Lecture 29 – Outline

• Transient on a transmission line
• Reflection coefficient
• Impulse response
• Bounce diagram
• Examples

Reading assignment
Prof. Kudeki’s ECE 329 Lecture Notes on Fields and Waves:
29) Bounce Diagram examples
Transient in a Transmission Line

We look for the voltage and current, \( V(z, t) \) and \( I(z, t) \) after the switch is closed and a certain input signal \( f_i(t) \) is injected.

Remember: for a lossless line, the characteristic impedance \( Z_0 \) is Real.
Close the Switch

After the switch is closed, the voltage at the input of the TL varies to a value $V^+$ and a current $I^+$ begins to flow into the line.

The load voltage remains zero until the wavefront reaches the end of the line.
After the switch is closed, positive charges start flowing into the top wire (that is, electrons are being pulled in by the generator).

Electrons are pushed into the bottom wire (as if positive charges are entering the generator), so that the same current flows.
Propagation toward the load

Until the wavefront reaches the load, the input impedance of the transmission line appears to be the same as the characteristic impedance $Z_0$ because the current cannot yet sense the load.

The voltage front $V^+$ propagates with current $I^+$ where

$$V^+ = Z_0 I^+ = V_G \frac{Z_0}{R_g + Z_0}$$

$$I^+ = \frac{V^+}{Z_0} = \frac{f_i(0)}{R_g + Z_0}$$

The wave fronts travel with a phase velocity equal to the speed of light for the dielectric medium surrounding the wires.
The wavefront has reached the load

If the load does not match exactly the characteristic impedance of the line, the voltage $V^+$ and the current $I^+$ are not compatible with the load $R_L$ because

$$V^+ \neq R_L I^+$$

Voltage and current adjust themselves to the load by reflecting back a wavefront with voltage $V^-$ and current $I^-$ such that

$$V^+ + V^- = (I^+ + I^-)R_L$$
Since also the reflected front encounters an impedance $Z_0$, we have

\[ V^+ = Z_0 I^+ \quad \text{and} \quad V^- = -Z_0 I^- \]

\[ V^+ + V^- = (I^+ + I^-)R_L \]

\[ V^+ + V^- = \left( \frac{V^+}{Z_0} - \frac{V^-}{Z_0} \right) R_L \]

\[ V^- = V^+ \frac{R_L - Z_0}{R_L + Z_0} \]

where we have the **Load Reflection Coefficient** $\Gamma_L$

\[ \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \]
Load Reflection Coefficient $\Gamma$ – Analogy with EM waves

Transmission Line

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$

EM wave at an interface

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$
Reflected wavefront

The wave reflected by the load propagates in the negative direction and interferes with voltage and current found along the transmission line, which continue to be injected by the generator.

When the reflected wave reaches the input of the transmission line, it terminates on the generator impedance $R_g$.

If $R_g$ does not match the line characteristic impedance $Z_0$, reflection back into the line occurs, generating an additional forward wave

$$V_2^+ = V - \frac{R_G - Z_0}{R_G + Z_0}$$

and the cycle repeats, while the generator may continue to inject a forward wave...

Remember, the ideal voltage source part of the generator behaves simply as a short for the reflected wave attempting to exit the line from the input.
Special case: Load Matched to the Transmission Line

Assume that the load is matched to the TL: \( R_L = Z_0 \)

At the load \( z = \ell \)

\[
\frac{V(\ell, t)}{I(\ell, t)} = \frac{V_L}{I_L} = R_L = Z_0
\]

\[
V(\ell, t) = f(t - \frac{\ell}{v}) + g(t + \frac{\ell}{v})
\]

\[
I(\ell, t) = \frac{f(t - \frac{\ell}{v}) - g(t + \frac{\ell}{v})}{Z_o}
\]

\[
\frac{V(\ell, t)}{I(\ell, t)} = Z_o \quad \text{only if} \quad g(t + \frac{\ell}{v}) = 0
\]

\[
V(z, t) = f(t - \frac{z}{v})
\]

\[
I(z, t) = \frac{1}{Z_o} f(t - \frac{z}{v})
\]

No Reflection
Transient in a Transmission Line

At the input

\[ z = 0 \]

\[ V(0, t) = f(t) \]

\[ I(0, t) = \frac{1}{Z_o} f(t) \]

\[ Z_o = \frac{V(0, t)}{I(0, t)} \]

Voltage Divider

\[ f(t) = \frac{Z_o}{R_g + Z_o} f_i(t) \]

\[ f(t) = \tau_g f_i(t) \]

Generator

\[ V_G = f_i(t) \]

input impedance

\[ 0 \]
Transient in a Transmission Line

Therefore, the distribution of voltage and current along a transmission line circuit **terminated by a matched load** is

\[ V(z, t) = \tau_g f_i(t - \frac{z}{v}) \]

\[ I(z, t) = \frac{\tau_g}{Z_o} f_i(t - \frac{z}{v}) \]

There are no interference patterns caused by load reflections, because there is only one forward wave propagating.
Impulse response for the matched case

A delta function input

\[ f_i(t) = \delta(t) \]

generates the impulse response

\[ V(z, t) = \tau_g \delta(t - \frac{z}{u}) \equiv h_z(t) \]

(From ECE 210) Convolution of a system’s impulse response with any input function provides the system’s response to that function

\[ Y(t) = \int_{-\infty}^{+\infty} X(t) h(t - \tau) \, d\tau \]
Arbitrary Load  $R_L$

The impulse response needs to be obtained again when the load is changed, with the new constraint

$$\frac{V(\ell, t)}{I(\ell, t)} = \frac{V_L}{I_L} = R_L$$

from which a new reflection coefficient is obtained each time

$$\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o}$$

In principle, one can construct the impulse response by adding the forward and reflected pulses, going back and forth in a series.
Arbitrary Load $R_L$

For the first roundtrip $0 < t < \frac{2\ell}{v}$, the load voltage and current expressions are

\[
V(\ell, t) = \tau_g \left[ \delta(t - \frac{\ell}{v}) + \Gamma_L \delta(t + \frac{\ell}{v} - \frac{2\ell}{v}) \right]
\]

\[
= \tau_g \delta(t - \frac{\ell}{v}) \left[ 1 + \Gamma_L \right]
\]

\[
I(\ell, t) = \frac{\tau_g}{Z_0} \left[ \delta(t - \frac{\ell}{v}) - \Gamma_L \delta(t + \frac{\ell}{v} - \frac{2\ell}{v}) \right]
\]

\[
= \frac{\tau_g}{Z_0} \delta(t - \frac{\ell}{v}) \left[ 1 - \Gamma_L \right]
\]

Written as for generic $z$ for convenience.
Bookkeeping of pulses

The first pulse starts at $t = 0$ from the input and it arrives to the load after a time $\ell/v$

$$\tau_g \delta(t) \quad \tau_g \delta(t - \frac{z}{v}) \quad \tau_g \delta(t - \frac{\ell}{v})$$

The first reflected pulse starts at $t = \ell/v$ from the load and returns to the input at time $2\ell/v$

$$\tau_g \Gamma_L \delta(t - \frac{2\ell}{v}) \quad \tau_g \Gamma_L \delta(t + \frac{z}{v} - \frac{2\ell}{v}) \quad \tau_g \Gamma_L \delta(t - \frac{\ell}{v})$$

and then

$$\tau_g \Gamma_L \Gamma_g \delta(t - \frac{z}{v} - \frac{2\ell}{v})$$

and so on...
Bookkeeping of pulses

The complete process can be written formally with summations. In many practical cases the series converges rapidly.

\[
V(z, t) = \tau_g \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t - \frac{z}{v} - n\frac{2\ell}{v})
\]

\[
+ \tau_g \Gamma_L \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t + \frac{z}{v} - (n + 1)\frac{2\ell}{v})
\]

\[
I(z, t) = \frac{\tau_g}{Z_o} \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t - \frac{z}{v} - n\frac{2\ell}{v})
\]

\[- \frac{\tau_g}{Z_o} \Gamma_L \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t + \frac{z}{v} - (n + 1)\frac{2\ell}{v})\]
Bookkeeping of pulses

For \( n = 0 \) we have the first forward wave

\[
V(z, t) = \tau_g \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t - \frac{z}{v} - n\frac{2\ell}{v})
\]

reflected by generator

\[
+\tau_g \Gamma_L \sum_{n=0}^{\infty} (\Gamma_L \Gamma_g)^n \delta(t + \frac{z}{v} - (n + 1)\frac{2\ell}{v})
\]

reflected by load

For \( n = 0 \) we have the first reflected wave

\[
\tau_g \Gamma_L \delta(t + \frac{z}{v} - \frac{2\ell}{v})
\]
Bounce diagram

Analysis is often done in graphical form. Each bounce adds a $\Gamma$. 
Example 1

Transmission Line
Length = 900 m
phase velocity \( v = c \)
Load \( R_L = 2Z_0 \)

Generator
\[ f_i(t) = \sin(\omega t)u(t) \]
\[ R_g = Z_0 \]
Frequency = 1 MHz

\[
\begin{align*}
T &= 1 \mu s \\
V_g &
\end{align*}
\]
Example 1

Transmission Line
Length = 900 m
phase velocity $v = c$
Load $R_L = 2Z_0$

Generator
$\mathbf{f}_i(t) = \sin(\omega t)u(t)$
$R_g = Z_0$
Frequency = 1 MHz

Injection coefficient
$\tau_g = \frac{Z_o}{R_g + Z_o} = \frac{Z_o}{Z_o + Z_o} = \frac{1}{2}$

Reflection coefficients
$\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o} = \frac{2Z_o - Z_o}{2Z_o + Z_o} = \frac{2 - 1}{2 + 1} = \frac{1}{3}$
$\Gamma_g = \frac{R_g - Z_o}{R_g + Z_o} = \frac{Z_o - Z_o}{Z_o + Z_o} = 0$

time-delay (roundtrip)
$\frac{2\ell}{v} = \frac{2 \cdot 900 \text{ m}}{300 \text{ m/\mu s}} = 6 \text{ \mu s}$
Example 1

\[ \tau_g = \frac{1}{2} \]

\[ \Gamma_L = \frac{1}{3} \]

\[ \Gamma_g = 0 \]

Bouncing stops after one roundtrip

Impulse response

\[ h_z(t) = \frac{1}{2} \delta(t - \frac{z}{c}) + \frac{1}{6} \delta(t + \frac{z}{v} - 6\mu) \]
Example 1

**input signal**

\[ f_i(t) = \sin(\omega t)u(t) \]

**impulse response**

\[ h_\z(t) = \frac{1}{2}\delta(t - \frac{\z}{v}) + \frac{1}{6}\delta(t + \frac{\z}{v} - 6\mu) \]

**voltage wave from convolution**

\[ V(\z, t) = h_\z(t) * \sin(\omega t)u(t) \]

\[ = \frac{1}{2}\sin \omega(t - \frac{\z}{v})u(t - \frac{\z}{v}) + \frac{1}{6}\sin \omega(t + \frac{\z}{v} - 6)u(t + \frac{\z}{v} - 6) \]

**Graphs**

- **Left graph**: \( t = 1 \mu s \)
- **Right graph**: \( t = 2 \mu s \)
Example 2

Transmission Line
Length = 2400 m
phase velocity \( v = c \)
Load \( R_L = 100 \, \Omega \)
\( Z_0 = 50 \, \Omega \)

Generator
\( f_i(t) = u(t) \) (unit step function)
\( R_g = 0 \)

Determine and plot \( V(1200, t) \)
Example 2

Transmission Line
Length = 2400 m
phase velocity \( v = c \)
Load \( R_L = 100 \Omega \)

Generator
\( f_i(t) = u(t) \) (unit step function)
\( R_g = 0 \)

Determine and plot \( V(1200, t) \)

\[
\tau_g = \frac{Z_o}{R_g + Z_o} = 1 \quad \text{(ideal source)}
\]

\[
\Gamma_g = \frac{R_g - Z_o}{R_g + Z_o} = -1 \quad \text{(like a short circuit)}
\]

\[
\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o} = \frac{1}{3}
\]

Time-delay (one way)
\[
\ell = \frac{2400 \text{ m}}{300 \times 10^6 \text{ m/s}} = 8 \mu s
\]
Example 2

Impulse response

\[ V(1200, t) = \delta(t - 4) + \frac{1}{3}\delta(t - 12) - \frac{1}{3}\delta(t - 20) - \frac{1}{9}\delta(t - 28) + \frac{1}{9}\delta(t - 36) + \cdots \]

Unit-step response

\[ V(1200, t) = u(t - 4) + \frac{1}{3}u(t - 12) - \frac{1}{3}u(t - 20) - \frac{1}{9}u(t - 28) + \frac{1}{9}u(t - 36) + \cdots \]
Transient Plots

- **Step**
- **Pulse**

Voltage $V(t)$

$V(t)$ [V] Cursor Location

-2.0
0 4T 8T 12T

$V_{g\text{max}} = 1.0$ V

$Z_0 = 50.0$ Ω

$\varepsilon_r = 1.0$

$R_L = 100.0$ Ω

$\Gamma_g = -1.0$

$\Gamma_L = 0.3333$

$z = 1.2$ [km]

$l = 2400000.0$ mm

0.0 Ω
Transient Animation Snapshot

Transient Response

- Step
- Pulse

Transient Voltage

$V(z,t) \ [V]$

$V_{ss} = 1.0$

$-2.0$

$2.0$

$0$

$1$

START  STOP  RESET

<< SLOW  1.54  T  12.33  μs  FAST >>
Example 3

Injection coefficient

\[ \tau_g = \frac{Z_o}{R_g + Z_o} = 0.6 \]

Reflection coefficients

\[ \Gamma_g = \frac{R_g - Z_o}{R_g + Z_o} = -\frac{1}{5} \]
\[ \Gamma_L = \frac{R_L - Z_o}{R_L + Z_o} = \frac{1}{3} \]

Current

\[ I(t = 0) = 1A \]
\[ \Gamma_{g_c} = \frac{1}{5} \]
\[ \Gamma_{L_c} = -\frac{1}{3} \]

Time-delay (one way)

\[ \frac{\ell}{V} = 1 \mu s \]
Example 3

\[ \Gamma_g = -\frac{1}{5} \quad \Gamma_L = \frac{1}{3} \]

\[ \begin{align*}
0 & \rightarrow 0 & 1 & \rightarrow 1 \\
60 & \rightarrow 60 & 20 & \rightarrow 20 \\
76 & \rightarrow 76 & -4/3 & \rightarrow -4/3 \\
9/15 & \rightarrow 9/15 & 4/15 & \rightarrow 4/15 \\
1124/15 & \rightarrow 1124/15 & 224/3 & \rightarrow 224/3 \\
16876/225 & \rightarrow 16876/225 & \frac{3376}{45} & \rightarrow \frac{3376}{45} \\
141/225 & \rightarrow 141/225 & 1/225 & \rightarrow 1/225 \\
2109/3375 & \rightarrow 2109/3375 & \frac{422}{675} & \rightarrow \frac{422}{675} \\
\end{align*} \]
Example 3

\[ z = 0 \]

\[ z = l \]

\[ z = \frac{l}{2} \]
Example 3

Find the voltage as a function of $z$ at $t = 2.5 \, \mu s$
Example 3

Find the current as a function of $z$ at $t = 1.3 \mu s$
Example 3  

**Steady State Analysis**

\[
\sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{15} \right)^n = \frac{1}{1 + 1/15} = 0.9375
\]

**Forward Path**

\[
V_{SS}^+ = 60 - 4 + \frac{4}{15} - \cdots = 60 \left( 1 - \frac{1}{15} + \frac{1}{15^2} - \cdots \right) = 56.25 \text{ V}
\]

\[
I_{SS}^+ = 1 - \frac{1}{15} + \frac{1}{225} - \cdots = 1 - \frac{1}{15} + \frac{1}{15^2} - \cdots = 0.9375 \text{ A}
\]

**Backward Path**

\[
V_{SS}^- = 20 - \frac{4}{3} + \frac{4}{45} - \cdots = 20 \left( 1 - \frac{1}{15} + \frac{1}{15^2} - \cdots \right) = 18.75 \text{ V}
\]

\[
I_{SS}^- = -\frac{1}{3} + \frac{1}{45} - \frac{1}{675} + \cdots = -\frac{1}{3} \left( 1 - \frac{1}{15} + \frac{1}{15^2} - \cdots \right) = -0.3125 \text{ A}
\]

\[
V_{SS} = V_{SS}^+ + V_{SS}^- = 75 \text{ V}
\]

\[
I_{SS} = I_{SS}^+ + I_{SS}^- = 0.625 \text{ A}
\]
Example 3

**Steady State Analysis**

\[ V_{SS} = V_{SS}^+ + V_{SS}^- = 75 \text{ V} \]

\[ I_{SS} = I_{SS}^+ + I_{SS}^- = 0.625 \text{ A} \]
How can we analyze a pulse signal?

Rectangular pulse as superposition of two step function signals
Questions

What is the input impedance $Z_{in}$ of these transmission lines?

(a) $V_g$ $Z_{in}$ $Z_0$ $Z_g$

(b) $V_g$ $Z_{in}$ $Z_0$ $Z_g$