Lecture 28 – Outline

- Transmission line equations
- Power Transmission
- Transient on a transmission line
- Impulse response
- Bounce diagram
- Examples

Reading assignment
Prof. Kudeki’s ECE 329 Lecture Notes on Fields and Waves:
28) Distributed Circuits and Bounce Diagrams
TEM propagation

\[ \mathbf{E} = \hat{x} E_x(z, t) \]
\[ \mathbf{H} = \hat{y} H_y(z, t) \]

Obtain scalar forms (1D model)

**Faraday’s Law**

\[ \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \Rightarrow \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \]

**Ampere’s Law**

\[ \nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \Rightarrow -\frac{\partial H_y}{\partial z} = \sigma E_x + \epsilon \frac{\partial E_x}{\partial t} \]

conductivity of dielectric between plates
Transform directly into circuit equations

• Multiply by $d$ and $W$ both equations
• Define the voltage from plate 2 to 1

$$V = E_x d$$

• Define the current on plate 2

$$I = J_{sz} W = H_y W$$

• Obtain

$$W \frac{\partial V}{\partial z} = -\mu d \frac{\partial I}{\partial t}$$

$$-d \frac{\partial I}{\partial z} = \epsilon W \frac{\partial V}{\partial t} + \sigma W V$$
\[ W \frac{\partial V}{\partial z} = -\mu d \frac{\partial I}{\partial t} \]

Circuit parameters for the parallel plate structure

\[ \mathcal{L} = \mu \frac{d}{W} \]
inductance per unit length

\[ \mathcal{C} = \epsilon \frac{W}{d} \]
capacitance per unit length

\[ \mathcal{G} = \sigma \frac{W}{d} \]
conductance per unit length
Telegrapher’s equations

\[- \frac{\partial V}{\partial z} = L \frac{\partial I}{\partial t} \]
\[- \frac{\partial I}{\partial z} = C \frac{\partial V}{\partial t} + G V \]

For good conductor plates with resistivity \( \rho \) there would be additional loss term in the first equation

\[RI \quad \text{where} \quad R = \text{resistance per unit length} \]

In the ideal case of perfect dielectric and perfect conductor we can neglect \( R \) and \( G \)

Telegrapher’s equations for the lossless transmission line
Geometrical factor $GF$  

For the parallel plates line we have a geometrical factor  

$$GF = \frac{W}{d}$$

so that  

$$L = \mu \frac{d}{W} = \frac{\mu}{GF} \quad \quad C = \varepsilon \frac{W}{d} = \varepsilon GF$$

These are general expressions. Different structures will have a different geometric factor

$$GF = \frac{2\pi}{\ln \frac{b}{a}} \quad \quad GF = \frac{\pi}{\cosh^{-1} \frac{D}{2a}}$$

coaxial cable \quad two-wire (twin-lead) line
Telephonist’s wave equations

Wave equations for voltage and current can be obtained in complete analogy to electric and magnetic field.

\[
\frac{\partial^2 V}{\partial z^2} = LC \frac{\partial^2 V}{\partial t^2}
\]

\[
\frac{\partial^2 I}{\partial z^2} = LC \frac{\partial^2 I}{\partial t^2}
\]

These are now equivalent decoupled equations. We only need one of the two to solve a problem.
d’Alembert wave solutions

\[ \frac{\partial^2 V}{\partial z^2} = \mathcal{L}C \frac{\partial^2 V}{\partial t^2} \]

This equation has solutions of the form

\[ V(z, t) = f(t \mp \frac{z}{\nu}) \]

with

\[ \mathcal{L} C = \frac{\mu}{GF} \varepsilon GF = \mu \varepsilon \]

\[ \nu \equiv \frac{1}{\sqrt{\mathcal{L}C}} = \frac{1}{\sqrt{\mu \varepsilon}}. \]
d’Alembert wave solutions

From the second telegrapher’s equation

\[-\frac{\partial I}{\partial z} = \mathcal{C} \frac{\partial V}{\partial t} = \mathcal{C} \frac{\partial}{\partial t} f \left( t \mp \frac{z}{v} \right)\]

\[\partial I = -\mathcal{C} \frac{\partial z}{\partial t} \partial f \left( t \mp \frac{z}{v} \right)\]

\[I(z, t) = \pm \mathcal{C} v f \left( t \mp \frac{z}{v} \right)\]

\[\mathcal{C} v = \frac{\mathcal{C}}{\sqrt{\mathcal{L} \mathcal{C}}} = \sqrt{\frac{\mathcal{C}}{\mathcal{L}}} = \text{GF} \sqrt{\frac{\varepsilon}{\mu}} = \frac{1}{Z_0}\]

characteristic impedance

\[I(z, t) = \pm \frac{f(t \mp \frac{z}{v})}{Z_0}\]

\[Z_0 \equiv \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\]
In a circuit we have boundary conditions

(Generator $\rightarrow$ Forward wave) (Load $\rightarrow$ Reflected wave)

\[ V(z, t) = f(t - \frac{z}{v}) + g(t + \frac{z}{v}) \]

\[ I(z, t) = \frac{f(t - \frac{z}{v})}{Z_o} - \frac{g(t + \frac{z}{v})}{Z_o} \]

\[ v = \frac{1}{\sqrt{LC}} \]

\[ Z_o = \sqrt{\frac{L}{C}} \]
The real quantity

\[ Z_0 = \sqrt{\frac{L}{C}} \]

is the “characteristic impedance” of the loss-less transmission line.
Never interpret the characteristic impedance as a lumped impedance that can replace the transmission line in an equivalent circuit. **This is a very common mistake!**

The line is rather a “two-port” network with input impedance which depends on its length, besides the load and $Z_0$. 
EXTRA

Phasor Solution of Transmission Line as a distributed circuit

The following seven slides present the single frequency phasor solution for you to review. In the notes, the phasor solution is simply invoked by transformation of the time-dependent solution.
A uniform transmission line behaves as a “distributed circuit” described by a cascade of identical cells with infinitesimal length.

$L = \text{series inductance per unit length}$

$R = \text{series resistance per unit length}$

$C = \text{shunt capacitance per unit length}$

$G = \text{shunt conductance per unit length}$

To lighten up the notation, we will drop the wavy hat on top of the phasor variables. When voltage and current are notated only as function of space, they are phasors.
For ideal conductors and perfectly insulating medium it is possible to neglect resistive effects. In this approximation we have the loss-less transmission line characterized only by reactive circuit parameters.
\[ L \, dz \]

\[ V(z) \quad I(z) \quad V(z) + dV \]

\[(V + dV) - V = -j \omega L \, dz \, I \]

\[ \frac{dV}{dz} = -j \omega L \, I \]

The series inductance determines the variation of the voltage from input to output of the cell.
\[ I(z) \quad \rightarrow \quad I(z) + dI \]

\[ C \, dz \]

\[ V(z) + dV \]

\[ dI = -j \omega C \, dz \, (V + dV) \]

\[ = -j \omega CV \, dz - j \omega C \frac{dV}{dz} \rightarrow 0 \]

\[ \frac{dI}{dz} = -j \omega CV \]

The current flowing through the shunt capacitance determines the variation of the current from input to output of the cell.
The elementary cell circuit is described by two coupled first order differential equations, which can be transformed into independent wave equations for voltage and current.

Phasor Telegrapher’s equations

\[
\begin{align*}
\frac{dV}{dz} &= -j\omega LI \\
\frac{dI}{dz} &= -j\omega CV
\end{align*}
\]

Phasor Telephonist’s equations

\[
\begin{align*}
\frac{d^2V}{dz^2} &= -\omega^2 LC V \\
\frac{d^2I}{dz^2} &= -\omega^2 LC I
\end{align*}
\]
The general solution of the wave equation has two terms corresponding to forward and backward travelling waves. The solution for the voltage has the form

\[ V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z} \]

\[ \beta = \omega \sqrt{LC} \]

propagation constant
The current distribution on the transmission line can be readily obtained by differentiation of the result for the voltage

\[
\frac{dV}{dz} = -j \beta V^+ e^{-j \beta z} + j \beta V^- e^{j \beta z} = -j \omega L I
\]

which gives, using \( \beta = \omega \sqrt{LC} \)

\[
I(z) = \sqrt{\frac{C}{L}} \left( V^+ e^{-j \beta z} - V^- e^{j \beta z} \right)
\]

\[
= \frac{1}{Z_0} \left( V^+ e^{-j \beta z} - V^- e^{j \beta z} \right)
\]

Note sign
Power Transmission

Let’s consider again the parallel plate transmission line prototype.

Poynting vector (density of power flow)

\[ \mathbf{E} \times \mathbf{H} = E_x H_y \hat{z} \]

Power flow in \( z \)-direction

\[ p(z, t) = W d E_x(z, t) H_y(z, t) \]

\[ = (E_x(z, t) d)(H_y(z, t) W) \]

\[ = V(z, t) I(z, t) \]

Power flow in \( z \)-direction – phasor formalism

\[ P = \frac{1}{2} \text{Re}\{\tilde{V} \tilde{I}^*\} \]
In a circuit we have boundary conditions

(Generator $\rightarrow$ Forward wave)  (Load $\rightarrow$ Reflected wave)

\[
V(z, t) = f(t - \frac{z}{v}) + g(t + \frac{z}{v})
\]

\[
I(z, t) = \frac{f(t - \frac{z}{v})}{Z_o} - \frac{g(t + \frac{z}{v})}{Z_o}
\]

\[
v = \frac{1}{\sqrt{LC}}
\]

\[
Z_o \equiv \sqrt{\frac{L}{C}}
\]
Transient in a Transmission Line

We look for the voltage and current, $V(z, t)$ and $I(z, t)$ after the switch is closed and a certain input signal $f_i(t)$ is injected.

Remember: for a lossless line, the characteristic impedance $Z_0$ is Real
Close the Switch

After the switch is closed, the voltage at the input of the TL varies to a value $V^+$ and a current $I^+$ begins to flow into the line.

The load voltage remains zero until the wavefront reaches the end of the line.
After the switch is closed, positive charges start flowing into the top wire (that is, electrons are being pulled in by the generator).

Electrons are pushed into the bottom wire (as if positive charges are entering the generator), so that the same current flows.
Propagation toward the load

Until the wavefront reaches the load, the input impedance of the transmission line appears to be the same as the characteristic impedance $Z_0$ because the current cannot yet sense the load.

The voltage front $V^+$ propagates with current $I^+$ where

$$V^+ = Z_0 I^+ = V_G \frac{Z_0}{R_g + Z_0}$$

$$I^+ = \frac{V^+}{Z_0} = \frac{f_i(0)}{R_g + Z_0}$$

The wave fronts travel with a phase velocity equal to the speed of light for the dielectric medium surrounding the wires.
The wavefront has reached the load

If the load does not match exactly the characteristic impedance of the line, the voltage $V^+$ and the current $I^+$ are not compatible with the load $R_L$ because

$$V^+ \neq R_L I^+$$

Voltage and current adjust themselves to the load by reflecting back a wavefront with voltage $V^-$ and current $I^-$ such that

$$V^+ + V^- = (I^+ + I^-)R_L$$
Since also the reflected front encounters an impedance $Z_0$, we have

$$V^+ = Z_0 I^+ \quad \text{and} \quad V^- = -Z_0 I^-$$

$$V^+ + V^- = (I^+ + I^-)R_L$$

$$V^+ + V^- = \left(\frac{V^+}{Z_0} - \frac{V^-}{Z_0}\right)R_L$$

where we have the Load Reflection Coefficient $\Gamma_L$

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$
Reflected wavefront

The wave reflected by the load propagates in the negative direction and interferes with voltage and current found along the transmission line, which continue to be injected by the generator.

When the reflected wave reaches the input of the transmission line, it terminates on the generator impedance $R_g$.

If $R_g$ does not match the line characteristic impedance $Z_0$, reflection back into the line occurs, generating an additional forward wave

$$V_2^+ = V - \frac{R_G - Z_0}{R_G + Z_0}$$

and the cycle repeats, while the generator may continue to inject a forward wave...

Remember, the ideal voltage source part of the generator behaves simply as a short for the reflected wave attempting to exit the line from the input.
Special case: Load Matched to the Transmission Line

Assume that the load is matched to the TL: \( R_L = Z_0 \)

At the load

\[
\frac{V(\ell, t)}{I(\ell, t)} = \frac{V_L}{I_L} = R_L = Z_0
\]

\[
V(\ell, t) = f(t - \frac{\ell}{v}) + g(t + \frac{\ell}{v})
\]

\[
I(\ell, t) = \frac{f(t - \frac{\ell}{v}) - g(t + \frac{\ell}{v})}{Z_0}
\]

\[
\frac{V(\ell, t)}{I(\ell, t)} = Z_0 \quad \text{only if} \quad g(t + \frac{\ell}{v}) = 0 \quad \text{No Reflection}
\]

\[
V(z, t) = f(t - \frac{z}{v})
\]

\[
I(z, t) = \frac{1}{Z_0} f(t - \frac{z}{v})
\]
Transient in a Transmission Line

At the input \[ z = 0 \]

\[ V(0, t) = f(t) \]

\[ I(0, t) = \frac{1}{Z_0} f(t) \]

\[ Z_0 = \frac{V(0, t)}{I(0, t)} \]

Voltage Divider

\[ f(t) = \frac{Z_0}{R_g + Z_0} f_i(t) \]

\[ f(t) = \tau_g f_i(t) \]
Transient in a Transmission Line

Therefore, the distribution of voltage and current along a transmission line circuit **terminated by a matched load** is

\[
V(z, t) = \tau_g f_i(t - \frac{z}{v})
\]

\[
I(z, t) = \frac{\tau_g}{Z_0} f_i(t - \frac{z}{v})
\]

There are no interference patterns caused by load reflections, because there is only one forward wave propagating.
Impulse response for the matched case

A delta function input

$$f_i(t) = \delta(t)$$

generates the impulse response

$$V(z, t) = \tau_g \delta(t - \frac{z}{V}) \equiv h_z(t)$$

(From ECE 210) Convolution of a system’s impulse response with any input function provides the system’s response to that function

$$Y(t) = \int_{-\infty}^{+\infty} X(t) h(t - \tau) \, d\tau$$
Arbitrary Load $R_L$

The impulse response needs to be obtained again when the load is changed, with the new constraint

$$\frac{V(\ell, t)}{I(\ell, t)} = \frac{V_L}{I_L} = R_L$$

from which a new reflection coefficient is obtained each time

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$

In principle, one can construct the impulse response by adding the forward and reflected pulses, going back and forth in a series.
For the first roundtrip $0 < t < \frac{2\ell}{v}$, the load voltage and current expressions are

$$V(\ell, t) = \tau_g \left[ \delta(t - \frac{\ell}{v}) + \Gamma_L \delta(t + \frac{\ell}{v} - \frac{2\ell}{v}) \right]$$

$$= \tau_g \delta(t - \frac{\ell}{v}) \left[ 1 + \Gamma_L \right]$$

$$I(\ell, t) = \frac{\tau_g}{Z_0} \left[ \delta(t - \frac{\ell}{v}) - \Gamma_L \delta(t + \frac{\ell}{v} - \frac{2\ell}{v}) \right]$$

$$= \frac{\tau_g}{Z_0} \delta(t - \frac{\ell}{v}) \left[ 1 - \Gamma_L \right]$$
Bookkeeping of pulses

The first pulse starts at $t = 0$ from the input and it arrives to the load after a time $\ell/v$

$$\tau_g \delta(t) \quad \tau_g \delta(t - \frac{z}{v}) \quad \tau_g \delta(t - \frac{\ell}{v})$$

The first reflected pulse starts at $t = \ell/v$ from the load and returns to the input at time $2\ell/v$

$$\tau_g \Gamma_L \delta(t - \frac{2\ell}{v}) \quad \tau_g \Gamma_L \delta(t + \frac{z}{v} - \frac{2\ell}{v}) \quad \tau_g \Gamma_L \delta(t - \frac{\ell}{v})$$

and then

$$\tau_g \Gamma_L \Gamma_g \delta(t - \frac{z}{v} - \frac{2\ell}{v})$$

and so on...
Bookkeeping of pulses

The complete process can be written formally with summations. In many practical cases the series converges rapidly.

\[ V(z, t) = \tau_g \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t - \frac{z}{v} - n \frac{2\ell}{v}) \]

\[ + \tau_g \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t + \frac{z}{v} - (n + 1) \frac{2\ell}{v}) \]

\[ I(z, t) = \frac{\tau_g}{Z_o} \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t - \frac{z}{v} - n \frac{2\ell}{v}) \]

\[ - \frac{\tau_g}{Z_o} \Gamma_L \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t + \frac{z}{v} - (n + 1) \frac{2\ell}{v}) \]
Bookkeeping of pulses

For $n = 0$ we have the first forward wave

$$V(z, t) = \tau_g \sum_{n=0}^{\infty} \left( \Gamma_L \Gamma_g \right)^n \delta \left( t - \frac{z}{v} - n \frac{2\ell}{v} \right)$$

reflected by load

$$+ \tau_g \Gamma_L \sum_{n=0}^{\infty} \left( \Gamma_L \Gamma_g \right)^n \delta \left( t + \frac{z}{v} - (n + 1) \frac{2\ell}{v} \right)$$

reflected by generator

For $n = 0$ we have the first reflected wave

$$\tau_g \Gamma_L \delta \left( t + \frac{z}{v} - \frac{2\ell}{v} \right)$$
Bounce diagram

Analysis is often done in graphical form. Each bounce adds a $\Gamma$. 

\[ \frac{2l}{v} \]

\[ \frac{4l}{v} \]

\[ \frac{3l}{v} \]

\[ \frac{l}{v} \]

\[ \tau_g \Gamma_L \Gamma_g \]

\[ \tau_g \Gamma_L^2 \Gamma_g \]

\[ \tau_g \Gamma_L^3 \Gamma_g \]