# ECE 329 – Fall 2022

## Prof. Ravaioli – Office: 2062 ECEB

Lecture 31

# Lecture 31 – Outline

- **Monochromatic (single frequency) excitation of transmission lines**
- **Phasor solution (steady-state regime)**
- **Periodicity in transmission lines**
- **Resonances**
- **Standing waves and periodic oscillations**
- **Realization of reactance (inductance or capacitance) with short-circuited or with open-circuited transmission lines**

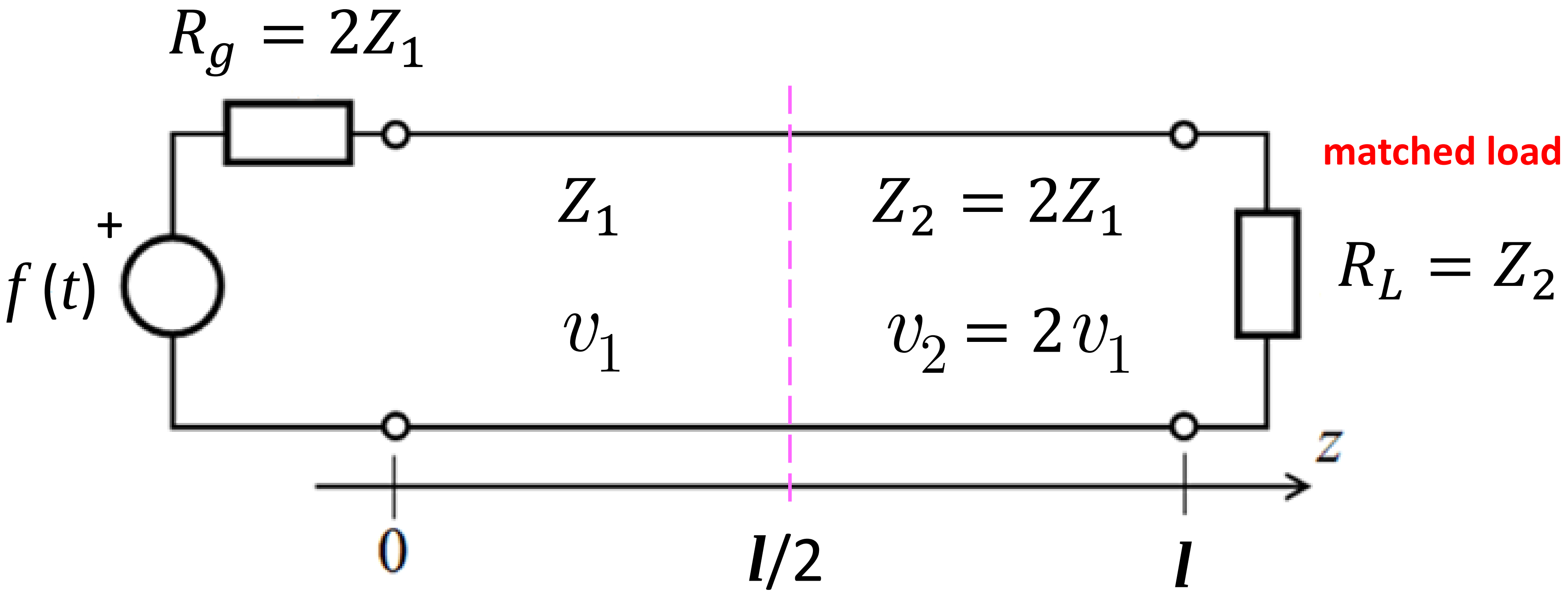
**Reading assignment**

**Prof. Kudeki's ECE 329 Lecture Notes on Fields and Waves:**

**31) Periodic oscillations in lossless Transmission Line circuits**

# Consider now the same circuit we saw last lecture, with input

$$f_i(t) = \sin(\omega t)u(t)$$



**matched load**

$$l = 2400 \text{ m}$$

$$Z_1 = 25\,\Omega$$

$$v_1 = 150\,\text{m}/\mu\text{s}$$

# Impulse response from the bounce diagram (voltage)

**Reflected by generator**

**Reflected by interface**

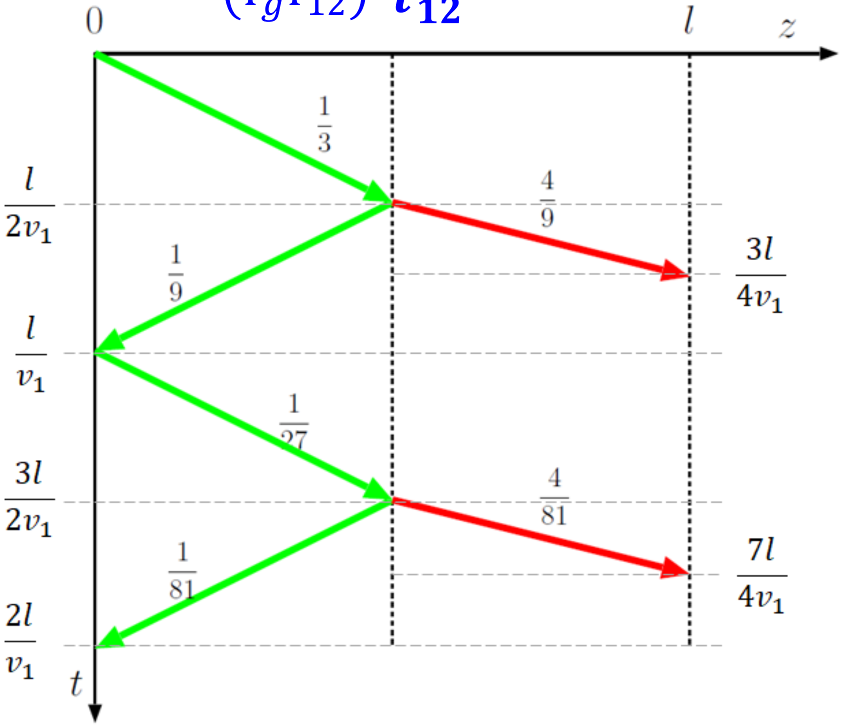
$\boxed{z < \frac{l}{2}}$

$$V(z,t) = \underset{\tau_g}{\frac{1}{3}} \sum_{n=0}^{\infty} \underbrace{(\frac{1}{3})^{2n}}_{(\Gamma_g\Gamma_{12})^n} [\delta(t - \frac{z}{v_1} - n\frac{l}{v_1}) + \underset{\Gamma_{12}}{\frac{1}{3}}\delta(t + \frac{z}{v_1} - (n+1)\frac{l}{v_1})]$$

**Injected at interface**

$\boxed{\frac{l}{2} < z < l}$

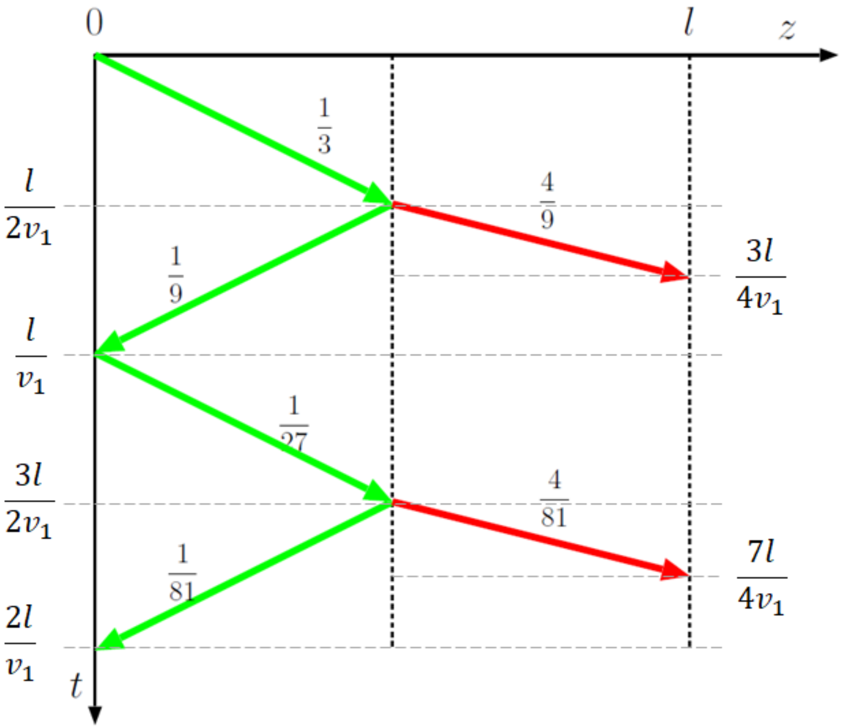
$$V(z,t) = \underset{\tau_g}{\frac{1}{3}} \sum_{n=0}^{\infty} \underbrace{(\frac{1}{3})^{2n}\frac{4}{3}}_{(\Gamma_g\Gamma_{12})^n \tau_{12}} \delta(t - \frac{z}{v_2} - (4n+1)\frac{l/2}{v_2})$$



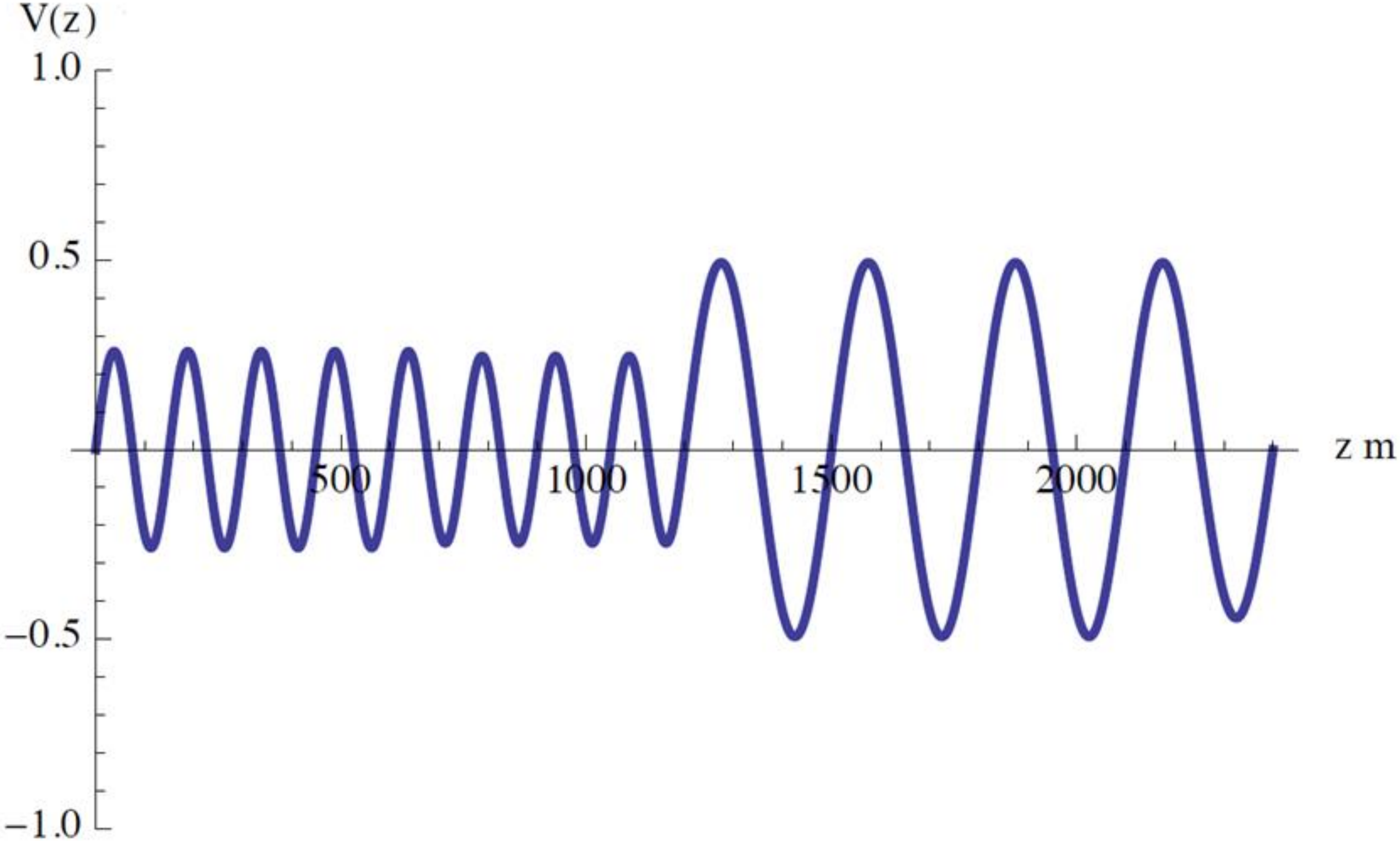
# Impulse response from the bounce diagram (current)

$$\boxed{z < \frac{l}{2}} \quad I(z,t) = \frac{1}{3Z_1}\sum_{n=0}^{\infty}(\frac{1}{3})^{2n}[\delta(t - \frac{z}{v_1} - n\frac{l}{v_1}) - \frac{1}{3}\delta(t + \frac{z}{v_1} - (n+1)\frac{l}{v_1})]$$

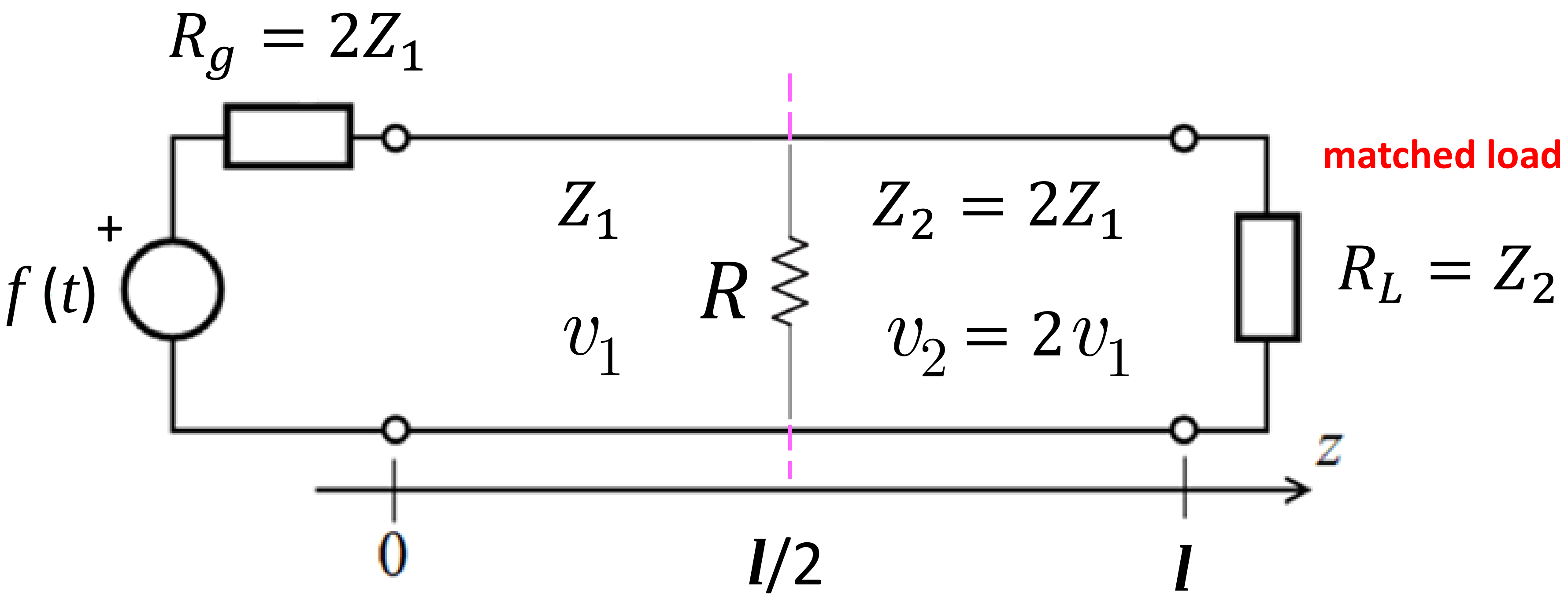
$$\boxed{\frac{l}{2} < z < l} \quad I(z,t) = \frac{1}{3Z_2}\sum_{n=0}^{\infty}(\frac{1}{3})^{2n}\frac{4}{3}\delta(t - \frac{z}{v_2} - (4n+1)\frac{l/2}{v_2})$$



# Solution Example



**A “shunt” resistance $R$ is placed at the junction. Determine reflection and transmission coefficient there.**



The wavefront reaching the junction is going to see two impedances in parallel, $R$ and $Z_2$, which correspond to an equivalent impedance

$$Z_{eq} \equiv \frac{RZ_2}{R + Z_2}$$

$$\boxed{\Gamma_{12} = \frac{Z_{eq} - Z_1}{Z_{eq} + Z_1}}$$

$$\boxed{\tau_{12} = \frac{2Z_{eq}}{Z_{eq} + Z_1}}$$

For a wave coming from the right

$$Z_{eq} \equiv \frac{RZ_1}{R + Z_1}$$

$$\boxed{\Gamma_{21} = \frac{Z_{eq} - Z_2}{Z_{eq} + Z_2}}$$

$$\boxed{\tau_{21} = \frac{2Z_{eq}}{Z_{eq} + Z_2}}$$

*The phasor steady-state solution for single frequency generator source can be obtained by applying phasor transformation*

## Phasor wave solution in a uniform transmission line

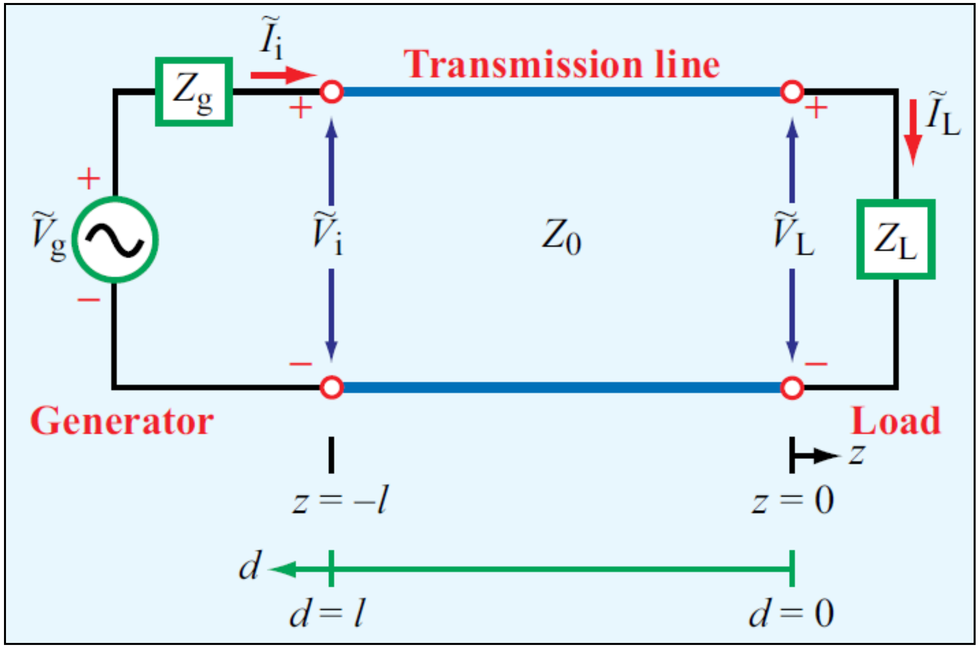
$$\tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}$$

**Mathematically these are the same as EM plane wave solutions**

$V_0^-$ and $V_0^+$ are in general complex and are determined from the boundary conditions imposed by the load and the generator.

***We mentioned last time that load location is the best space reference for steady-state analysis. We will explain why through examples.***



Phasor wave solution

$$\tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}$$

Coordinate transformation $\boxed{d = -z}$

$$\tilde{V}(z) = V_0^+ e^{j\beta d} + V_0^- e^{-j\beta d}$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{j\beta d} - \frac{V_0^-}{Z_0} e^{-j\beta d}$$

# Line with a generic load impedance

**Since the reflection coefficient is**

$$\Gamma_L = \frac{V_0^-}{V_0^+}$$

**we can derive**

$$V(d) = V_0^+ e^{j\beta d}\left(1+\Gamma_L e^{-2j\beta d}\right) = V_0^+ e^{j\beta d}\left(1+\Gamma(d)\right)$$

$$I(d) = \frac{V_0^+ e^{j\beta d}}{Z_0}\left(1-\Gamma_L e^{-2j\beta d}\right) = \frac{V_0^+ e^{j\beta d}}{Z_0}\left(1-\Gamma(d)\right)$$

$$\Gamma(d) = \Gamma_L e^{-2j\beta d}$$

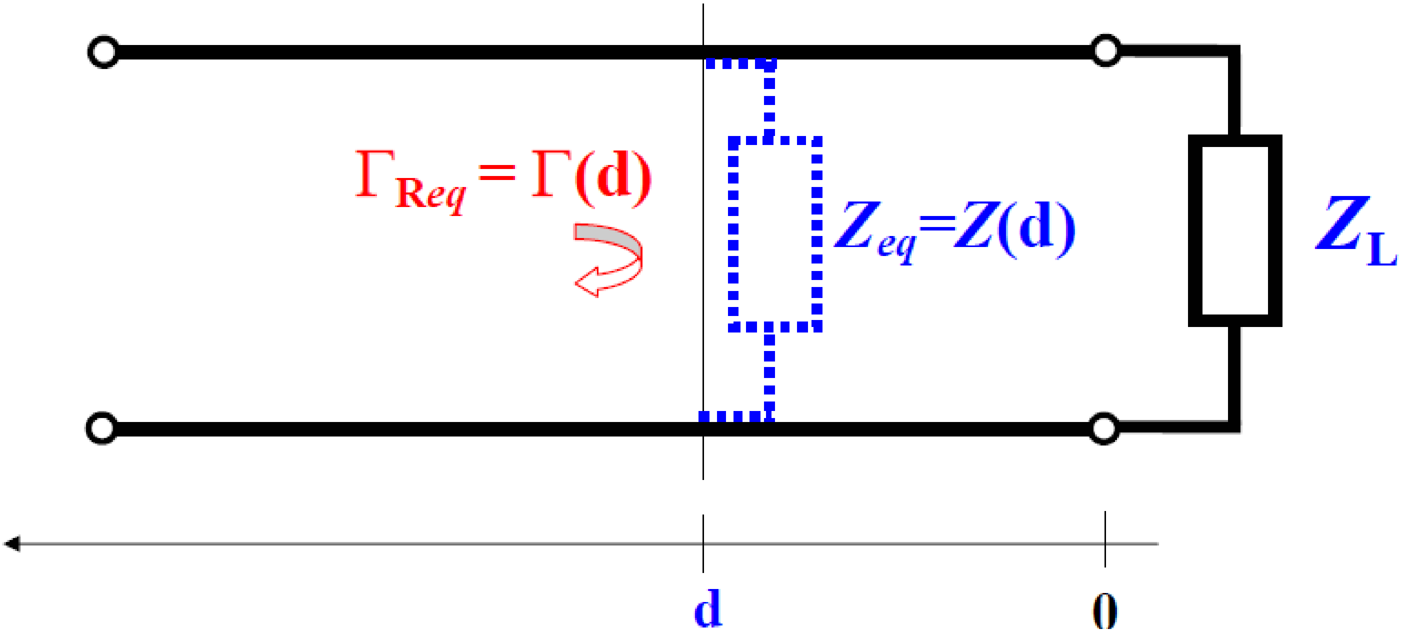
**generalized reflection coefficient**

$$Z(d) = \frac{V(d)}{I(d)} = Z_0 \frac{1+\Gamma(d)}{1-\Gamma(d)}$$

**line impedance**

# Significance of line impedance

Every line location is characterized by a **line impedance** $Z(d)$ and a **reflection coefficient** $\Gamma(d)$



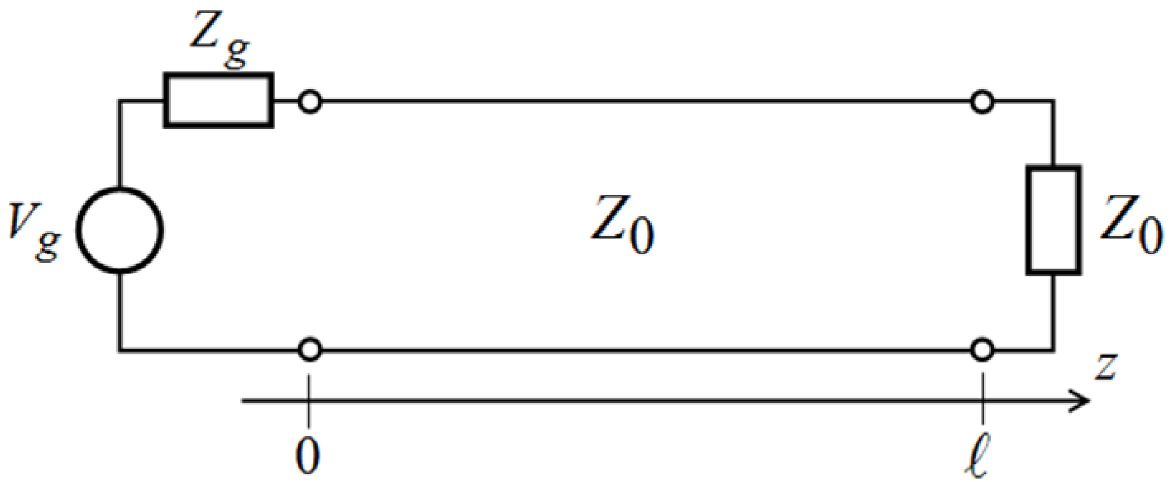
Imagine to cut the line at location d. The input impedance of the portion of line terminated by the load is the same as the line impedance at that location "before the cut". The behavior of the line on the left of location d is the same if an equivalent impedance with value $Z$(d) replaces the cut out portion.

# Periodicity of transmission lines

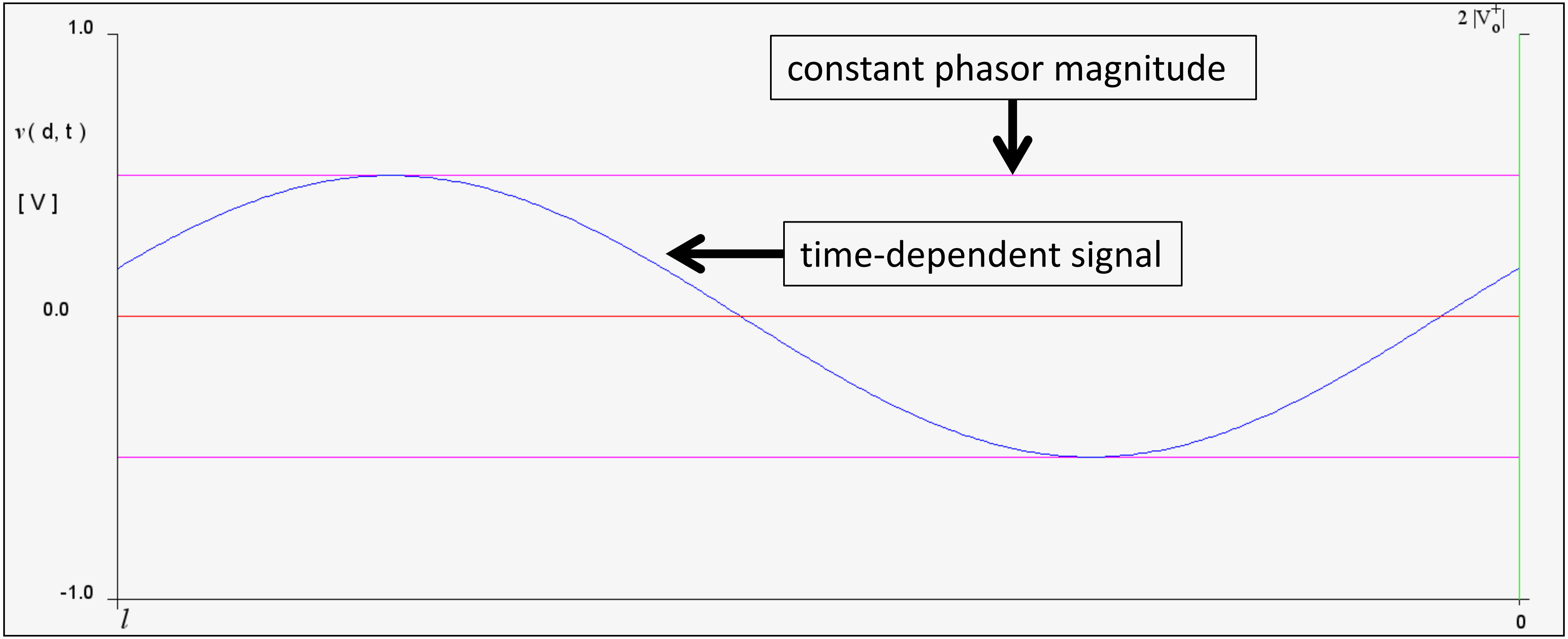
**Consider monochromatic excitation of a transmission line at a specific frequency.**

**Let's assume that we can change the length of the line, keeping the generator and the load unchanged.**

**We will exclude the case of a load matched to the line's characteristic impedance, since there are no reflections and the length does not affect the patterns of voltage and current.**



# Behavior of matched line ($Z_L = Z\_0$)

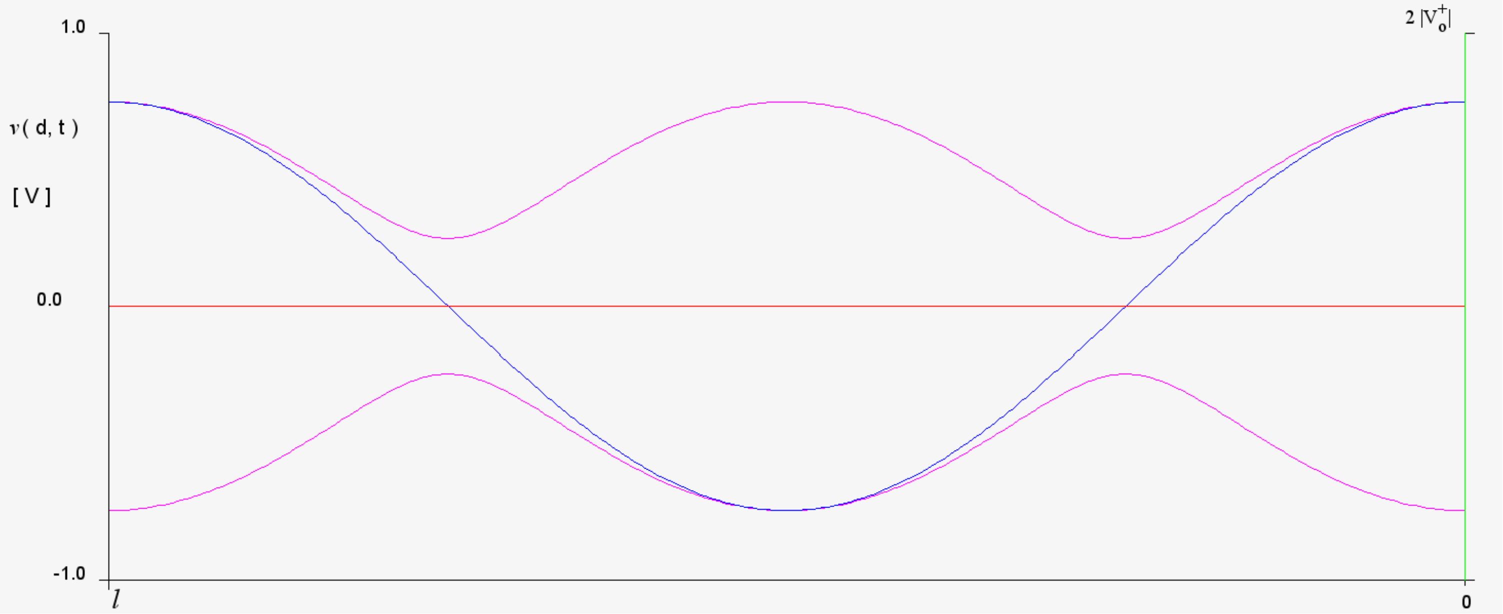


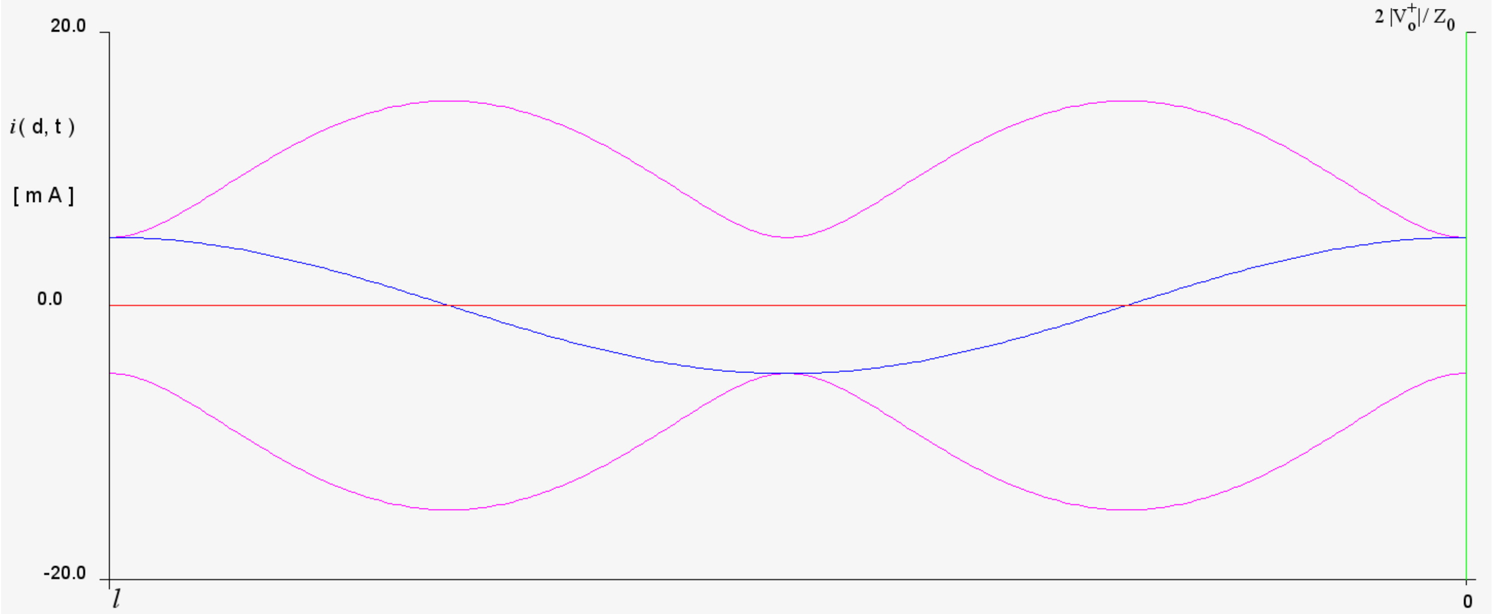
constant phasor magnitude

time-dependent signal

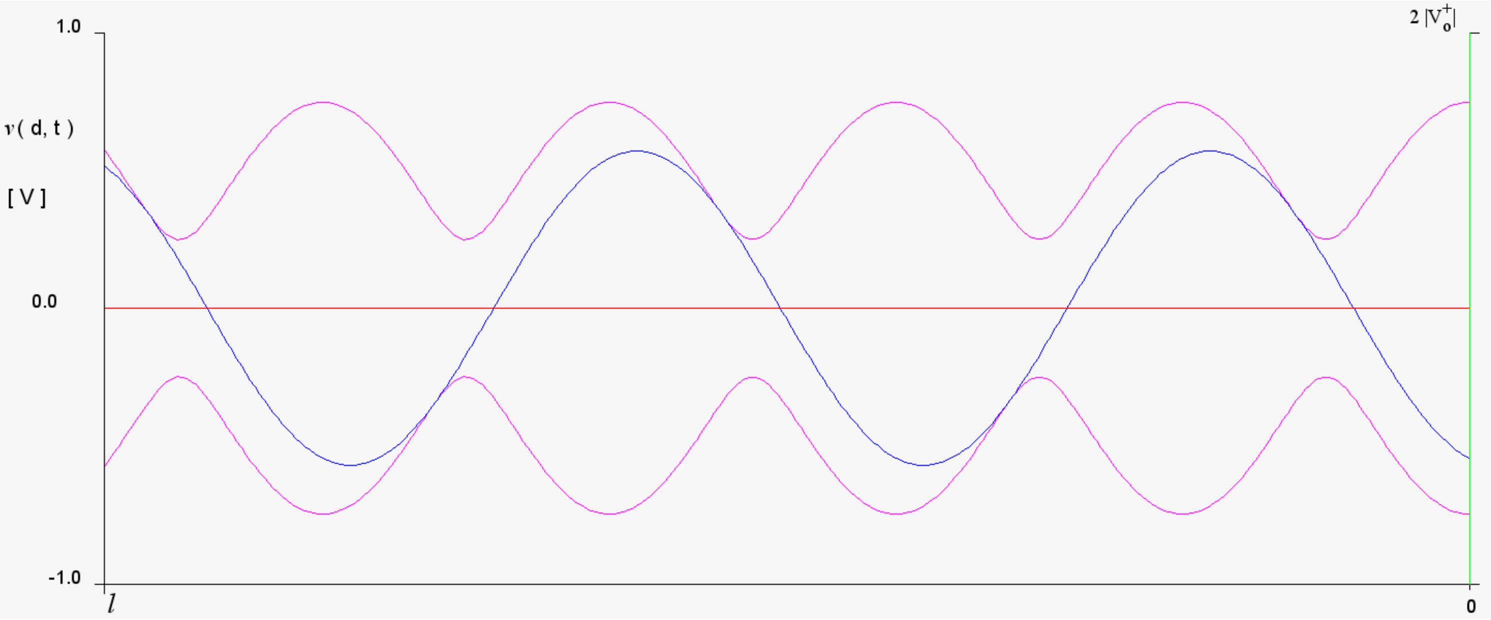
**The current has a similar pattern along the line.**

# Line with arbitrary load $Z_0 = 50\,\Omega$ and $Z_L = 150\,\Omega$ $l = 1.0\,\lambda$

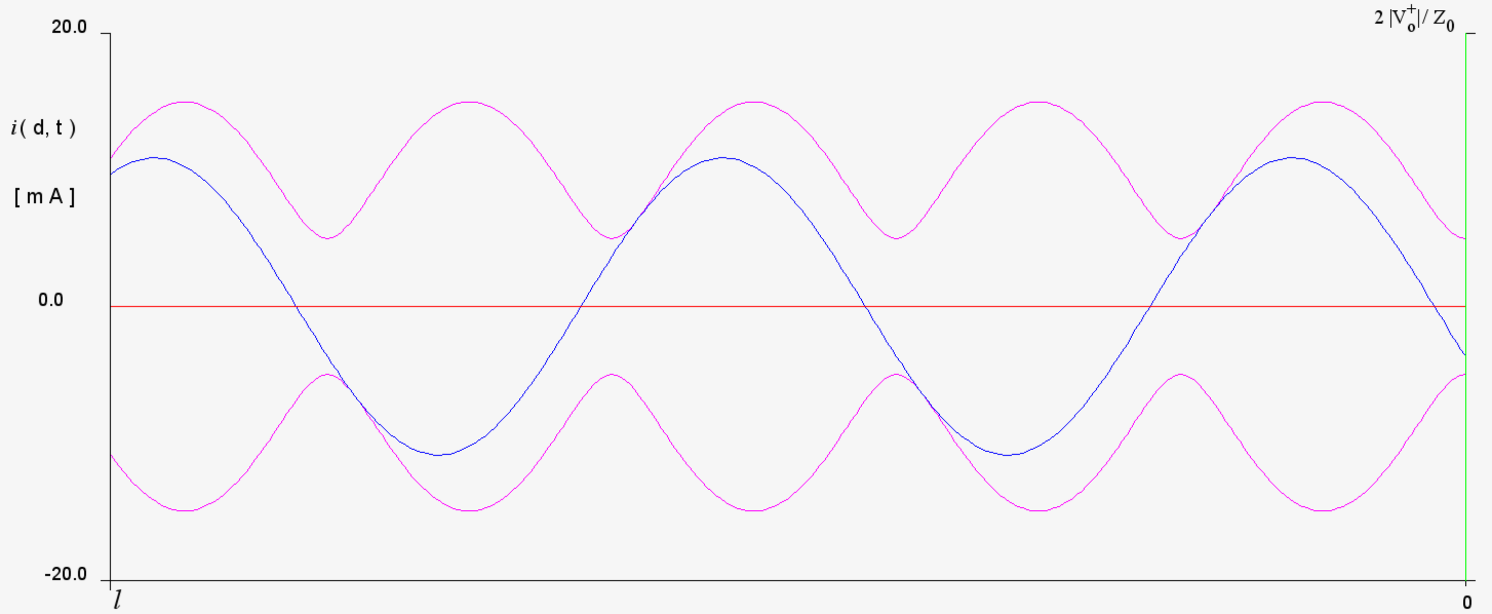




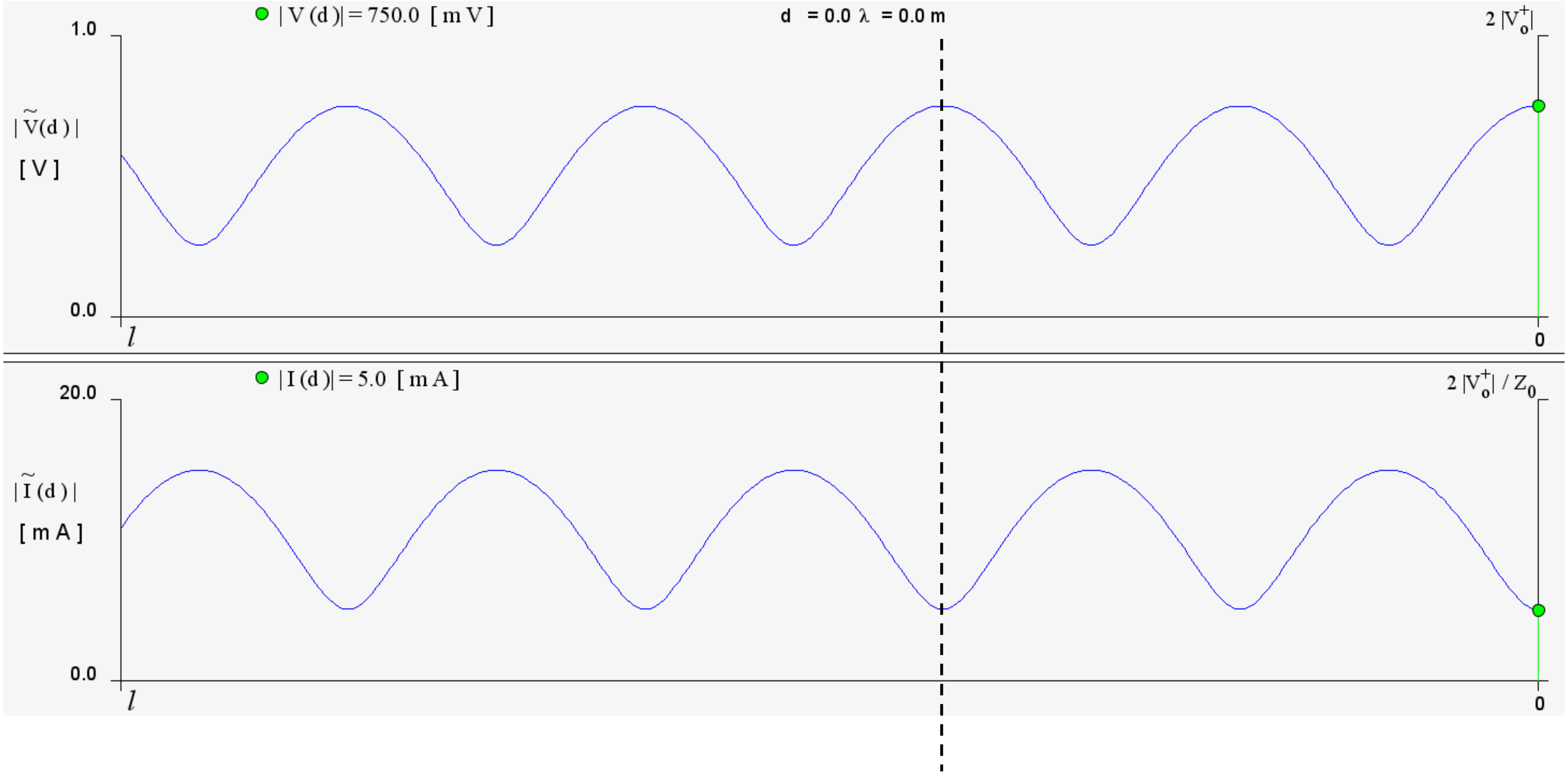
# Line with arbitrary load $Z_0 = 50\,\Omega$ and $Z_L = 150\,\Omega$ $l = 2.38\,\lambda$



same behavior at load



# Line with arbitrary load $Z_0 = 50\,\Omega$ and $Z_L = 150\,\Omega$ $l = 2.38\,\lambda$

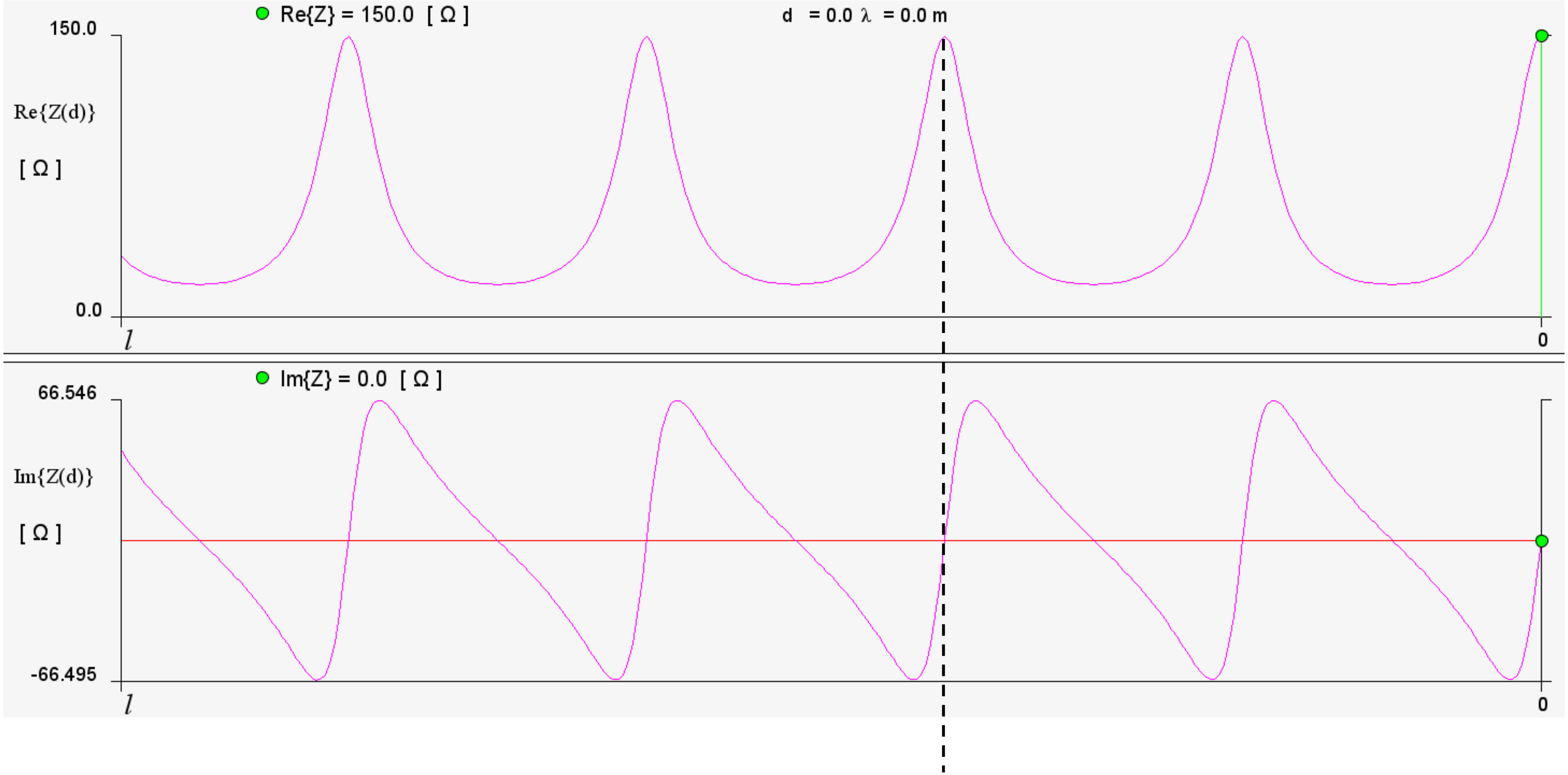


$d = 1.0\,\lambda$

## Standing wave patterns

(Space-dependent magnitudes of the phasors for voltage and current)

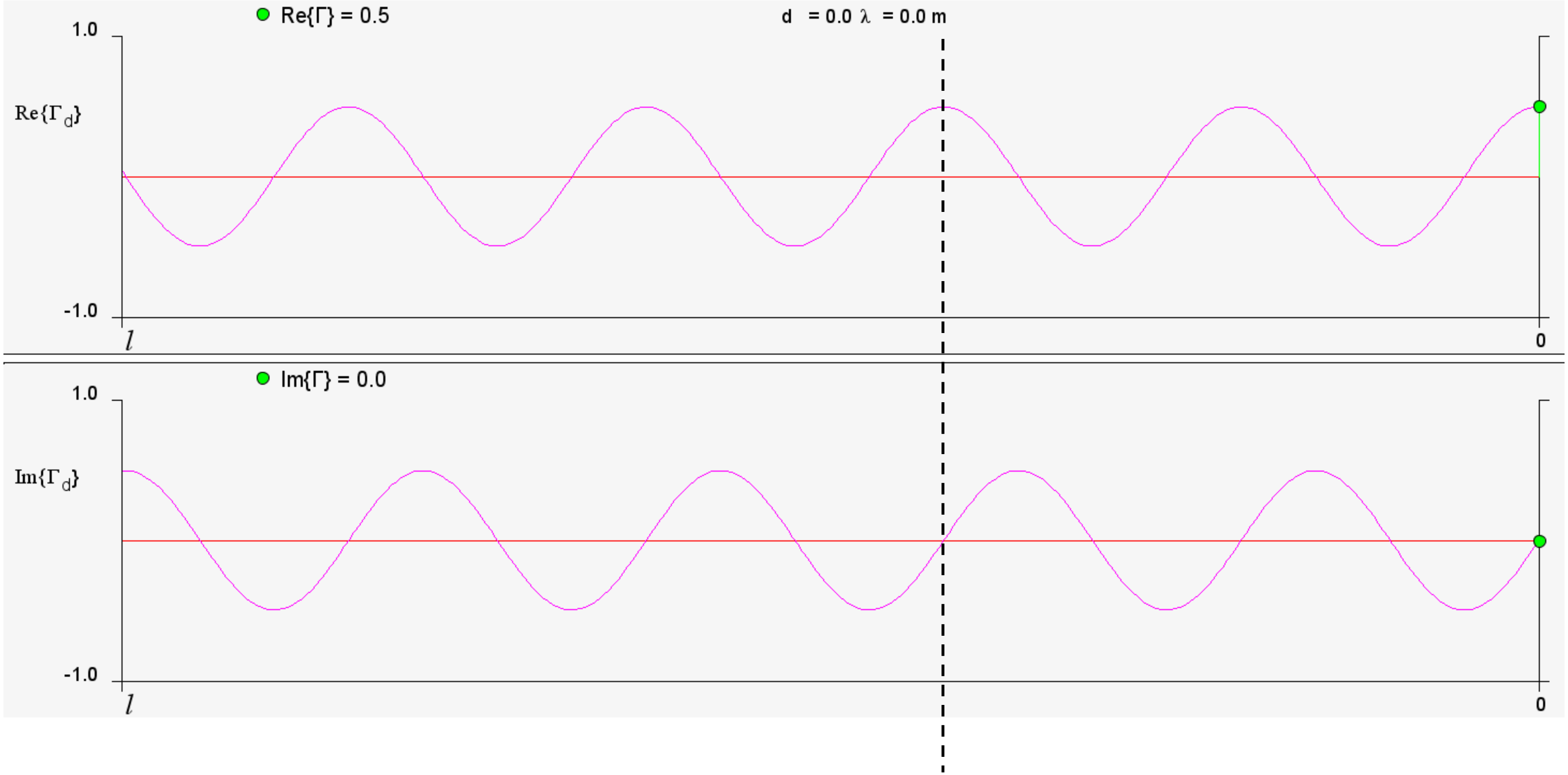
## Line with arbitrary load $Z_0 = 50\,\Omega$ and $Z_L = 150\,\Omega$ $l = 2.38\,\lambda$



$d = 1.0\,\lambda$

## Line impedance

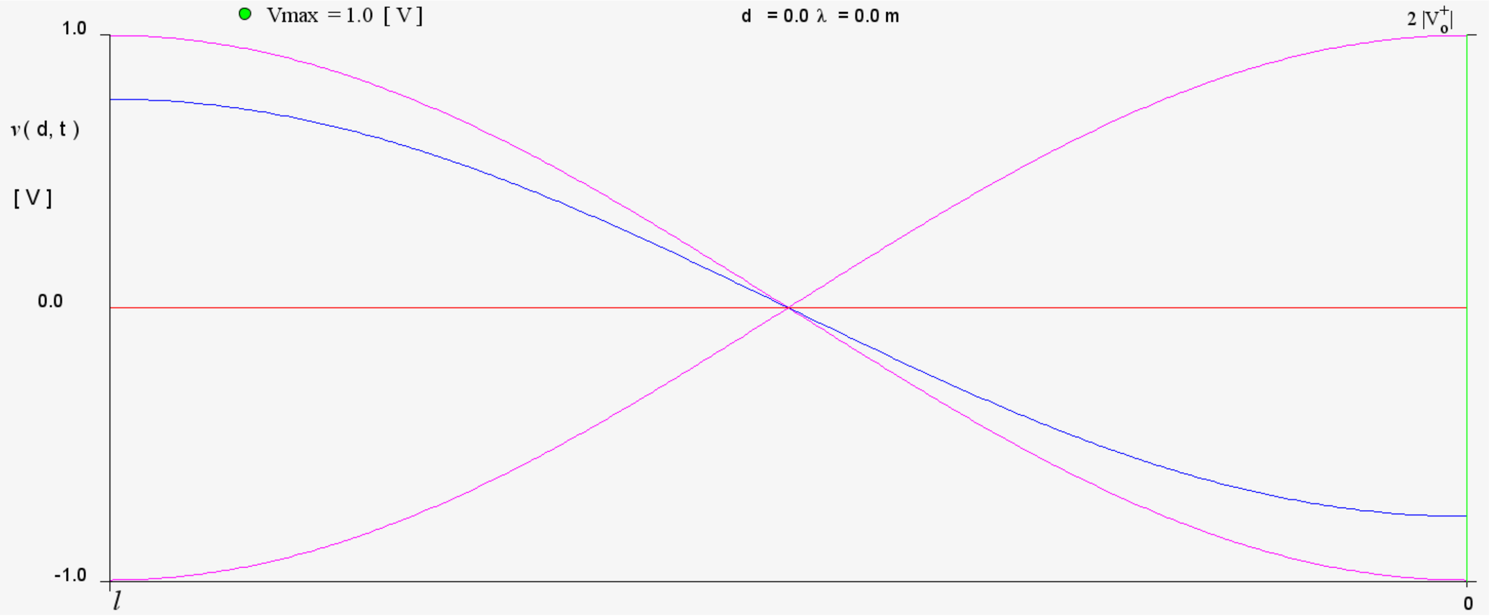
# Line with arbitrary load $Z_0 = 50\,\Omega$ and $Z_L = 150\,\Omega$ $l = 2.38\,\lambda$

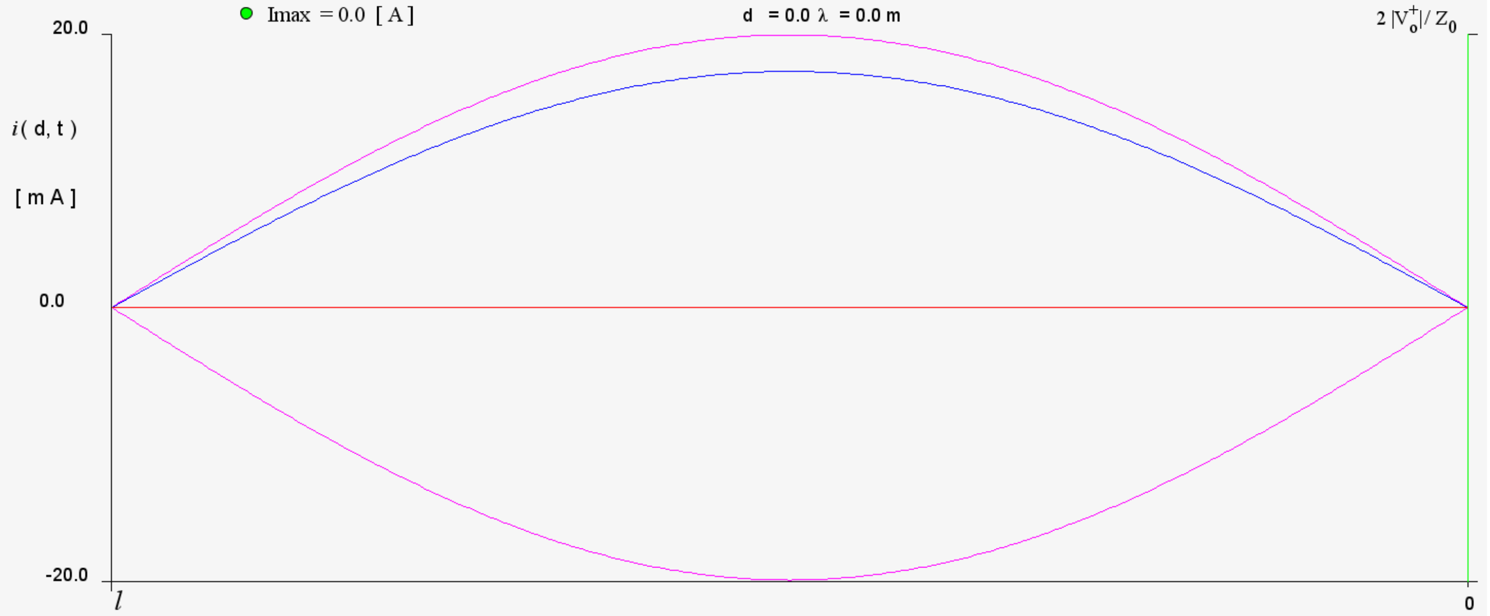


$d = 1.0\,\lambda$

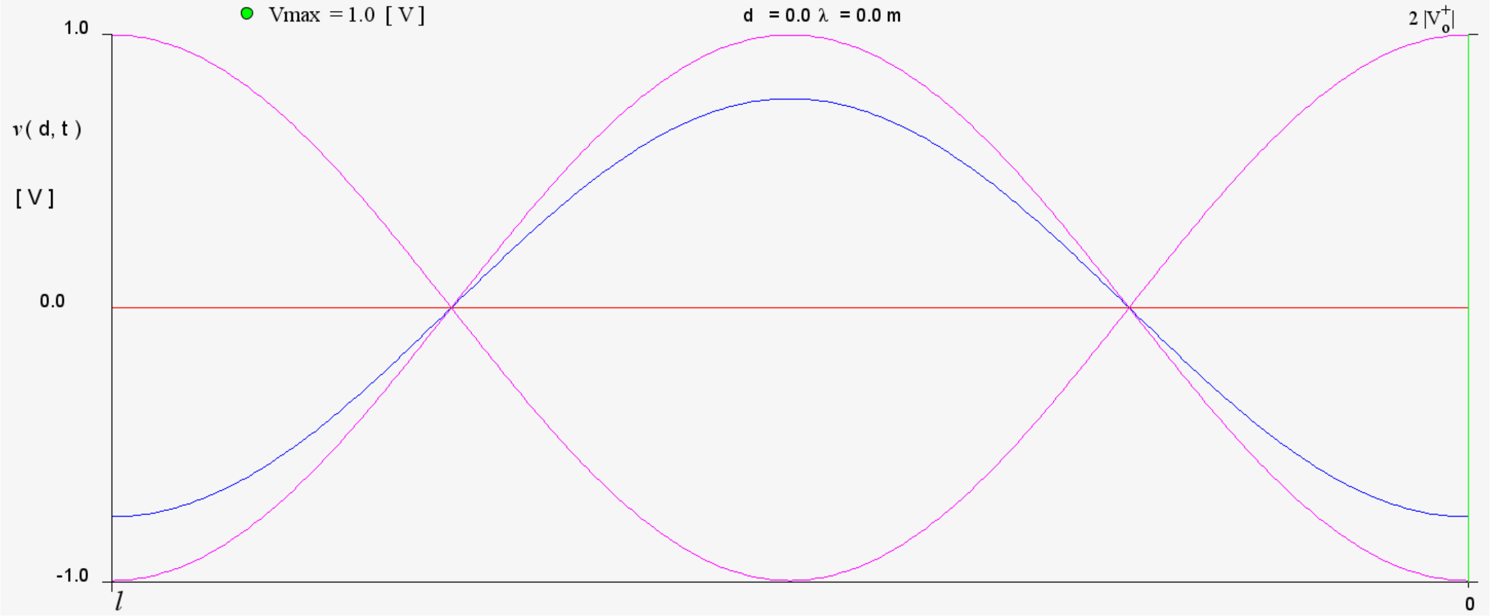
## Reflection coefficient

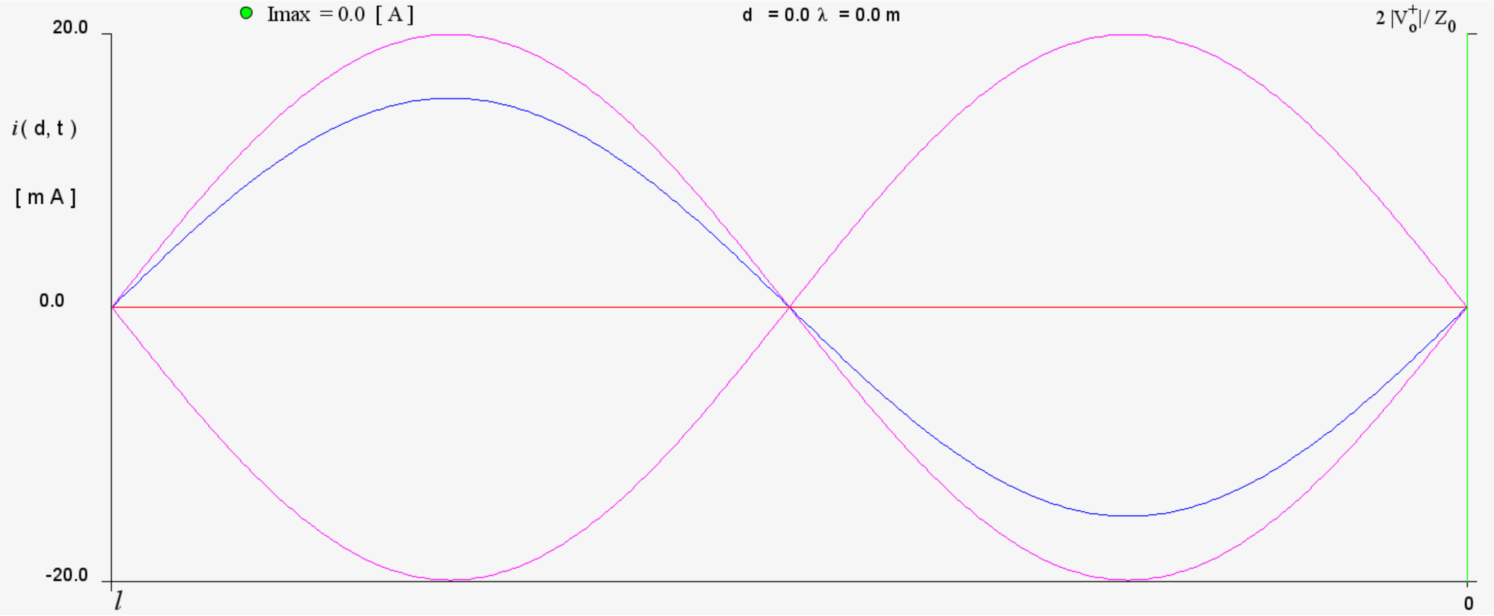
# Line with open circuit load $Z_0 = 50\,\Omega$ $l = 0.5\,\lambda$



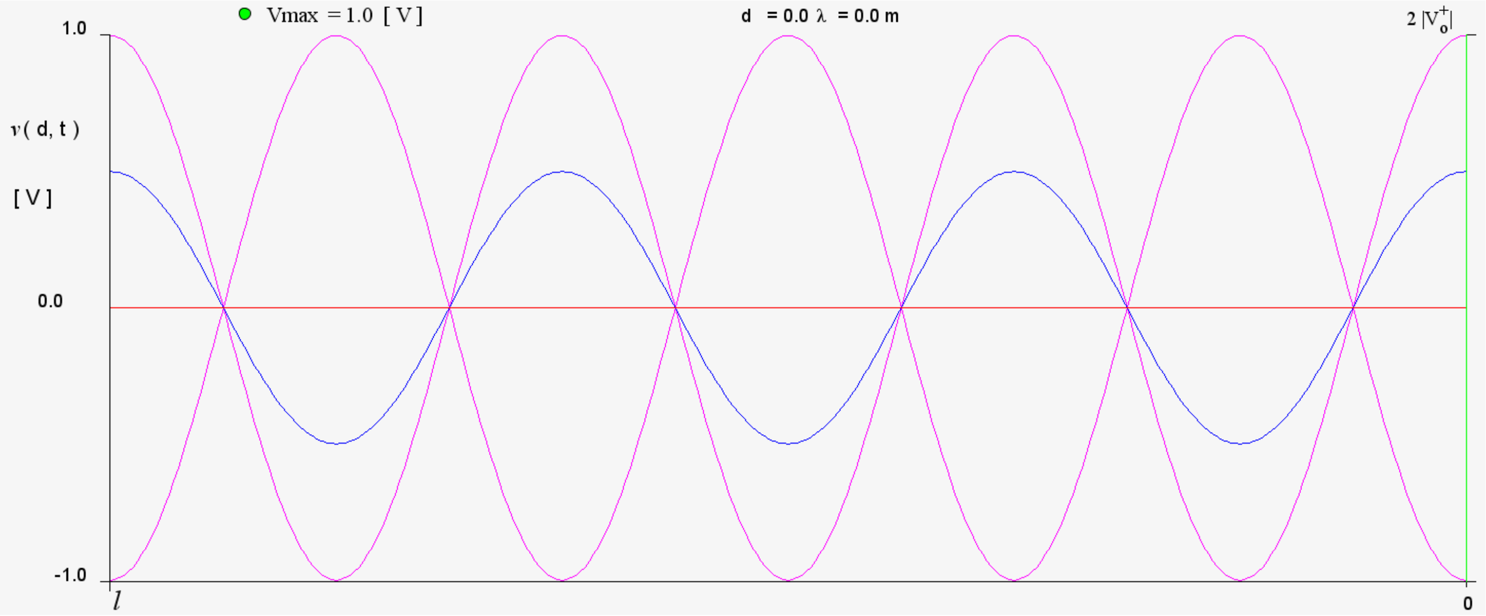


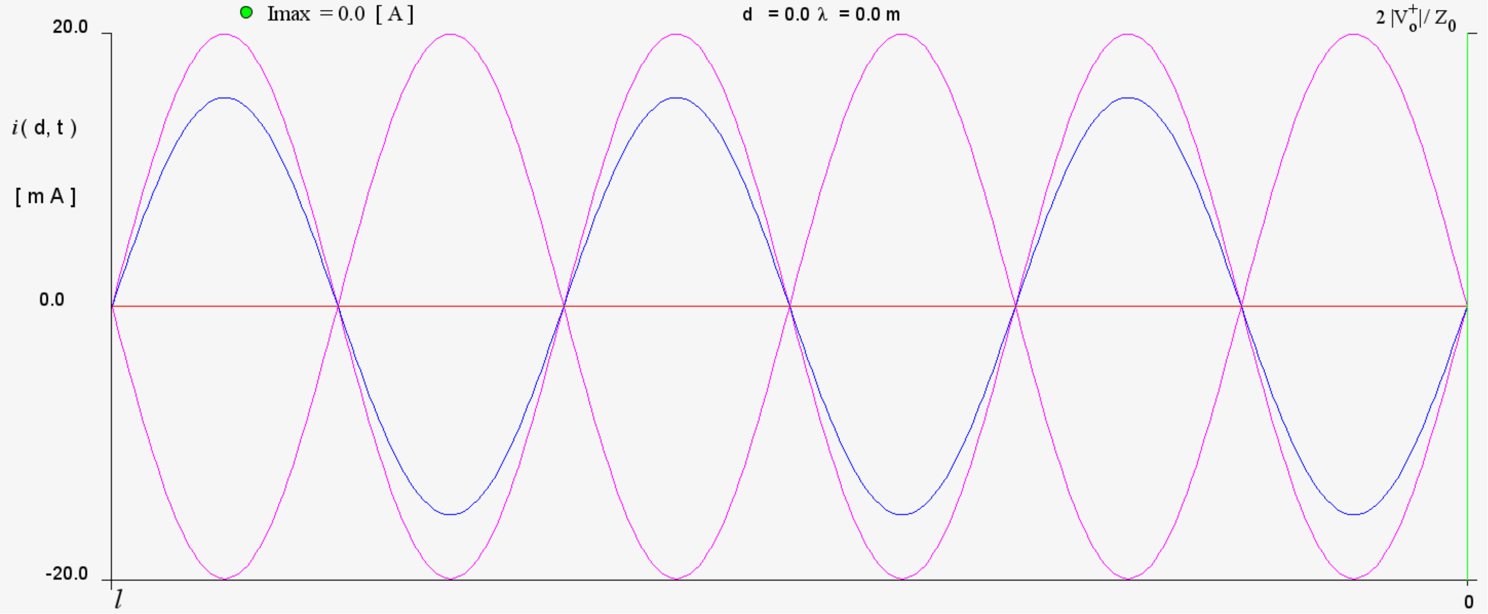
# Line with open circuit load $Z_0 = 50\,\Omega$ $l = 1.0\,\lambda$



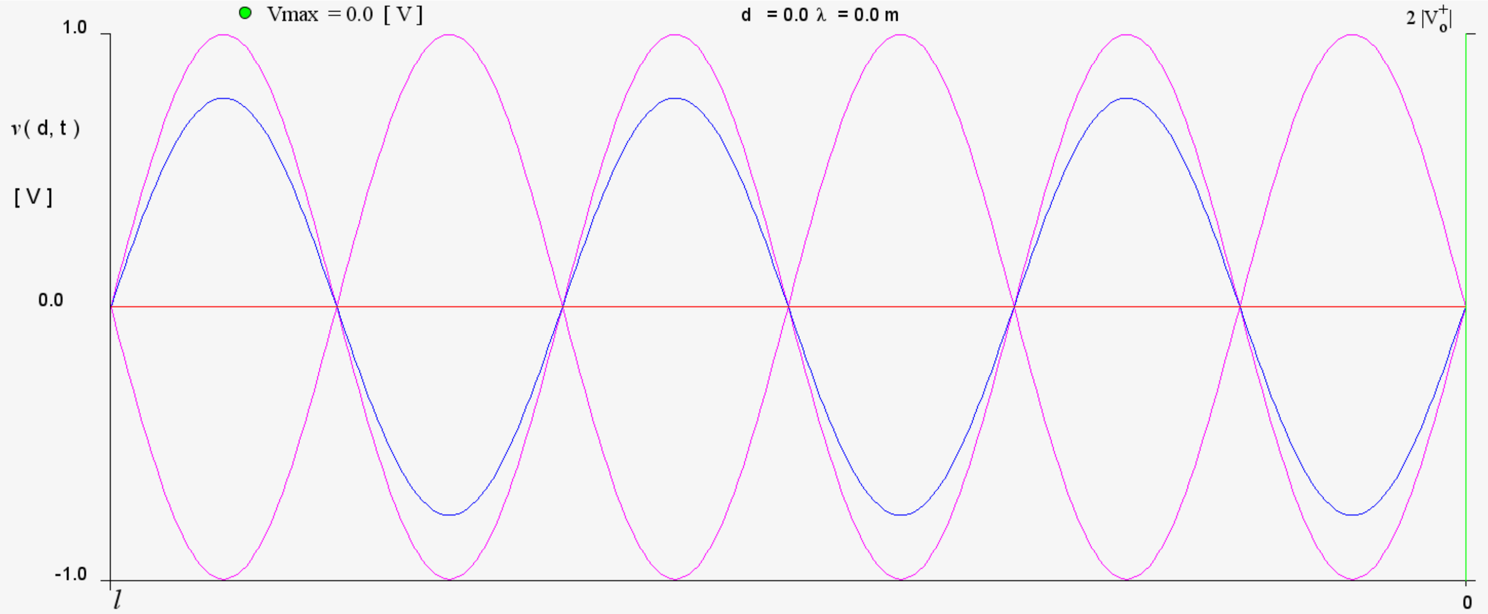


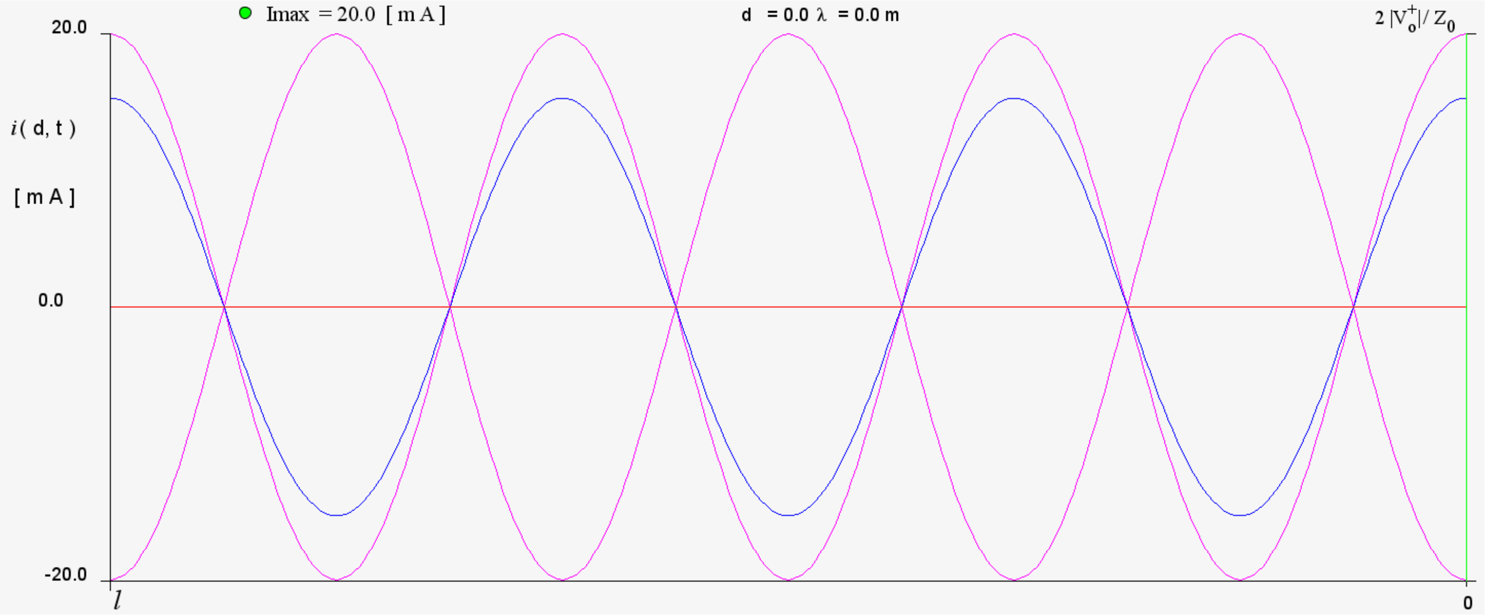
# Line with open circuit load $Z_0 = 50\,\Omega$ $l = 3.0\,\lambda$



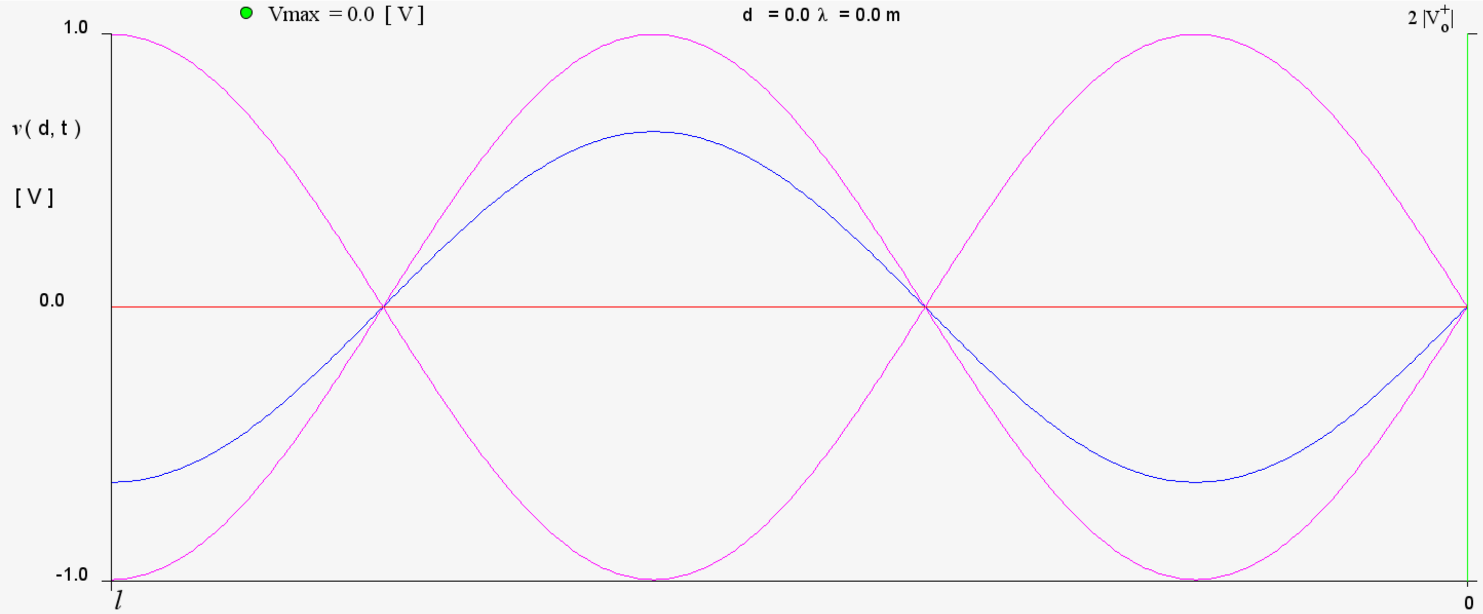


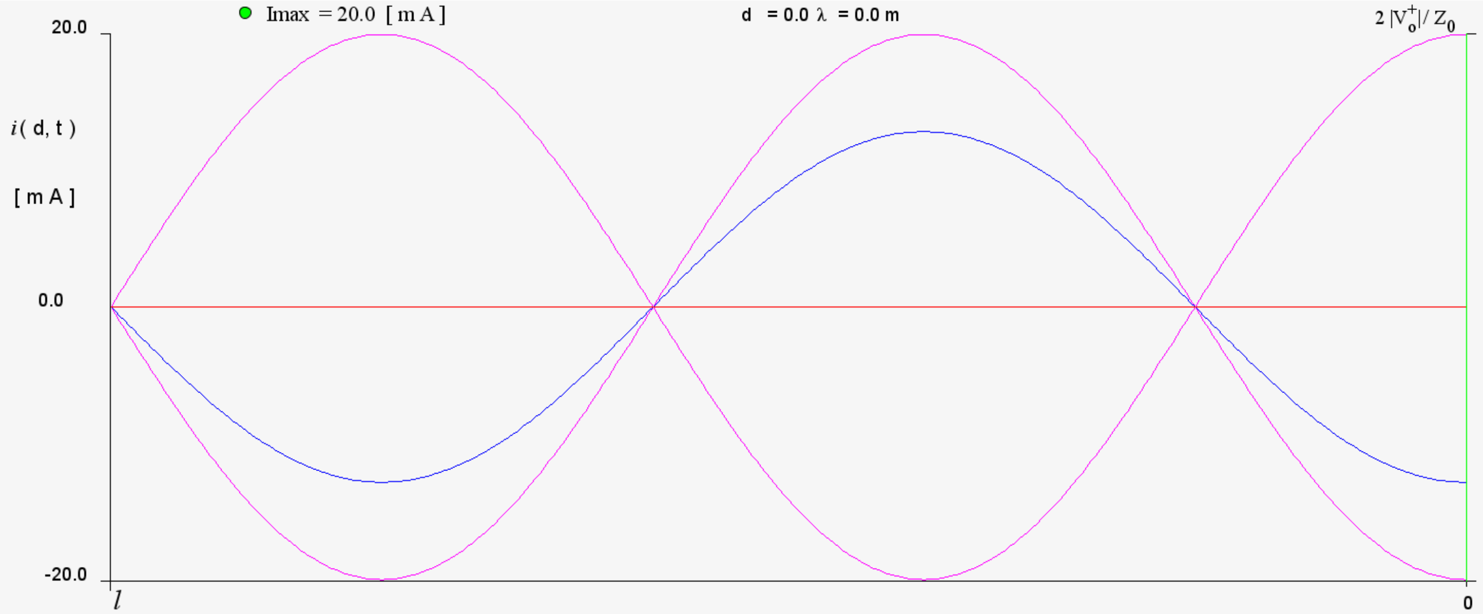
# Line with short circuit load $Z_0 = 50\,\Omega$ $l = 3.0\,\lambda$





# short circuit load, open circuit input $Z_0 = 50\,\Omega$ $l = 1.25\,\lambda$





**These explorations show that periodicity of line properties are established by the reflection coefficient which repeats every $\lambda/2$.**

**For a given length of line we can identify resonant modes (complete standing waves) for frequencies which correspond to multiples of $\lambda/2$, when the ends of the line are both *open circuits* or *short circuits*.**

**resonant frequency**

$$\boxed{\omega = \frac{\pi v}{\ell} n}$$

$$\boxed{f = \frac{v}{2\ell} n \ \mathrm{Hz}}$$

**resonant wavelength**

$$\boxed{\lambda = \frac{v}{f} = \frac{2\ell}{n}}$$

$$\downarrow$$

**implying that resonances occur at frequencies for which the physical length corresponds to an integer number of $\lambda/2$.**

$$\boxed{\ell = n\frac{\lambda}{2}}$$

**Consider a line section open circuited at both ends. The current is expressed by forward and reflected waves as**

$$I(z,t) = \frac{f(t - \frac{z}{v})}{Z_o} - \frac{g(t + \frac{z}{v})}{Z_o}$$

**with zero boundary conditions at the ends**

**waveforms are the same**

$$I(0,t) = \frac{f(t)}{Z_o} - \frac{g(t)}{Z_o} = 0 \quad \longrightarrow \quad g(t) = f(t)$$

$$I(l,t) = \frac{f(t - \frac{\ell}{v})}{Z_o} - \frac{g(t + \frac{\ell}{v})}{Z_o} = 0 \quad \longrightarrow \quad f(t - \frac{\ell}{v}) = f(t + \frac{\ell}{v})$$

$$f(t) = f(t + \frac{2\ell}{v})$$

**periodicity**

**period** $T = \frac{2\ell}{v}$

**fundamental frequency** $\omega_o = \frac{2\pi}{T} = \frac{\pi v}{\ell}$

## From Fourier analysis

$$f(t) = F_o + \sum_{n=1}^{\infty} F_n \cos(n\omega_o t + \theta_n)$$

## with zero boundary conditions at the ends

$$I(z,t) = \frac{f(t - \frac{z}{v}) - f(t + \frac{z}{v})}{Z_o}$$

$$= \sum_{n=1}^{\infty} \frac{F_n}{Z_o}[\cos(n\omega_o t + \theta_n - n\beta_o z) - \cos(n\omega_o t + \theta_n + n\beta_o z)]$$

**fundamental wavenumber**

$$\boxed{\beta_o \equiv \omega_o / v = \pi / \ell}$$

## In phasor form

$$\tilde{I}(z) = \sum_{n=1}^{\infty} \frac{F_n}{Z_o} e^{j\theta_n} [e^{-jn\beta_o z} - e^{jn\beta_o z}]$$

$$= \sum_{n=1}^{\infty} \frac{F_n}{Z_o} e^{j\theta_n} (-2j) \sin(n\beta_o z)$$

## Back to the time domain

$$I(z,t) = \sum_{n=1}^{\infty} \frac{2F_n}{Z_o} \sin(n\omega_o t + \theta_n) \sin(n\beta_o z)$$

$\boxed{2\sin(A)\sin(B) = \cos(A - B) - \cos(A + B)}$

**compare with original form**

Similarly for the voltage

$$V(z,t) = \sum_{n=1}^{\infty} 2F_n \cos(n\omega_o t + \theta_n)\cos(n\beta_o z)$$

from the phasor form

$$\tilde{V}(z) = \sum_n F_n e^{j\theta_n}[e^{-jn\beta_o z} + e^{jn\beta_o z}]$$

If the transmission line is terminated by a ***short circuit*** and has an ***open circuit*** on the other end then resonant frequencies are obtained when the length $\ell$ corresponds to an odd multiple of λ/4

$$\frac{\lambda}{4} = \frac{2\pi/\beta}{4} = \frac{\pi}{2\beta}$$

resonant condition

$$\ell = \frac{\lambda}{4}(2n+1), \quad n \geq 0$$

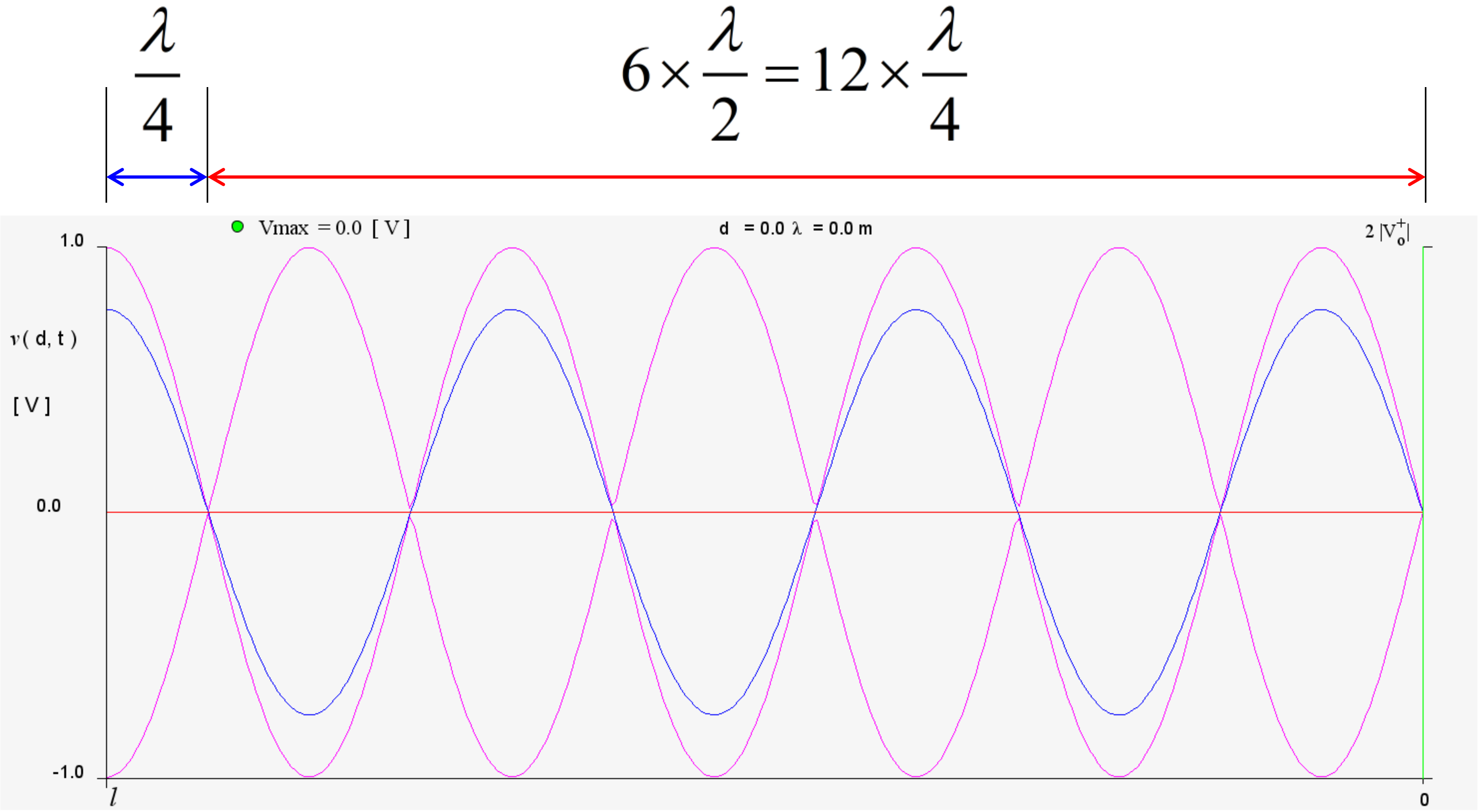
resonant frequency

$$\boxed{\omega = \frac{\pi v}{\ell}(\frac{1}{2}+n)}$$

$$\boxed{f = \frac{v}{2\ell}(\frac{1}{2}+n)}$$

$$\text{for} \quad n \geq 0$$

# Example



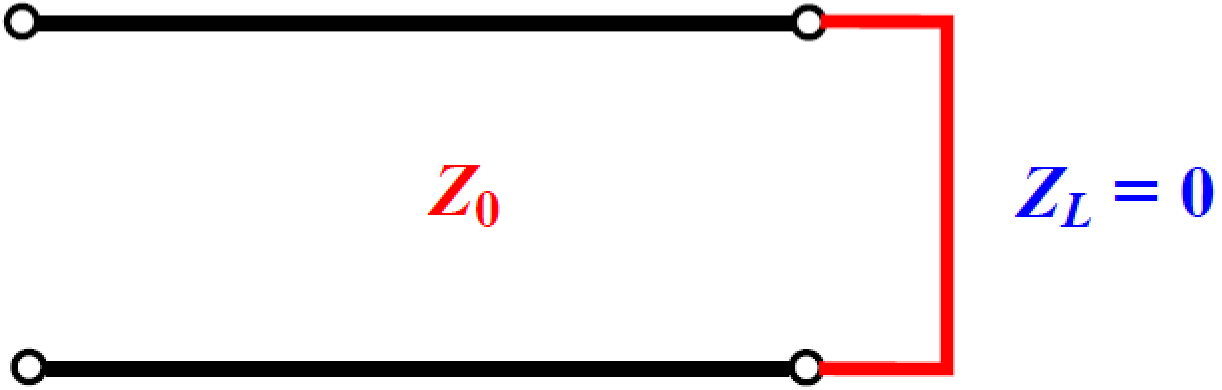
$$\frac{\lambda}{4}$$

$$6\times\frac{\lambda}{2}=12\times\frac{\lambda}{4}$$

open
circuit

short
circuit

# Realize any imaginary impedance with a short-circuited line



$$V(d=0) = V_0^+ e^{j\beta 0}\left[1+\Gamma_L e^{j2\beta 0}\right] = V_0^+\left[1+\Gamma_L\right] = 0$$

$$\Rightarrow \quad \Gamma_L = -1$$

$$\Gamma_L = \frac{V_0^-}{V_0^+} \quad \Rightarrow \quad V_0^- = -V_0^+$$

## short-circuited line

**line voltage**

$$V\left(d\right)=V_0^+e^{j\beta d}+V_0^-e^{-j\beta d}=V_0^+e^{j\beta d}-V_0^+e^{-j\beta d}$$

$$=V_0^+\left[e^{j\beta d}-e^{-j\beta d}\right]=2jV_0^+\sin\left(\beta d\right)$$

**line current**

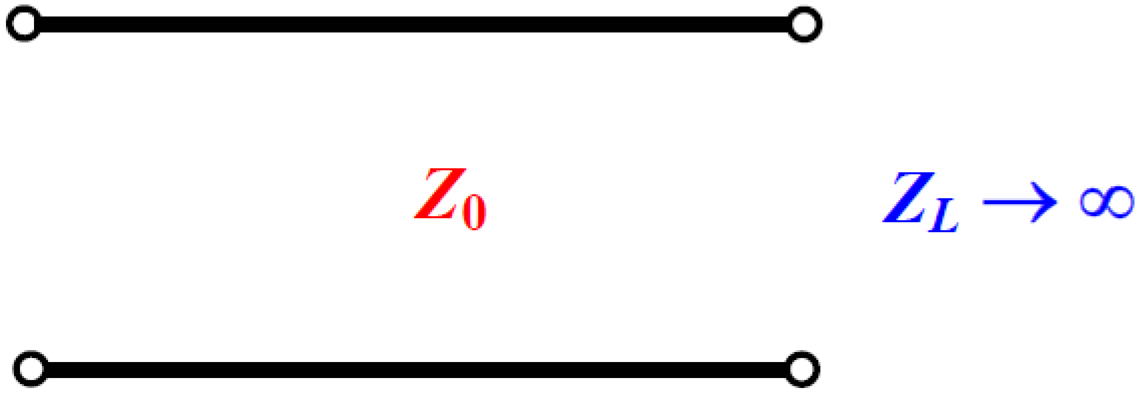
$$I\left(d\right)=\frac{1}{Z_0}\left[V_0^+e^{j\beta d}-V_0^-e^{-j\beta d}\right]=\frac{1}{Z_0}\left[V_0^+e^{j\beta d}+V_0^+e^{-j\beta d}\right]$$

$$=\frac{V_0^+}{Z_0}\left[e^{j\beta d}+e^{-j\beta d}\right]=\frac{2V_0^+}{Z_0}\cos\left(\beta d\right)$$

**line impedance**

$$Z\left(d\right)=\frac{V\left(d\right)}{I\left(d\right)}=\frac{2jV_0^+\sin\left(\beta d\right)}{2V_0^+\cos\left(\beta d\right)/Z_0}=jZ_0\tan\left(\beta d\right)$$

# Realize any imaginary impedance with an open-circuited line



$Z_0$

$Z_L \to \infty$

$$I(d=0)=\frac{V_0^+}{Z_0}e^{j\beta 0}\left[1-\Gamma_L e^{j2\beta 0}\right]=\frac{V_0^+}{Z_0}\left[1-\Gamma_L\right]=0$$

$$\Rightarrow \quad \Gamma_L = 1$$

$$\Gamma_L = \frac{V_0^-}{V_0^+} \quad \Rightarrow \quad V_0^- = V_0^+$$

## open-circuited line

**line voltage**

$$V(d) = V_0^+ e^{j\beta d} + V_0^- e^{-j\beta d} = V_0^+ e^{j\beta d} + V_0^+ e^{-j\beta d}$$

$$= V_0^+ \left[ e^{j\beta d} + e^{-j\beta d} \right] = 2V_0^+ \cos(\beta d)$$

**line current**

$$I(d) = \frac{1}{Z_0}\left[ V_0^+ e^{j\beta d} - V_0^- e^{-j\beta d} \right] = \frac{1}{Z_0}\left[ V_0^+ e^{j\beta d} - V_0^+ e^{-j\beta d} \right]$$

$$= \frac{V_0^+}{Z_0}\left[ e^{j\beta d} - e^{-j\beta d} \right] = \frac{2jV_0^+}{Z_0}\sin(\beta d)$$

**line impedance**

$$Z(d) = \frac{V(d)}{I(d)} = \frac{2V_0^+ \cos(\beta d)}{2jV_0^+ \sin(\beta d)/Z_0} = -j\frac{Z_0}{\tan(\beta d)}$$

Reactive impedances can be realized with transmission lines terminated by a short or by an open circuit. The input impedance of a loss-less transmission line of length L terminated by a short circuit is purely imaginary

$$Z_{in} = jZ_0 \tan(\beta \mathrm{L}) = jZ_0 \tan\left(\frac{2\pi}{\lambda}\mathrm{L}\right) = jZ_0 \tan\left(\frac{2\pi f}{v_p}\mathrm{L}\right)$$

For a specified frequency $f$, any reactance value (positive or negative!) can be obtained by changing the length of the line from $0$ to $\lambda/2$. An inductance is realized for $\mathrm{L} < \lambda/4$ (positive tangent) while a capacitance is realized for $\lambda/4 < \mathrm{L} < \lambda/2$ (negative tangent).

When $\mathrm{L} = 0$ and $\mathrm{L} = \lambda/2$ the tangent is zero, and the input impedance corresponds to a short circuit. However, when $\mathrm{L} = \lambda/4$ the tangent is infinite and the input impedance corresponds to an open circuit.

Since the tangent function is periodic, the same impedance behavior of the impedance will repeat identically for each additional line increment of length $\lambda/2$. A similar periodic behavior is also obtained when the length of the line is fixed and the frequency of operation is changed.

At zero frequency (infinite wavelength), the short circuited line behaves as a short circuit for any line length. When the frequency is increased, the wavelength shortens and one obtains an inductance for $L < \lambda/4$ and a capacitance for $\lambda/4 < L < \lambda/2$, with an open circuit at $L = \lambda/4$ and a short circuit again at $L = \lambda/2$.

Note that the frequency behavior of lumped elements is very different. Consider an ideal inductor with inductance $L$ assumed to be constant with frequency, for simplicity. At zero frequency the inductor also behaves as a short circuit, but the reactance varies monotonically and linearly with frequency as

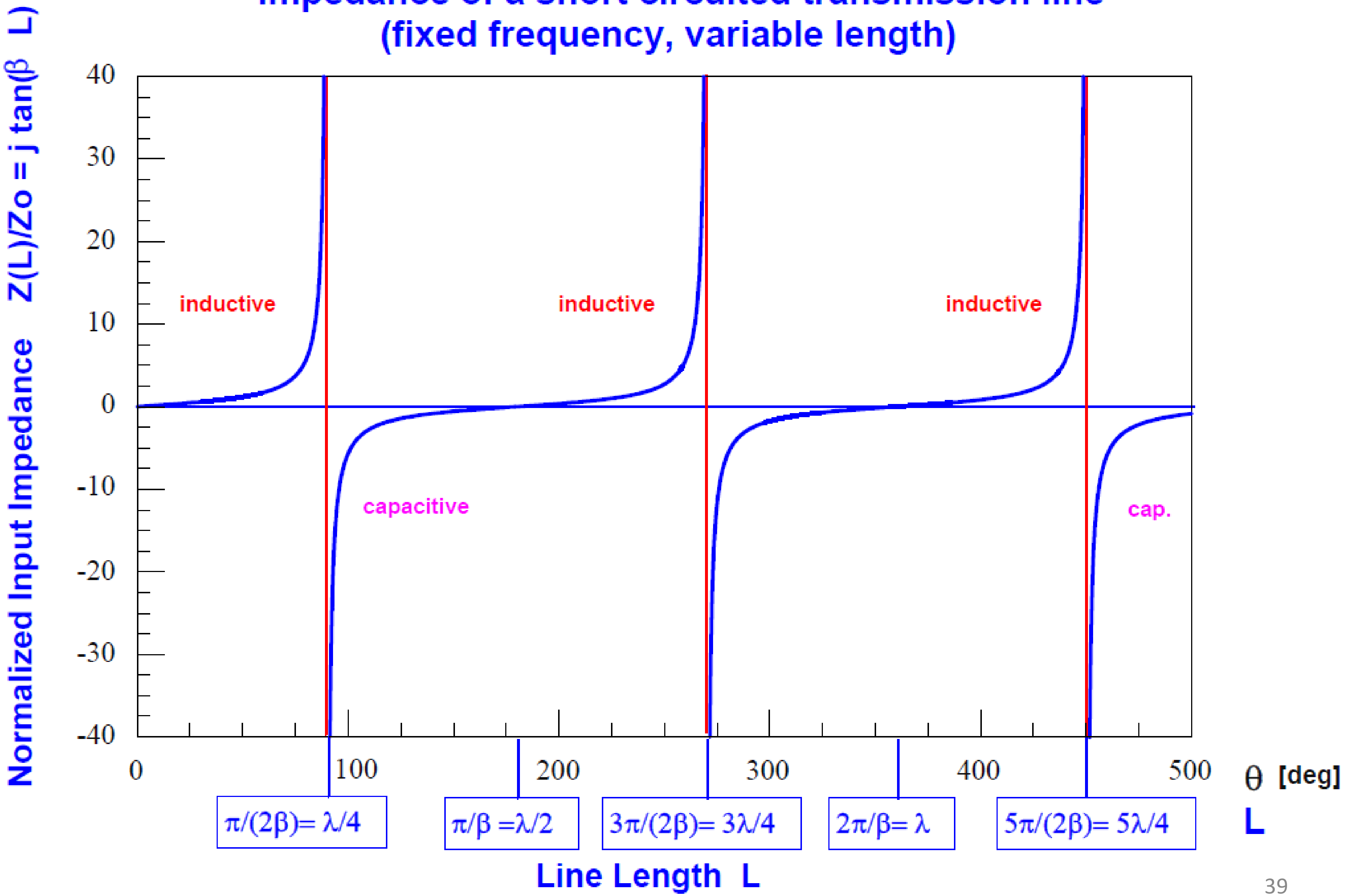
$$X = \omega L \quad \text{(always an inductance)}$$

# Short circuited transmission line – Fixed frequency

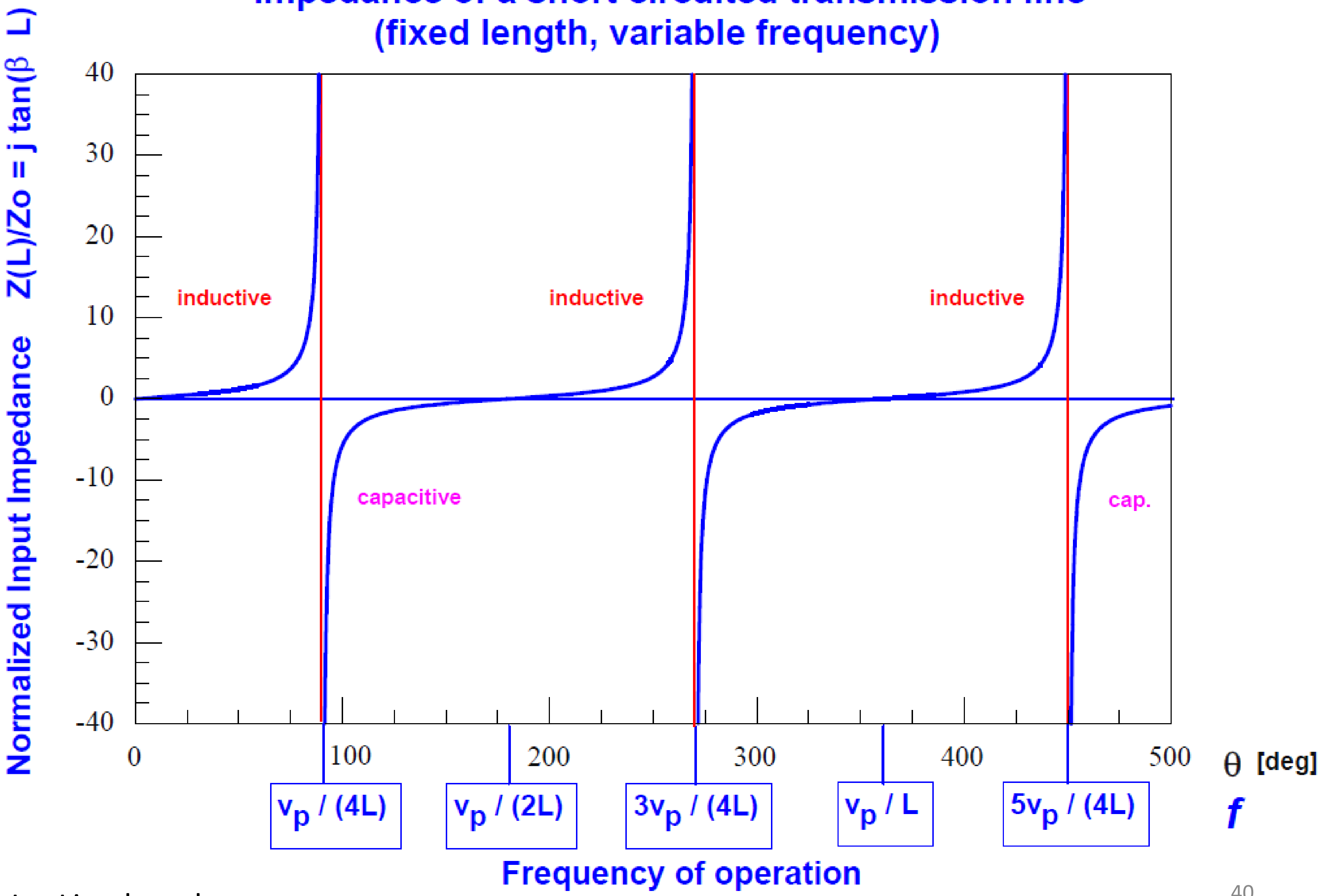
$L$

| | | |
|---|---|---|
| $L = 0$ | $Z_{in} = 0$ | short circuit |
| $0 < L < \frac{\lambda}{4}$ | $\mathrm{Im}\{Z_{in}\} > 0$ | inductance |
| $L = \frac{\lambda}{4}$ | $Z_{in} \to \infty$ | open circuit |
| $\frac{\lambda}{4} < L < \frac{\lambda}{2}$ | $\mathrm{Im}\{Z_{in}\} < 0$ | capacitance |
| $L = \frac{\lambda}{2}$ | $Z_{in} = 0$ | short circuit |
| $\frac{\lambda}{2} < L < \frac{3\lambda}{4}$ | $\mathrm{Im}\{Z_{in}\} > 0$ | inductance |
| $L = \frac{3\lambda}{4}$ | $Z_{in} \to \infty$ | open circuit |
| $\frac{3\lambda}{4} < L < \lambda$ | $\mathrm{Im}\{Z_{in}\} < 0$ | capacitance |

# Impedance of a short circuited transmission line (fixed frequency, variable length)



# Impedance of a short circuited transmission line
# (fixed length, variable frequency)



L = Line length

For a transmission line of length $\mathrm{L}$ terminated by an open circuit, the input impedance is again purely imaginary

$$Z_{in} = -j\frac{Z_0}{\tan(\beta\mathrm{L})} = -j\frac{Z_0}{\tan\left(\dfrac{2\pi}{\lambda}\mathrm{L}\right)} = -j\frac{Z_0}{\tan\left(\dfrac{2\pi f}{v_p}\mathrm{L}\right)}$$

We can also use the open circuited line to realize any reactance, but starting from a capacitive value when the line length is very short.

Note once again that the frequency behavior of a corresponding lumped element is different. Consider an ideal capacitor with capacitance $C$ assumed to be constant with frequency. At zero frequency the capacitor behaves as an open circuit, but the reactance varies monotonically and linearly with frequency as

$$X = \frac{1}{\omega C}$$ (always a capacitance)

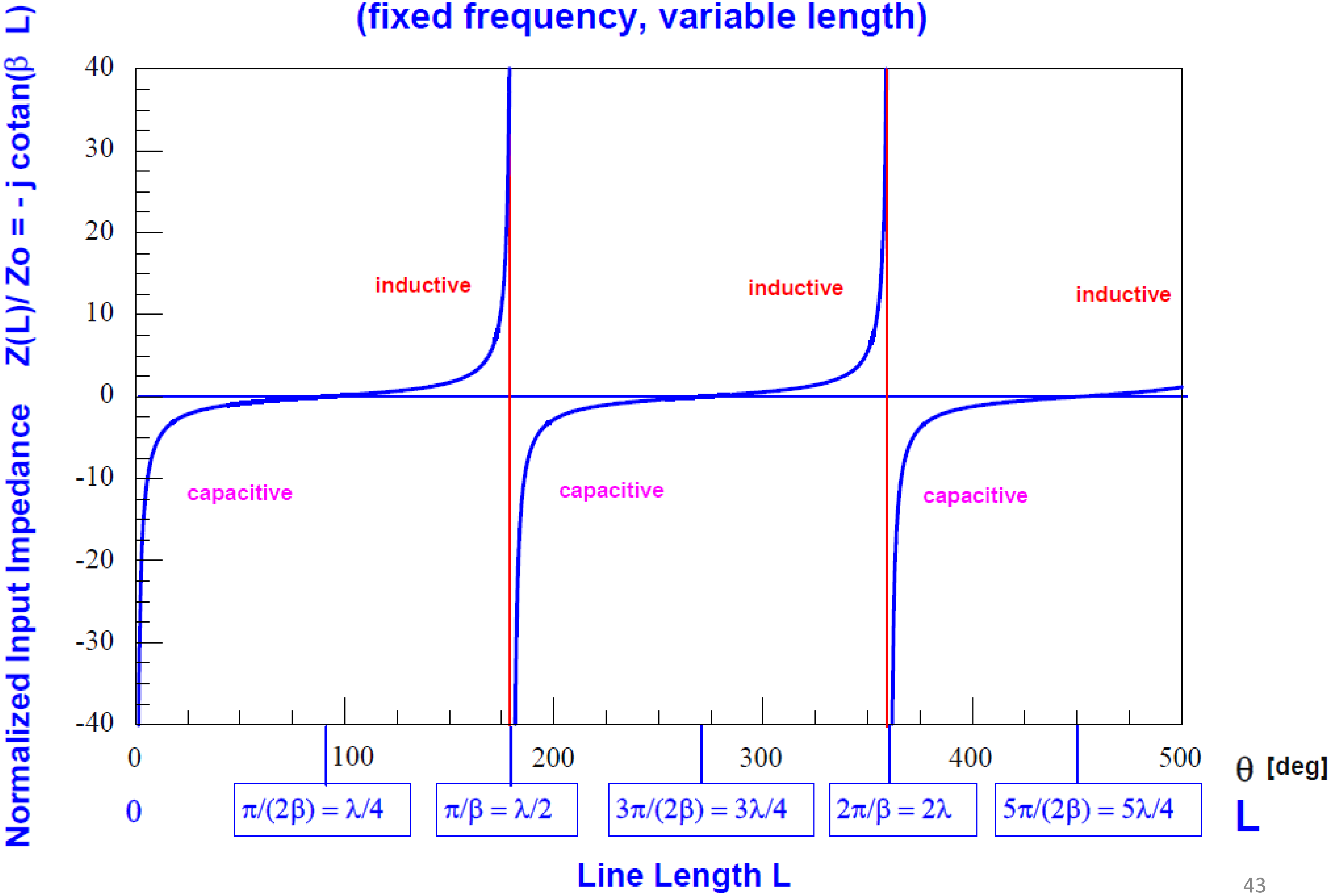
# Open circuit transmission line – Fixed frequency

$L$

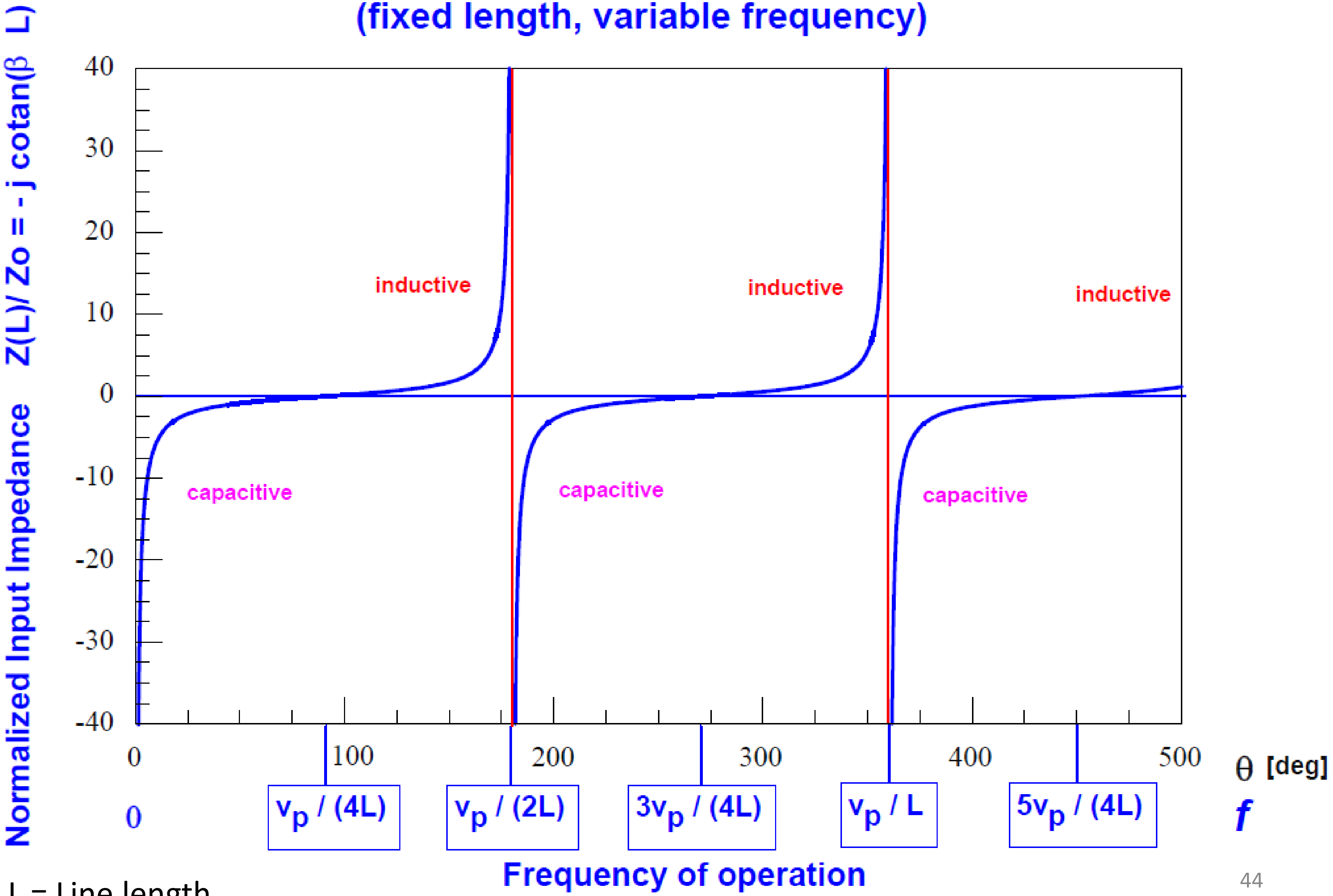
| | | |
|---|---|---|
| $L = 0$ | $Z_{in} \to \infty$ | open circuit |
| $0 < L < \frac{\lambda}{4}$ | $\mathrm{Im}\{Z_{in}\} < 0$ | capacitance |
| $L = \frac{\lambda}{4}$ | $Z_{in} = 0$ | short circuit |
| $\frac{\lambda}{4} < L < \frac{\lambda}{2}$ | $\mathrm{Im}\{Z_{in}\} > 0$ | inductance |
| $L = \frac{\lambda}{2}$ | $Z_{in} \to \infty$ | open circuit |
| $\frac{\lambda}{2} < L < \frac{3\lambda}{4}$ | $\mathrm{Im}\{Z_{in}\} < 0$ | capacitance |
| $L = \frac{3\lambda}{4}$ | $Z_{in} = 0$ | short circuit |
| $\frac{3\lambda}{4} < L < \lambda$ | $\mathrm{Im}\{Z_{in}\} > 0$ | inductance |

...

# Impedance of an open circuited transmission line
## (fixed frequency, variable length)



# Impedance of an open circuited transmission line (fixed length, variable frequency)



L = Line length

## You can also use

$$Y_o \equiv \frac{1}{Z_o} \quad \text{Characteristic admittance.}$$

## Short circuited line

### Input Impedance

$$Z(l) = jZ_o \tan(\beta l)$$

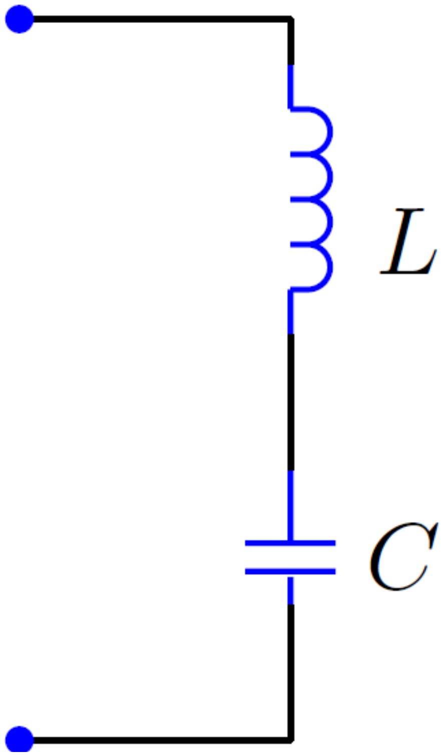
### Input Admittance

$$Y(l) = \frac{1}{Z(l)} = \frac{1}{jZ_o \tan(\beta l)} = -jY_o \cot(\beta l)$$

# Resonant circuits

$$\omega = \frac{1}{\sqrt{LC}} \equiv \omega_o$$

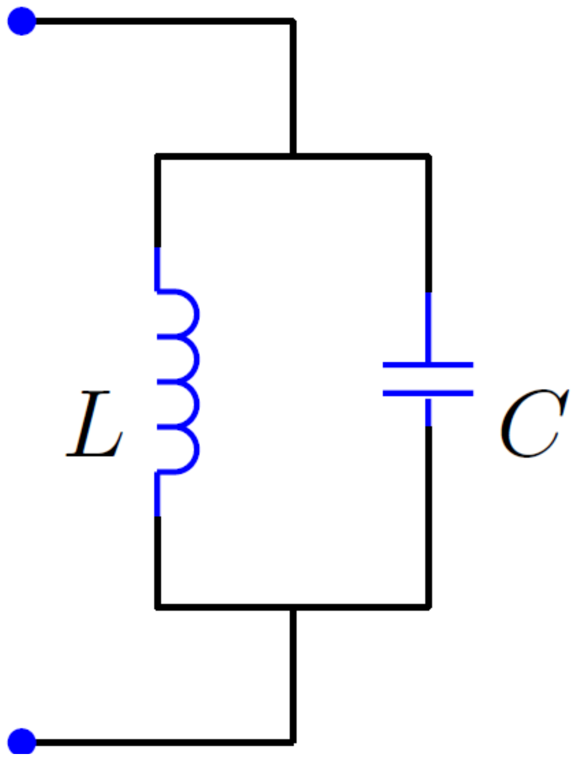
Series



$$Z_s = j(\omega L - \frac{1}{\omega C})$$

**short circuit at resonance**

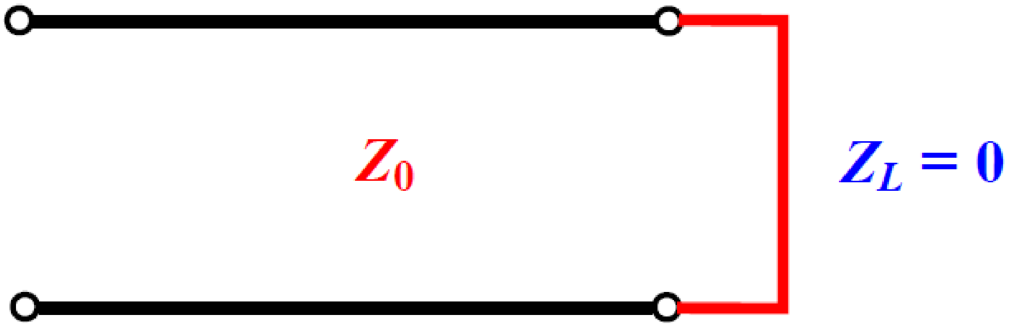
Parallel



$$Y_p = j(\omega C - \frac{1}{\omega L})$$

**parallel circuit at resonance**

# Resonant circuits



A short line ***stub*** is equivalent to an open circuit when the length is an odd multiple of λ/4 with resonant frequencies

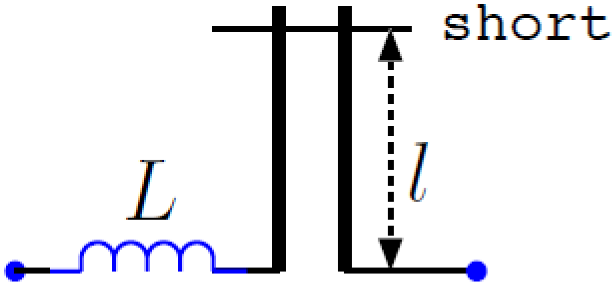
$$f = \frac{v}{2\ell}(\frac{1}{2} + n) \;\; \text{for} \;\; n = 0, 1, 2, 3, \cdots$$

A short line ***stub*** is equivalent to a short circuit when the length is an even multiple of λ/4 (integer number of λ/2) with resonant frequencies

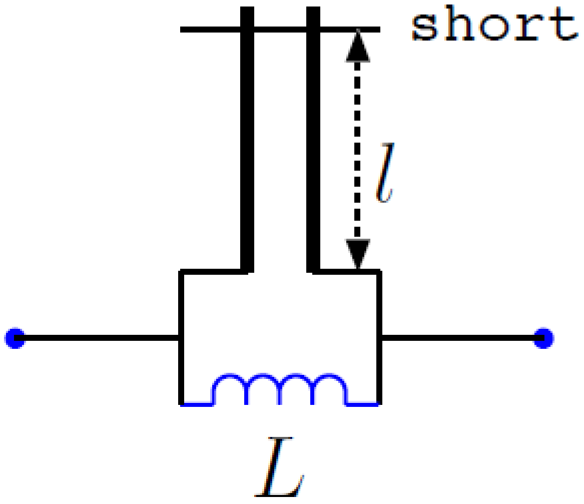
$$f = \frac{v}{2\ell} n \;\; \text{for} \;\; n = 1, 2, 3, \cdots$$

# You can build circuits with short TL stubs as reactive elements

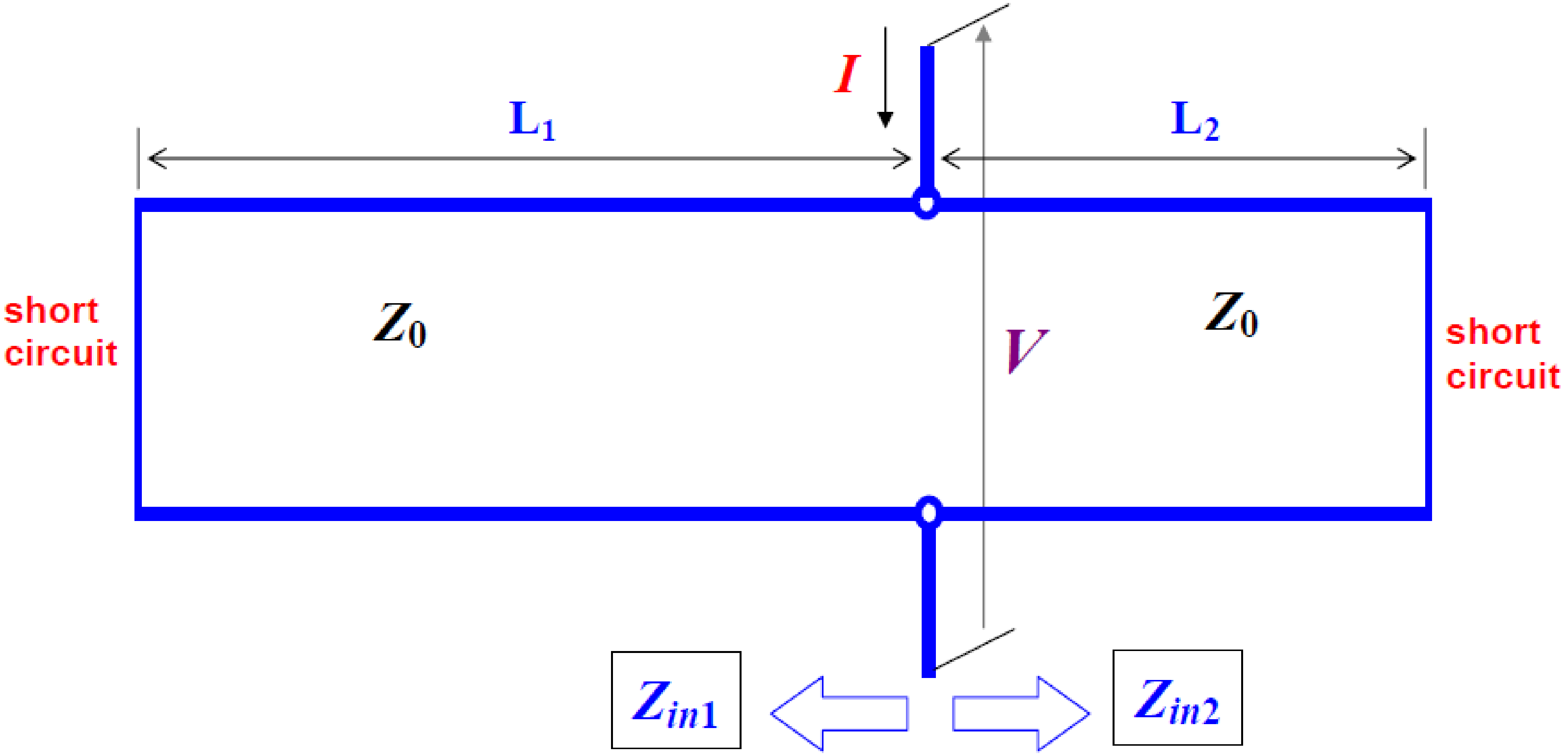
Series network:



Parallel network:



It is possible to realize resonant circuits by using transmission lines as reactive elements. For instance, consider the circuit below realized with lines having the same characteristic impedance:



$$Z_{in1} = j\,Z_0 \tan\left(\beta L_1\right) \qquad Z_{in2} = j\,Z_0 \tan\left(\beta L_2\right)$$

The circuit is resonant if $L_1$ and $L_2$ are chosen such that an inductance and a capacitance are realized.

A resonance condition is established when the total input impedance of the parallel circuit is infinite or, equivalently, when the input admittance of the parallel circuit is zero

$$\frac{1}{j Z_0 \tan\left(\beta_r L_1\right)} + \frac{1}{j Z_0 \tan\left(\beta_r L_2\right)} = 0$$

or

$$\tan\left(\frac{\omega_r}{v_p} L_1\right) = -\tan\left(\frac{\omega_r}{v_p} L_2\right) \qquad \text{with} \qquad \beta_r = \frac{2\pi}{\lambda_r} = \frac{\omega_r}{v_p}$$

Since the tangent is a periodic function, there is a multiplicity of possible resonant angular frequencies $\omega_r$ that satisfy the condition above. The values can be found by using a numerical procedure to solve the trascendental equation above.