

$$-d\frac{\partial I}{\partial z} = \epsilon W\frac{\partial V}{\partial t} + \sigma W V$$

$$\boxed{-\frac{\partial I}{\partial z} = \mathcal{C}\frac{\partial V}{\partial t} + \mathcal{G}V}$$

Circuit parameters for the parallel plate structure

$$\mathcal{L} = \mu\frac{d}{W}$$

inductance per unit length

$$\mathcal{C} = \epsilon\frac{W}{d}$$

capacitance per unit length

$$\mathcal{G} = \sigma\frac{W}{d}$$

conductance per unit length

Telegrapher's equations

$$-\frac{\partial V}{\partial z} = \mathcal{L}\frac{\partial I}{\partial t} \qquad -\frac{\partial I}{\partial z} = \mathcal{C}\frac{\partial V}{\partial t} + \mathcal{G}V$$

For good conductor plates with resistivity ρ there would be additional loss term in the first equation

$\mathcal{R}I$ **where** $\mathcal{R}$ = resistance per unit length

In the ideal case of perfect dielectric and perfect conductor we can neglect $\mathcal{R}$ and $\mathcal{G}$

$$-\frac{\partial V}{\partial z} = \mathcal{L}\frac{\partial I}{\partial t} \qquad -\frac{\partial I}{\partial z} = \mathcal{C}\frac{\partial V}{\partial t}$$

Telegrapher's equations for the lossless transmission line

Geometrical factor $\mathbf{GF}$

For the parallel plates line we have a geometrical factor

$$\mathbf{GF} = \frac{W}{d}$$

so that

$$\mathcal{L} = \mu \frac{d}{W} = \frac{\mu}{\mathbf{GF}} \qquad \mathcal{C} = \epsilon \frac{W}{d} = \epsilon \mathbf{GF}$$

These are general expressions. Different structures will have a different geometric factor

$$\mathbf{GF} = \frac{2\pi}{\ln \frac{b}{a}} \qquad \mathbf{GF} = \frac{\pi}{\cosh^{-1} \frac{D}{2a}}$$

coaxial cable　　　two-wire (twin-lead) line

Telephonist's wave equations

Wave equations for voltage and current can be obtained in complete analogy to electric and magnetic field

$$\frac{\partial^2 V}{\partial z^2} = \mathcal{L}C\frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 I}{\partial z^2} = \mathcal{L}C\frac{\partial^2 I}{\partial t^2}$$

These are now equivalent decoupled equations. We only need one of the two to solve a problem.

d'Alembert wave solutions

$$\frac{\partial^2 V}{\partial z^2} = \mathcal{L}\mathcal{C}\frac{\partial^2 V}{\partial t^2}$$

This equation has solutions of the form

$$V(z,t) = f(t \mp \frac{z}{v})$$

with

$$\mathcal{L}\,\mathcal{C} = \frac{\mu}{\mathbf{GF}}\epsilon\mathbf{GF} = \mu\,\epsilon$$

$$\rightarrow \quad v \equiv \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} = \frac{1}{\sqrt{\mu\epsilon}}$$

d'Alembert wave solutions

From the second telegrapher's equation

$$-\frac{\partial I}{\partial z} = \mathcal{C}\frac{\partial V}{\partial t} = \mathcal{C}\frac{\partial}{\partial t} f\left(t \mp \frac{z}{v}\right)$$

$$\partial I = -\mathcal{C}\frac{\partial z}{\partial t} \quad \partial f\left(t \mp \frac{z}{v}\right)$$

$$I(z,t) = \pm \mathcal{C} v\, f\left(t \mp \frac{z}{v}\right)$$

$$\mathcal{C} v = \frac{\mathcal{C}}{\sqrt{\mathcal{L}\mathcal{C}}} = \sqrt{\frac{\mathcal{C}}{\mathcal{L}}} = \mathrm{GF}\sqrt{\frac{\epsilon}{\mu}} = \frac{1}{Z_o}$$

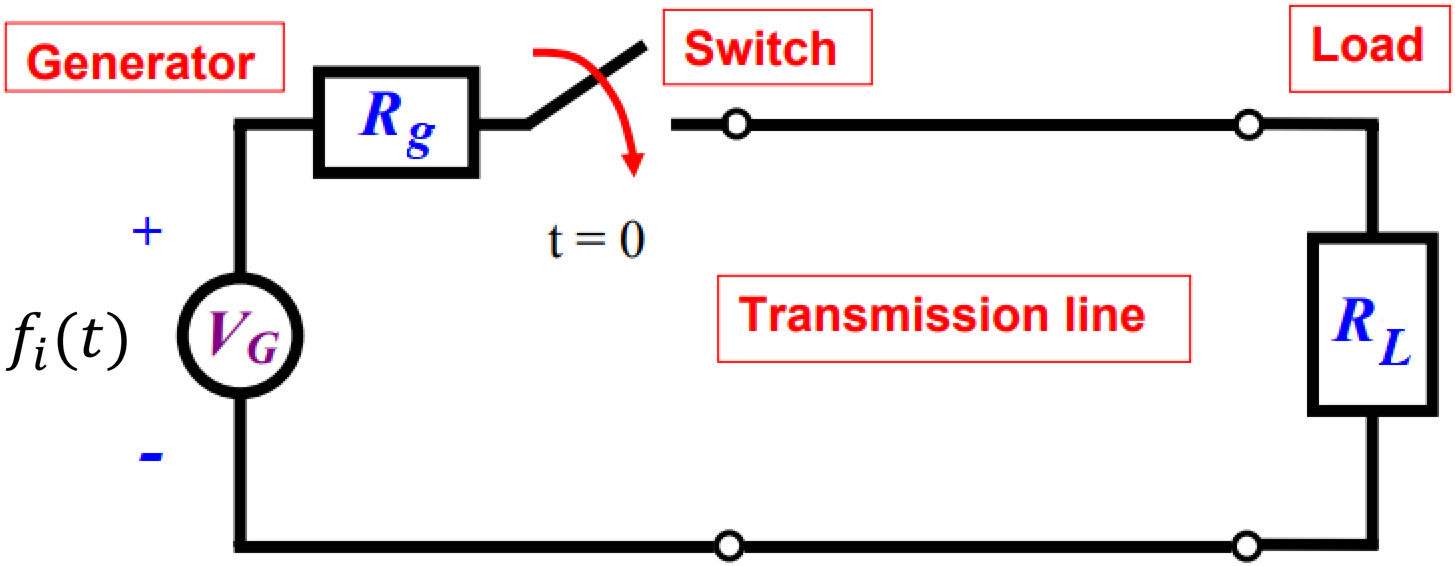
characteristic impedance

$$\boxed{I(z,t) = \pm \frac{f\left(t \mp \frac{z}{v}\right)}{Z_o}}$$

$$\boxed{Z_o \equiv \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}}$$

In a circuit we have boundary conditions

(Generator → Forward wave) **(Load → Reflected wave)**



$$\boxed{V(z,t) = f(t - \frac{z}{v}) + g(t + \frac{z}{v})}$$

$$\boxed{v = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}}}$$

$$\boxed{I(z,t) = \frac{f(t - \frac{z}{v})}{Z_o} - \frac{g(t + \frac{z}{v})}{Z_o}}$$

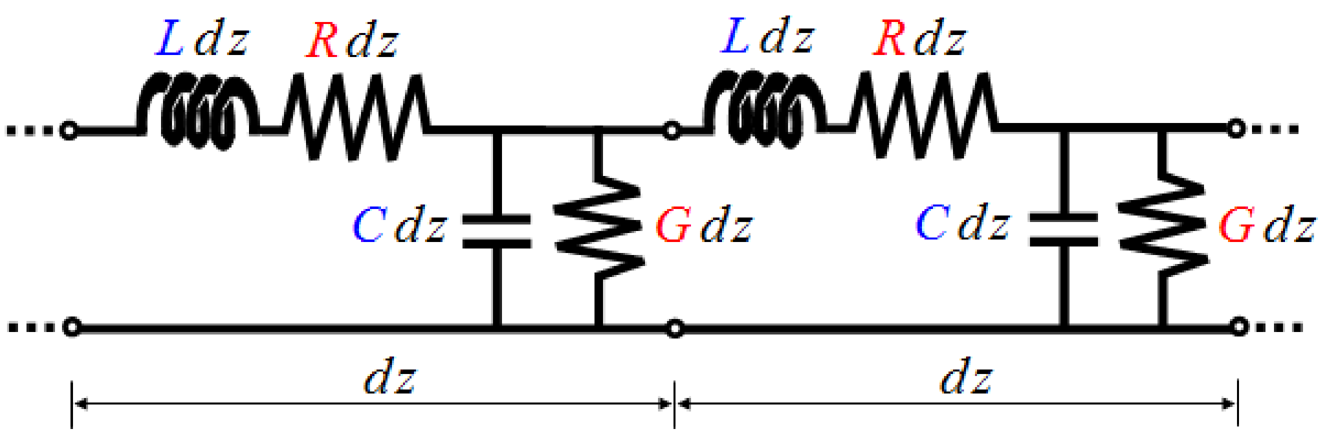
$$\boxed{Z_o \equiv \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}}$$

Complete Phasor Solution with TL as a Circuit

A uniform transmission line behaves as a "distributed circuit" described by a cascade of identical cells with infinitesimal length.

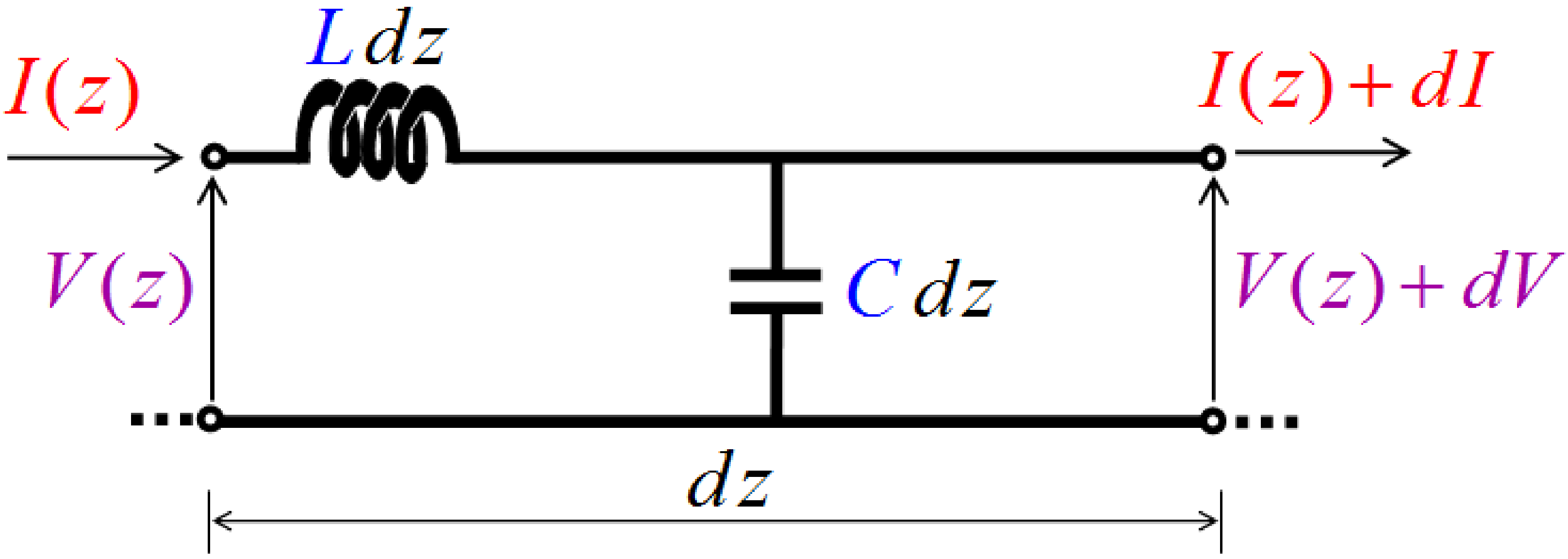
L = series inductance per unit length
R = series resistance per unit length

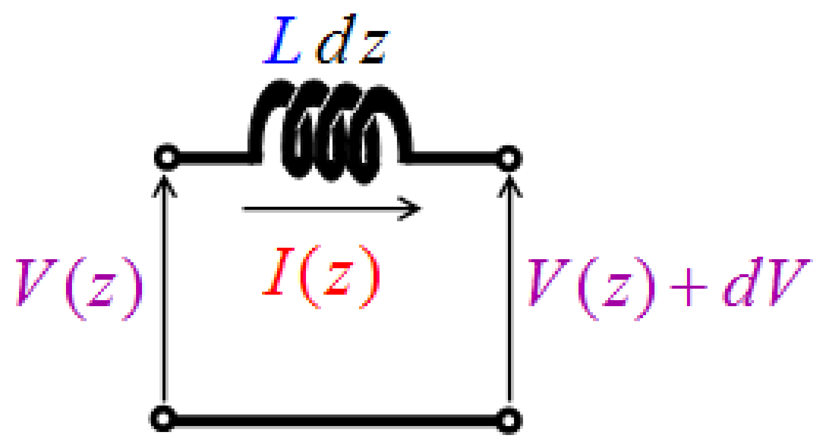
C = shunt capacitance per unit length
G = shunt conductance per unit length



To lighten up the notation, we will drop the wavy hat on top of the phasor variables. When voltage and current are notated only as function of space, they are phasors.

For ideal conductors and perfectly insulating medium it is possible to neglect resistive effects. In this approximation we have the ***loss-less transmission line*** *characterized only by reactive circuit parameters.*

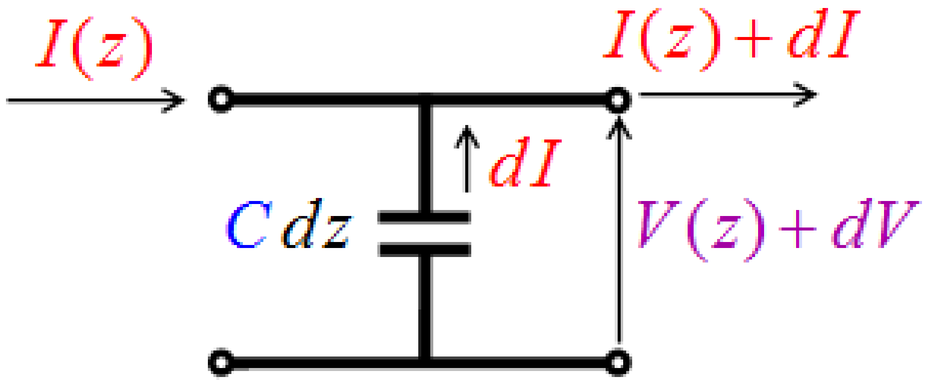




$$(V + \mathrm{d}V) - V = -j\,\omega L\,\mathrm{d}z\,I$$

$$\frac{\mathrm{d}V}{\mathrm{d}z} = -j\,\omega L\;I$$

The series inductance determines the variation of the voltage from input to output of the cell.



$$\mathrm{d}I = -j\omega C\,\mathrm{d}z(V + \mathrm{d}V)$$

$$= -j\omega C V\,\mathrm{d}z - \underbrace{j\omega C\,\mathrm{d}V\,\mathrm{d}z}_{\to 0}$$

$$\frac{\mathrm{d}I}{\mathrm{d}z} = -j\omega C V$$

The current flowing through the shunt capacitance determines the variation of the current from input to output of the cell.

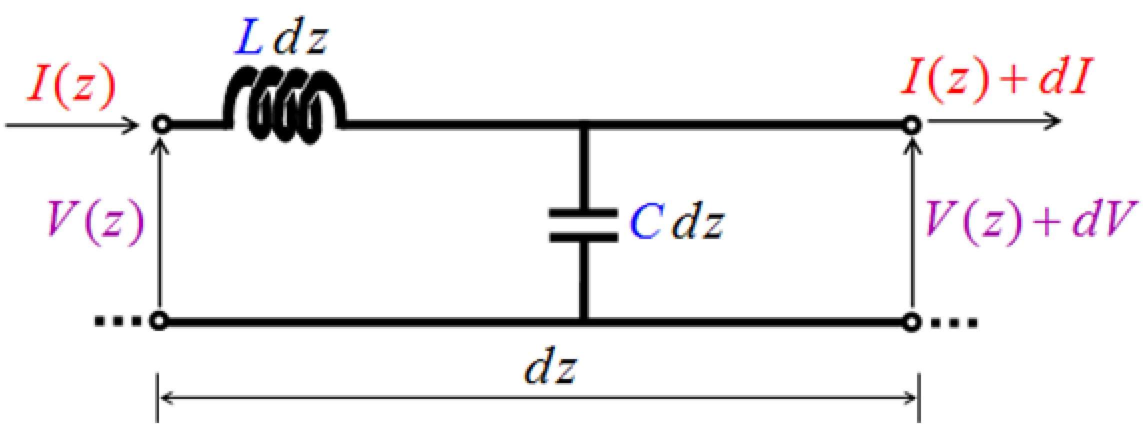
The elementary cell circuit is described by two coupled first order differential equations, which can be transformed into independent wave equations for voltage and current.

Phasor Telegrapher's equations

$$\begin{cases} \dfrac{\mathrm{d}V}{\mathrm{d}z} = -j\omega L\, I \\ \dfrac{\mathrm{d}I}{\mathrm{d}z} = -j\omega C\, V \end{cases}$$

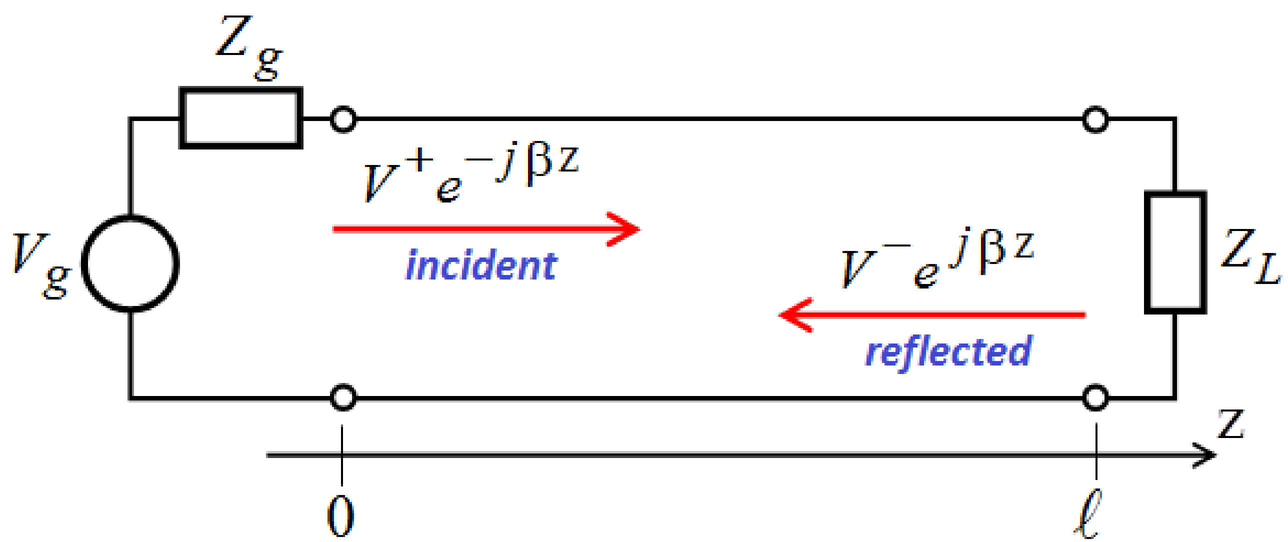
Phasor Telephonist's equations

$$\begin{cases} \dfrac{\mathrm{d}^2 V}{\mathrm{d}z^2} = -\omega^2\, LC\, V \\ \dfrac{\mathrm{d}^2 I}{\mathrm{d}z^2} = -\omega^2\, LC\, I \end{cases}$$



The general solution of the wave equation has two terms corresponding to forward and backward travelling waves. The solution for the voltage has the form

$$\boxed{V(z) = V^{+}e^{-j\beta z} + V^{-}e^{j\beta z}}$$



$$\beta = \omega\sqrt{LC}$$

propagation constant

The current distribution on the transmission line can be readily obtained by differentiation of the result for the voltage

$$\frac{\mathrm{d}V}{\mathrm{d}z} = -j\,\beta\,V^{+}e^{-j\beta z} + j\,\beta\,V^{-}e^{j\beta z} = -j\,\omega L\;I$$

which gives, using $\beta = \omega\sqrt{LC}$

$$I(z) = \sqrt{\frac{C}{L}}\left(V^{+}e^{-j\beta z} - V^{-}e^{j\beta z}\right)$$

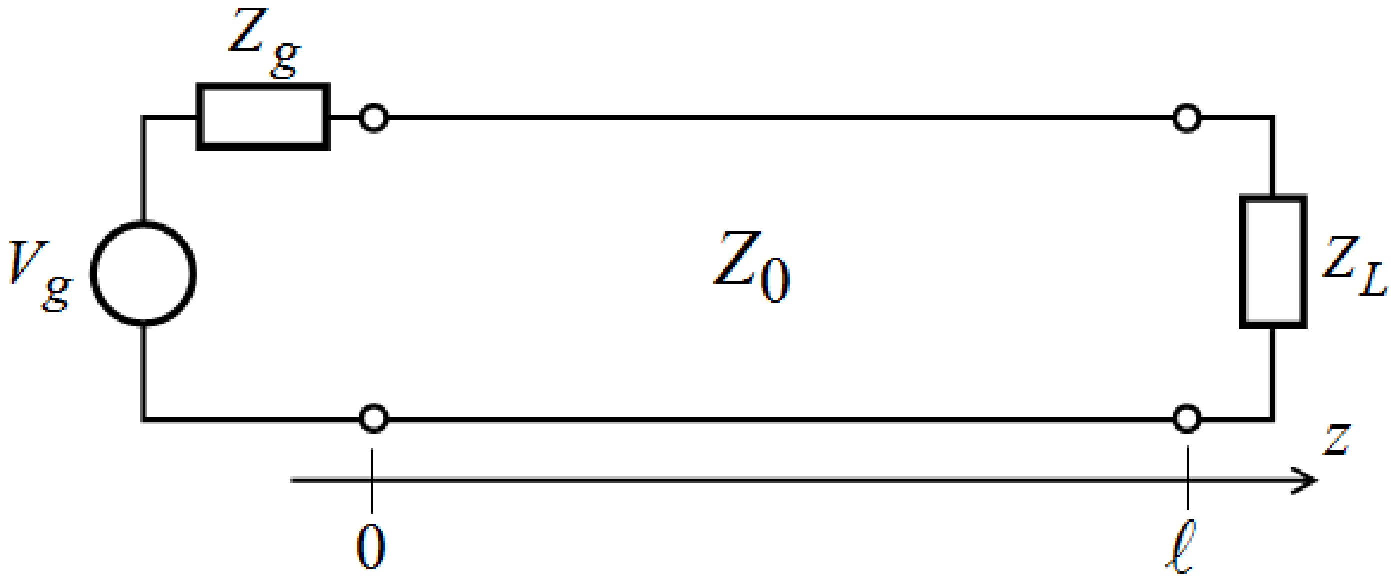
$$= \frac{1}{Z_0}\left(V^{+}e^{-j\beta z} - V^{-}e^{j\beta z}\right)$$

Note sign

The real quantity

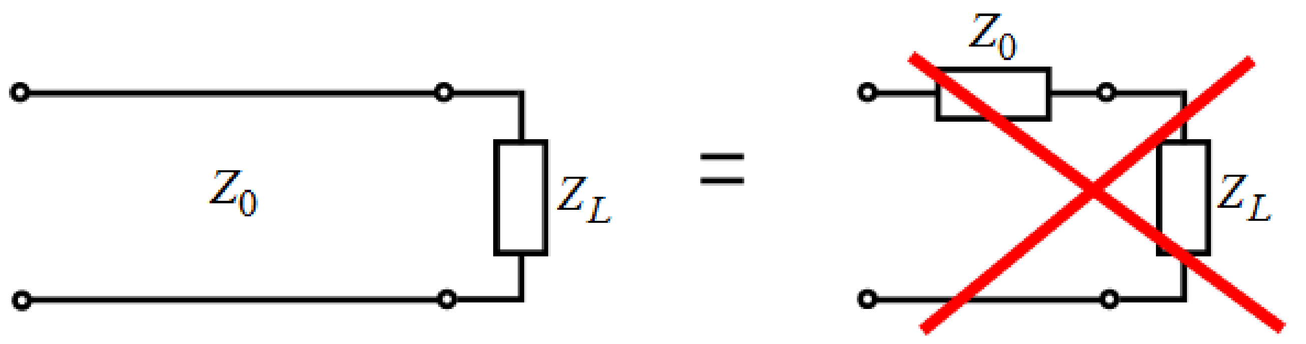
$$Z_0 = \sqrt{\frac{L}{C}}$$

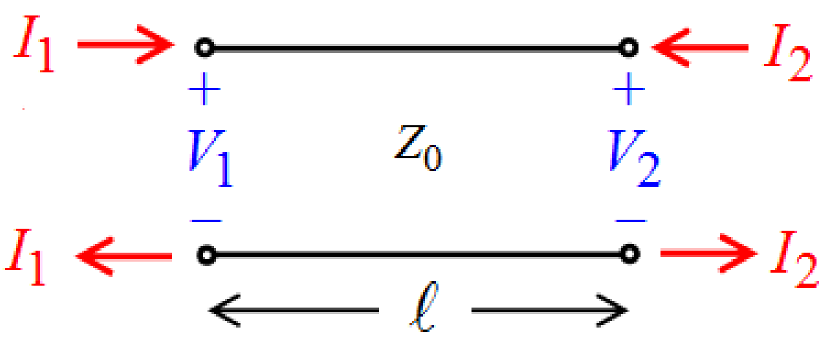
is the "characteristic impedance" of the loss-less transmission line.



Never interpret the characteristic impedance as a lumped impedance that can replace the transmission line in an equivalent circuit.

This is a very common mistake!





The line is rather a "two-port" network with input impedance which depends on its length, besides the load and Z_0.